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K-BESSEL FUNCTIONS IN TWO VARIABLES

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The Bessel-Muirhead hypergeometric system (or ${}_0F_1$-system) in two variables (and three variables) is solved using symmetric series, with an explicit formula for coefficients, in order to express the K-Bessel function as a linear combination of the J-solutions. Limits of this method and suggestions for generalizations to a higher rank are discussed.

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1. Introduction. The Bessel functions (of the first kind) defined on the space of real symmetric matrices appeared in the work of James [5] as an ingredient in the expression of some densities in multivariate statistics. At the same time, more systematic treatment was done by Herz [4]. In [8], Muirhead proved that they are solutions of a system of differential operators which will be designated here as Bessel-Muirhead operators following [6]. We can see [1, 3] for the generalization of this set of functions to a Jordan algebra. In what follows, we explicitly write down a fundamental set of solutions when the rank equals 2 or 3. Our approach is slightly different from [7] in the final form of the coefficients. Then (and this is our main result), we express the K-Bessel function defined in this context as a linear combination of the J-solutions in the rank-2 case, so answering a question in [4].

Definition 1.1. Bessel-Muirhead operators are defined by

$$B_i = x_i \frac{\partial^2}{\partial x_i^2} + (\nu+1)\frac{\partial}{\partial x_i} + 1 + \frac{d}{2}\sum_{j\neq i}\frac{1}{x_i - x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right), \quad 1 \le i \le r, \tag{1.1}$$

where r is the rank of the system. A symmetric function f is said to be a Bessel function if it is a solution of $B_i f = 0$, $i = 1, 2, \dots, r$.

Denote by $t_1, t_2, \dots, t_r$ the elementary symmetric functions, that is,

$$t_p = \sum_{1\le i_1 \le i_2 \le \cdots \le i_p \le r} x_{i_1} x_{i_2} \cdots x_{i_p} \tag{1.2}$$

with $t_0 = 1$ and $t_p = 0$ if $p < 0$ or $p > r$. The Bessel-Muirhead system is then equivalent to the system $Z_k g = 0$, $1 \le k \le r$, (see [1, 5]) where

$$Z_k = \sum_{i,j=1}^{r} A_{ij}^k \frac{\partial^2}{\partial t_i \partial t_j} + \left(\nu + 1 + \frac{r-k}{2} d\right) \frac{\partial}{\partial t_k} + \delta_k^1, \tag{1.3}$$

$$A_{ij}^k = \begin{cases} t_{i+j-k} & \text{if } i,j \ge k, \\ -t_{i+j-k} & \text{if } i,j < k,\ i+j \ge k, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.4}$$

Here, δ_k^1 is the Kronecker symbol and $g(t_1, t_2, \dots, t_r) = f(x_1, x_2, \dots, x_r)$.

2. Case $r = 2$. In this case, we have $A^1 = \begin{pmatrix} t_1 & t_2 \\ t_2 & 0 \end{pmatrix}$, and $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & t_2 \end{pmatrix}$, and the operators in the modified system (1.3) can be written as follows:

$$\begin{aligned} t_1 Z_1 &= \theta_1 \left(\theta_1 + 2\theta_2 + \nu + \frac{d}{2}\right) + t_1, \\ t_2 Z_2 &= \theta_2 (\theta_2 + \nu) - t_2 \frac{\partial^2}{\partial t_1^2}, \end{aligned} \tag{2.1}$$

where $\theta_1 = t_1(\partial/\partial t_1)$ and $\theta_2 = t_2(\partial/\partial t_2)$. The operators θ_i are used because their action on powers is easily checked by the rule $\theta_i t_i^\alpha = \alpha t_i^\alpha$. Now, putting in the system (2.1) a series of the form $S_{(\lambda_1,\lambda_2)}(t_1, t_2) = \sum_{m_1,m_2 \ge 0} c(m_1, m_2) t_1^{m_1+\lambda_1} t_2^{m_2+\lambda_2}$, we can write the following system of recurrence formulas:

$$\begin{aligned} &(m_1 + \lambda_1)\left(m_1 + 2m_2 + \lambda_1 + 2\lambda_2 + \nu + \frac{d}{2}\right) c(m_1, m_2) + c(m_1 - 1, m_2) = 0, \\ &\quad (m_2 + \lambda_2)(m_2 + \lambda_2 + \nu) c(m_1, m_2) \\ &\qquad\qquad - (m_1 + 2 + \lambda_1)(m_1 + 1 + \lambda_1) c(m_1 + 2, m_2 - 1) = 0. \end{aligned} \tag{2.2}$$

Then, we first obtain the system of critical exponents (λ_1, λ_2) when $(m_1, m_2) = (0, 0)$;

$$\begin{aligned} \lambda_1 \left(\lambda_1 + 2\lambda_2 + \nu + \frac{d}{2}\right) &= 0, \\ \lambda_2 (\lambda_2 + \nu) &= 0, \end{aligned} \tag{2.3}$$

which admits, as solutions, the set

$$\Lambda_{2,\nu} = \left\{(0,0); (0,-\nu); \left(-\nu - \frac{d}{2}, 0\right); \left(\nu - \frac{d}{2}, -\nu\right)\right\}. \tag{2.4}$$

Now, with the help of the second equation of (2.2), we can express $c(m_1, m_2)$ in terms of $c(m_1 + 2m_2, 0)$ and then in terms of $c(0,0)$ thanks to the first equation of (2.2). We obtain

$$c(m_1,m_2)=\frac{(-1)^{m_1+2m_2}c(0,0)}{(1+\lambda_1)_{m_1}(1+\lambda_2)_{m_2}(1+\lambda_2+\nu)_{m_2}(1+\lambda_1+2\lambda_2+\nu+d/2)_{m_1+2m_2}}. \tag{2.5}$$

Theorem 2.1. *For generic ν (i.e., $\nu\notin Z$ and $\nu\pm d/2\notin Z$), the series $S_{(\lambda_1,\lambda_2)}(t_1,t_2)$ with $c(m_1,m_2)$ as in (2.5) and $(\lambda_1,\lambda_2)\in\Lambda_{2,\nu}$ form a fundamental set of solutions of system (2.1).*

Remark 2.2. The convergence of this series is obvious.

3. Case $r=3$. As in the previous case, we have

$$A^1=\begin{pmatrix}t_1&t_2&t_3\\t_2&t_3&0\\t_3&0&0\end{pmatrix},\qquad A^2=\begin{pmatrix}-1&0&0\\0&t_2&t_3\\0&t_3&0\end{pmatrix},\qquad A^3=\begin{pmatrix}0&-1&0\\-1&-t_1&0\\0&0&t_3\end{pmatrix}, \tag{3.1}$$

the modified system (1.3) takes the form

$$\begin{aligned}t_1Z_1&=\theta_1(\theta_1+2\theta_2+2\theta_3+\nu+d)+t_1+t_1t_3\frac{\partial^2}{\partial t_2^2},\\ t_2Z_2&=\theta_2\left(\theta_2+2\theta_3+\nu+\frac{d}{2}\right)-t_2\frac{\partial^2}{\partial t_1^2},\\ t_3Z_3&=\theta_3(\theta_3+\nu)-2t_3\frac{\partial^2}{\partial t_1\partial t_2}-t_1t_3\frac{\partial^2}{\partial t_2^2},\end{aligned} \tag{3.2}$$

and we obtain the following system of recurrence formulas for the coefficients of a series of the form $\sum_{m_1,m_2,m_3\ge 0}c(m_1,m_2,m_3)t_1^{m_1+\lambda_1}t_2^{m_2+\lambda_2}t_3^{m_3+\lambda_3}$:

$$\begin{aligned}&I_1(\underline{\lambda}+\underline{m})c(\underline{m})+c(\underline{m}-e_1)+(m_2+2+\lambda_2)(m_2+1+\lambda_2)c(\underline{m}-e_1+2e_2-e_3)=0,\\ &I_2(\underline{\lambda}+\underline{m})c(\underline{m})-(m_1+2+\lambda_1)(m_1+1+\lambda_1)c(\underline{m}+2e_1-e_2)=0,\\ &I_3(\underline{\lambda}+\underline{m})c(\underline{m})-2(m_1+1+\lambda_1)(m_2+1+\lambda_2)c(\underline{m}+e_1+e_2-e_3)\\ &\qquad-(m_2+2+\lambda_2)(m_2+1+\lambda_2)c(\underline{m}-e_1+2e_2-e_3)=0,\end{aligned} \tag{3.3}$$

where $\underline{m}=(m_1,m_2,m_3)$, $\underline{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$, $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$, and

$$\begin{aligned}I_1(\underline{s})&=s_1(s_1+2s_2+2s_3+\nu+d),\\ I_2(\underline{s})&=s_2\left(s_2+2s_3+\nu+\frac{d}{2}\right),\\ I_3(\underline{s})&=s_3(s_3+\nu).\end{aligned} \tag{3.4}$$

The critical exponents set $\Lambda_{3,\nu}$ is obtained after solving $I_1(\underline{\lambda}) = I_2(\underline{\lambda}) = I_3(\underline{\lambda}) = 0$. Then we have

$$\Lambda_{3,\nu} = \begin{cases} (0,0,0); & (0,0,-\nu), \\ (-\nu-d,0,0); & (\nu-d,0,-\nu), \\ \left(0,-\nu-\frac{d}{2},0\right); & \left(0,\nu-\frac{d}{2},-\nu\right), \\ \left(\nu,-\nu-\frac{d}{2},0\right); & \left(-\nu,\nu-\frac{d}{2},-\nu\right). \end{cases} \tag{3.5}$$

Now, by the second equation of (3.3), we can express $c(\underline{m})$ in terms of $c(m_1+2m_2,0,m_3)$. The third equation of (3.3) allows us to express $c(m_1+2m_2,0,m_3)$ by $c(m_1+2m_2+3m_3,0,0)$, and finally, by the first equation, we regress to $c(0,0,0)$. After all reductions, we obtain

$$\begin{aligned} c(\underline{m}) = {} & \frac{(-1)^{m_1+2m_2+3m_3}c(0)}{(1+\lambda_1)_{m_1}(1+\lambda_2)_{m_2}(1+\lambda_3)_{m_3}(1+\lambda_3+\nu)_{m_3}(1+\lambda_2+2\lambda_3+\nu+d/2)_{m_2+2m_3}} \\ & \times \frac{(1+\lambda_1+2\lambda_2+4\lambda_3+2\nu+d)_{m_1+2m_2+4m_3}}{(1+\lambda_1+2\lambda_2+2\lambda_3+\nu+d)_{m_1+2m_2+3m_3}} \\ & \times \frac{1}{(1+\lambda_1+2\lambda_2+4\lambda_3+2\nu+d)_{m_1+2m_2+3m_3}} \end{aligned} \tag{3.6}$$

and all ingredients to write a theorem like Theorem 2.1.

4. K-Bessel function. As an application, we derive, in the case $r = 2$, the expansion of the K-Bessel function in the previous basis (J-functions) of the Bessel system. Recall the one-variable situation (small letters refer to special functions of one variable); the k-Bessel function can be defined by

$$k_\nu(x) = \int_0^{+\infty} \exp\left(-y-\frac{x}{y}\right) y^{-\nu-1}\,dy. \tag{4.1}$$

If we put

$$j_\nu(x) = {}_0f_1\left(\begin{matrix} - \\ \nu+1 \end{matrix};x\right) = \sum_{n\geq 0} \frac{(-1)^n}{n!(\nu+1)_n} x^n, \tag{4.2}$$

we have the formula

$$k_\nu(x) = \Gamma(-\nu)j_\nu(-x) + \Gamma(\nu)x^{-\nu}j_{-\nu}(-x). \tag{4.3}$$

Recall also the Mellin transform of $k_\nu(x)$,

$$M(k_\nu)(s) = \int_0^{+\infty} k_\nu(x)x^{s-1}\,dx = \Gamma(s)\Gamma(s-\nu). \tag{4.4}$$

Now, we write the two-variable situation in a Jordan algebra context. Take an n-dimensional Jordan algebra A of a rank 2, the generic case is $A = \mathbb{R} \times \mathbb{R}^{n-1}$, endowed with the product

$$x \cdot y = (\xi\eta + \langle u, v \rangle, \xi v + \eta u) \tag{4.5}$$

if $x = (\xi, u)$, $y = (\eta, v)$, and $\langle u, v \rangle = \sum_{1 \le i \le n-1} u_i v_i$. The unit is obviously $e = (1, 0)$. Then we have a Cayley-Hamilton-like theorem $x^2 - 2\xi x + (\xi^2 - \|u\|^2)e = 0$, and we can put $\operatorname{tr}(x) = 2\xi$ and $\det(x) = \xi^2 - \|u\|^2$. We consider the following scalar product on A:

$$(x, y) = \operatorname{tr}(x \cdot y) = 2\xi\eta + 2\langle u, v \rangle. \tag{4.6}$$

We can show that each x has a spectral decomposition $x = x_1\hat{e}_1 + x_2\hat{e}_2$, with $x_1, x_2 \in \mathbb{R}$ and $\{\hat{e}_1, \hat{e}_2\}$ is a pair of primitive strongly orthogonal idempotents. More precisely, $\hat{e}_1 = (1/2, u/2\|u\|)$ and $\hat{e}_2 = (1/2, -u/2\|u\|)$. Observe that $\sigma_x = u/\|u\| \in \mathbf{S}^{n-2}$. Any element y can be decomposed as follows: $y = k \cdot (y_1\hat{e}_1 + y_2\hat{e}_2)$ with $k \in \mathrm{SO}(n-1)$ acting on $\hat{e}_1$, for example, by $k \cdot \hat{e}_1 = (1/2, (1/2)k \cdot \sigma_x)$, where $k \cdot \sigma_x$ is the standard action of $\mathrm{SO}(n-1)$ on $\mathbb{R}^{n-1}$. The scalar product takes the form

$$(x, y) = \frac{1}{2}(x_1 + x_2)(y_1 + y_2) + \frac{1}{2}(x_1 - x_2)(y_1 - y_2)\langle \sigma_x, k \cdot \sigma_x \rangle. \tag{4.7}$$

Now, the K-Bessel function can be defined by

$$K_\nu(x) = \int_\Omega e^{-\operatorname{tr}(y^{-1}) - (x, y)} (\det y)^{\nu - n/2}\, dy, \tag{4.8}$$

where $\Omega = \{x \in A / \operatorname{tr}(x) > 0 \text{ and } \det x > 0\}$ is the cone of positivity of A. After a change of variables, we can show that

$$K_\nu(x) = (\det x)^{-\nu} K_{-\nu}(x). \tag{4.9}$$

So, following [1], where it is proved that K_ν is a solution of a differential system similar to (1.1), we can write

$$\begin{aligned} K_\nu(x) = {} & a_\nu S_{(0,0)}(-t_1, t_2) + b_\nu S_{(0,-\nu)}(-t_1, t_2) \\ & + c_\nu S_{(-\nu-d/2,0)}(-t_1, t_2) + d_\nu S_{(\nu-d/2,-\nu)}(-t_1, t_2) \end{aligned} \tag{4.10}$$

(here, $d = n-2$). According to (4.9), we have $a_\nu = b_{-\nu}$ and $c_\nu = d_{-\nu}$. For suitable ν, the following limit holds (see [2] for more information on Γ_Ω, the gamma function of the cone Ω):

$$\lim_{\substack{x\to 0\\ x\in\Omega}} K_\nu(x) = \Gamma_\Omega(-\nu) = (2\pi)^{(n-2)/2}\Gamma(-\nu)\Gamma\left(-\nu-\frac{n-2}{2}\right), \tag{4.11}$$

so

$$a_\nu = b_{-\nu} = (2\pi)^{(n-2)/2}\Gamma(-\nu)\Gamma\left(-\nu-\frac{n-2}{2}\right) \tag{4.12}$$

according to the behaviour of the solutions $S_{(\lambda_1,\lambda_2)}$. To determine c_ν (and then d_ν), we take $x \neq 0$ on the boundary of Ω. So if $x = 2\xi\hat{e}_1$, then the integral representation of K_ν takes the explicit form

$$\begin{aligned} K_\nu(2\xi\hat{e}_1) = C\int_{SO(n-1)}\int_{y_1>y_2>0} & e^{-(1/y_1+1/y_2+\xi(y_1+y_1))-\xi(y_1-y_2)\langle\sigma_x,k\cdot\sigma_x\rangle} \\ & \times (y_1y_2)^{\nu-n/2}(y_1-y_2)^{n-2}\,dk\,dy_1\,dy_2, \end{aligned} \tag{4.13}$$

where C is a constant (see [2, Theorem VI.2.3, page 104] for the integration formula in polar coordinates in Ω). In the particular case of rank-2 Jordan algebras, we have $C = 2^{2-n/2}\pi^{(n-1)/2}/\Gamma((n-1)/2)$. Now, after integration over $SO(n-1)$, we obtain

$$\begin{aligned} K_\nu(2\xi\hat{e}_1) = C\int_{y_1>y_2>0} & e^{-(1/y_1+1/y_2+\xi(y_1+y_1))}(y_1y_2)^{\nu-n/2} \\ & \times (y_1-y_2)^{n-2}{}_0f_1\left(\genfrac{}{}{0pt}{}{-}{\frac{n-1}{2}};\frac{\xi^2(y_1-y_2)^2}{4}\right)dy_1\,dy_2. \end{aligned} \tag{4.14}$$

Then, the evaluation of the (one variable) Mellin transform of $K_\nu(2\xi\hat{e}_1)$ gives

$$\begin{aligned} M(K_\nu(2(\cdot)\hat{e}_1))(s) &= \int_0^\infty K_\nu(2\xi e_1)\xi^{s-1}\,d\xi \\ &= C\Gamma(s)\int_{y_1>y_2>0} e^{-(1/y_1+1/y_2)}(y_1y_2)^{\nu-n/2}(y_1-y_2)^{n-2} \\ &\quad \times (y_1+y_2)^{-s}{}_2f_1\left(\genfrac{}{}{0pt}{}{\frac{s}{2},\frac{s+1}{2}}{\frac{n-1}{2}};\left(\frac{y_1-y_2}{y_1+y_2}\right)^2\right)dy_1\,dy_2. \end{aligned} \tag{4.15}$$

This last integral can be computed after making the change $y_1 = re^{\theta}$ and $y_2 = re^{-\theta}$ with $r, \theta > 0$; so

$$
\begin{aligned}
M(K_\nu)(s) &= 2^{n-1-s} C\Gamma(s) \int_0^\infty \int_0^\infty e^{2\cosh\theta/r} r^{2\nu-s-1} (\sinh\theta)^{n-2} (\cosh\theta)^{-s} \\
&\qquad \times {}_2f_1\left(\begin{matrix} \frac{s}{2}, \frac{s+1}{2} \\ \frac{n-1}{2} \end{matrix}; (\tanh\theta)^2\right) dr\, d\theta \\
&= 2^{n-1+2(\nu-s)} C\Gamma(s)\Gamma(s-2\nu) \int_0^\infty (\sinh\theta)^{n-2} (\cosh\theta)^{2(\nu-s)} {}_2f_1 \\
&\qquad \times \left(\begin{matrix} \frac{s}{2}, \frac{s+1}{2} \\ \frac{n-1}{2} \end{matrix}; (\tanh\theta)^2\right) d\theta \\
&= 2^{n-1+2(\nu-s)} C \frac{\Gamma(s)\Gamma(s-2\nu)\Gamma(s-\nu+1-n/2)\Gamma((n-1)/2)}{\Gamma(s-\nu+1/2)} {}_2f_1 \\
&\quad \times \left(\begin{matrix} \frac{s}{2}, \frac{s+1}{2} \\ s-\nu+\frac{1}{2} \end{matrix}; 1\right)
\end{aligned}
\tag{4.16}
$$

and finally

$$
M\left(K_\nu(2(\cdot)\hat{e}_1)\right)(s) = (2\pi)^{(n-2)/2} \Gamma(-\nu) \frac{\Gamma(s)\Gamma(s-\nu-(n-2)/2)}{2^s}. \tag{4.17}
$$

So, we can write

$$
K_\nu(\xi\hat{e}_1) = (2\pi)^{(n-2)/2} \Gamma(-\nu) k_{\nu+(n-2)/2}(\xi) \tag{4.18}
$$

according to (4.4). Finally, using (4.3) and the expression of $S_{(\lambda_1,\lambda_2)}$ in terms of j_ν when $x = 2\xi\hat{e}_1$, we obtain

$$
c_\nu = d_{-\nu} = (2\pi)^{(n-2)/2} \Gamma(-\nu) \Gamma\left(\nu + \frac{n-2}{2}\right). \tag{4.19}
$$

5. Conclusion. The resolution of the recurrence systems was possible because each one contains at least one equation with two coefficients of the series. Unfortunately, in the higher rank, such a situation does not occur. But we conjecture that a recurrence on the rank exists. We expect also that a similar situation is possible for the systems satisfied by the multivariate hypergeometric functions ${}_1F_1$ and ${}_2F_1$.

For the K-Bessel function in the case $r = 3$, there is four nonequivalent classes of the Euclidean Jordan algebra. So, we think that we have to perform case-by-case calculations, and the essential difficulty arises in the evaluation of the integral over the automorphism group of the Jordan algebra-like formulas (4.13) and (4.14). This will be the subject of another paper.

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