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RESEARCH NOTES

SUBMATRICES OF SUMMABILITY MATRICES

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ABSTRACT. It is proved that a matrix that maps ℓ^1 into ℓ^1 can be obtained from any regular matrix by the deletion of rows. Similarly, a conservative matrix can be obtained by deletion of rows from a matrix that preserves boundedness. These techniques are also used to derive a simple sufficient condition for a matrix to sum an unbounded sequence.

KEY WORDS AND PHRASES. *Regular matrix, ℓ-ℓ matrix, conservative matrix.*
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1. INTRODUCTION.

In [7] Knopp and Lorentz showed that the matrix summability transformation that maps the sequence x into Ax, given by

$$(Ax)_n = \Sigma_{k=0}^{\infty} a_{nk}x_k, \tag{1.1}$$

maps ℓ^1 into ℓ^1 if and only if

$$\sup_n \Sigma_{n=0}^{\infty} |a_{nk}| < \infty. \tag{1.2}$$

Such a matrix is called an ℓ-ℓ matrix [4]. This theorm is the analogue of the well-known theorem of Kojima and Schur [6, p. 43] that characterizes those matrices A that map the set c (convergent sequences) into c by the three conditions:

(i) for each k, $\lim_n a_{nk} = \alpha_k$;

(ii) $\lim_n \{\Sigma_{k=0}^{\infty} a_{nk}\} = S$;

(iii) $\sup_n \{\Sigma_{k=0}^{\infty} |a_{nk}|\} < \infty$.

Such a matrix is called a conservative matrix. A regular method preserves limit values as well as convergence, and such matrices are characterized by the Silverman-Toeplitz conditions (i), (ii), (iii) in which S=1 and $\alpha_k \equiv 0$.

Some of the well-known summability matrices are both ℓ-ℓ and regular methods [5]. The main purpose of this paper is to establish a general correspondence between regular matrices and ℓ-ℓ matrices by showing that every regular matrix gives rise to an ℓ-ℓ matrix by the deletion of an appropriate set of rows. A similar theorem is proved that asserts that a matrix that maps the set m (bounded sequences) into m contains a row-submatrix that is conservative. In the final section, the row-selection technique is replaced by a column-selection technique in order to prove a simple criterion for the summability of an unbounded sequence.

2. THE MAIN RESULTS.

Although our primary motivation is concerned with regular matrices, we can relax considerably the Silverman-Toeplitz conditions and still select the row-submatrix that we seek.

THEOREM 1. If A is a summability matrix in which each row and each column converge to zero and $\sup_{n,k} |a_{nk}| = \mu < \infty$, then A contains a row-submatrix that is an ℓ-ℓ matrix.

PROOF. First choose a positive integer $\nu(0)$ satisfying $|a_{\nu(0),0}| \leqq 1$; then, using the assumption that $\lim_k a_{\nu(0),k} = 0$, choose $\kappa(0)$ so that $k > \kappa(0)$ implies $|a_{\nu(0),k}| \leqq 1$. Having selected $\nu(i)$ and $\kappa(i)$ for $i < m$, we choose $\nu(m)$ greater than $\nu(m-1)$ so that

$$k \leqq \kappa(m-1) \quad \text{implies} \quad |a_{\nu(m),k}| \leqq 2^{-m};$$

then choose $\kappa(m)$ greater than $\kappa(m-1)$ so that

$$k > \kappa(m) \quad \text{implies} \quad |a_{\nu(m),k}| \leqq 2^{-m}.$$

Now define the submatrix B by $b_{mk} \equiv a_{\nu(m),k}$. The above construction guarantees that each column sequence of B is dominated, except for at most one term, by the sequence $\{2^{-m}\}$; i.e., if $\kappa(m-1) < k \leqq \kappa(m)$ and $i \neq m$, then $|b_{ik}| = |a_{\nu(i),k}| \leqq 2^{-i}$. Since $|a_{\nu(m),k}| \leqq \mu$, it is clear that for each k,

$$\Sigma_{m=0}^{\infty} |b_{mk}| \leqq 2 + \mu.$$

Hence, by (1.2), B is an ℓ-ℓ matrix.

We can now state our principle objective as an immediate consequence of this theorem.

COROLLARY 1. Every regular matrix contains a row-submatrix that is an ℓ-ℓ matrix.

It is easy to see that if A is regular, then the submatrix B of the preceding proof is both ℓ-ℓ and regular; for, any matrix method is included by a method determined by one of its row-submatrices. Also, it is obvious that in Corollary 1 it is not sufficient to assume only that A is conservative; for if $\alpha_k \neq 0$ for some k, then $\Sigma_{m=0}^{\infty} |a_{\nu(m),k}| = \infty$ for any choice of $\{\nu(m)\}_{m=0}^{\infty}$. Furthermore, it is easy to see that not every ℓ-ℓ matrix is a submatrix of a regular matrix; e.g., if $b_{0,k} = 1$ and $b_{mk} = 0$ (when $m > 0$) for every k, then B is ℓ-ℓ but $\sup_n \Sigma_{k=0}^{\infty} |b_{nk}| = \infty$.

Another way of ensuring that the hypotheses of Theorem 1 hold is to assume that A maps ℓ^p into ℓ^q, where $p \geqq 1$ and $q \geqq 1$. Although explicit row/column conditions that characterize such a matrix are not known, it is easy to see that the columns of A must be in ℓ^q and the rows must be uniformly bounded in $\ell^{p'}$, where $1/p + 1/p' = 1$. Thus we state this formally in the following result.

COROLLARY 2. If A maps ℓ^p into ℓ^q, where $p \geq 1$ and $q \geq 1$, then A contains a row-submatrix that is an ℓ-ℓ matrix.

For the next theorem, we prove a variant of Corollary 2 in which ℓ^p and ℓ^1 are replaced by m and c, respectively.

THEOREM 2. If A maps m into m, then A contains a row-submatrix B that is conservative.

PROOF. Since A maps m into m, we have $\sup_n \Sigma_{k=0}^{\infty} |a_{nk}| < \infty$. Therefore the sequence of row sums $\{\Sigma_{k=0}^{\infty} a_{nk}\}_{n=0}^{\infty}$ is bounded, so we can choose a convergent subsequence. This yields a row-submatrix A' of A that satisfies properties (ii) and (iii). It remains to choose a row-submatrix of A' whose columns are convergent sequences. But this is simply a special case of the familiar diagonal process that is used in the proof of the Helley Selection Principle (see, e.g., [2, p. 227]); for we have a family of functions (the rows of A') that are uniformly bounded by $\sup_n \Sigma_{k=0}^{\infty} |a_{nk}|$ on their countable domain $\{0, 1, 2, \ldots\}$. Therefore we can select a sequence of these "functions" that converges at each k. This sequence of rows of A' are then the rows of B.

3. SUMMABILITY OF UNBOUNDED SEQUENCES.

In [1], R. P. Agnew proved that if A is a regular matrix such that

$$\lim_{n,k\to\infty} |a_{nk}| = 0, \tag{3.1}$$

then there exists a nonconvergent sequence of zeros and ones that is summable by A. It then follows by the well-known theorem of Mazur and Orlicz [8] that A sums an unbounded sequence. Because the Mazur-Orlicz Theorem requires the development of Fk-spaces, it would be useful to have a direct construction of an unbounded sequence that is summed by such an A. By modifying the proof of Theorem 1 from row selection to column selection, we can prove a theorem in which we relax the regularity of A, weaken property (3.1), and construct an unbounded sequence that is summed by A.

THEOREM 2. If A is a summability matrix whose column sequences tend to zero and

$$\liminf_k \{\max_n |a_{nk}|\} = 0, \tag{3.2}$$

then A sums an unbounded sequence.

PROOF. Using (3.2), we choose an increasing sequence of column indices $\{\kappa(m)\}_{m=0}^{\infty}$ such that for each m,

$$\max_n |a_{n,\kappa(m)}| < 2^{-m}. \tag{3.3}$$

Then choose increasing row indices $\{\nu(m)\}_{m=0}^{\infty}$ so that if $k \leq \kappa(m)$ and $n > \nu(m)$, then $|a_{nk}| < 2^{-m}$. Now define the sequence x by

$$x_k = \begin{cases} m + 1, & \text{if } k = \kappa(m) \text{ for some } m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Then $n > \nu(m)$ implies

$$\begin{aligned} |(Ax)_n| &= |\Sigma_{j=0}^{\infty} a_{n,\kappa(j)} x_{\kappa(j)}| \\ &\leq \Sigma_{j=0}^{m} (j + 1) 2^{-m} + \Sigma_{j>m} (j + 1) 2^{-j} \\ &= (m + 1)\ (m + 2) 2^{-m-1} + R_m, \end{aligned}$$

where $\lim_m R_m = 0$. Hence, $\lim_n (Ax)_n = 0$.

In closing we note that if the row sequences of A tend to zero, then (3.1) implies $\lim_k \{\max_n |a_{nk}|\} = 0$, which is stronger than (3.2). Therefore Theorem 2 does have a weaker hypothesis than Agnew's theorem. Theorem 2 has been proved by Bennett [3, Theorem 29] and Tatchell [9], both using extensive functional analytic techniques. These proofs do not, however, provide a direct construction of the desired unbounded sequence.

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