THE EFFECT OF RANDOM SCALE CHANGES ON LIMITS

OF INFINITESIMAL SYSTEMS

PATRICK L. BROCKETT

Department of Mathematics
The University of Texas
Austin, Texas   78712   U.S.A.

ABSTRACT.   Suppose $S = \{\{X_{nj},\ j=1,2,\ldots,k_n\}\}$ is an infinitesimal system of

random variables whose centered sums converge in law to a (necessarily

infinitely divisible) distribution with Levy representation determined by

the triple (γ,σ^2,M).   If $\{Y_j,\ j=1,2,\ldots\}$ are independent indentically

distributed random variables independent of S, then the system

$S' = \{\{Y_j X_{nj},\ j=1,2,\ldots,k_n\}\}$ is obtained by randomizing the scale parameters

in S according to the distribution of Y_1.   We give sufficient conditions on

the distribution of Y in terms of an index of convergence of S, to insure

that centered sums from S' be convergent.   If such sums converge to a dis-

tribution determined by $(\gamma',(\sigma')^2,\Lambda)$, then the exact relationship between

(γ,σ^2,M) and $(\gamma',(\sigma')^2,\Lambda)$ is established.   Also investigated is when limit

distributions from S and S' are of the same type, and conditions insuring

products of random variables belong to the domain of attraction of a stable

law.

1. INTRODUCTION AND SUMMARY.

The classical linear model for the relationship between empirical data

Y and theoretical or "true" data X is to assume $Y = X + \varepsilon$ where ε (the error)

is a random variable independent of X with $E(\varepsilon) = 0$. In some cases however

the error tends to depend upon X. For example if X denotes the measurement

of some random phenomenon we may find the empirical data agrees well with X

for values of X which are small, but the error becomes increasingly greater as X

becomes larger. Such is the case when a measuring device has a constant per-

centage error. If we have selected a measuring device at random from a popu-

lation of devices whose constant percentage error Y follows the distribution

function G, then we may model the empirical data as YX where Y and X are

independent and $E(Y) = 1$. The empirical data can be considered as a random

scale change of the theoretical data X, or equivalently the scale parameter

of X has been subjected to a mixture with mixing distribution function G.

The problem we shall consider is the mathematical problem of limit dis-

tributions for sums when the scale parameter is mixed. Specifically, if

$S = \{\{X_{nj}, j=1,2,\ldots,k_n\}\}$ is an infinitesimal system of random variables

whose centered sums converge in distribution to some (infinitely divisible)

random variable X, and if $\{Y_j, j=1,2,\ldots\}$ is a sequence of independent iden-

tically distributed random variables which is independent of S, we seek

conditions on the distribution of Y_1 and the system S to insure that centered

sums from the randomly scale changed system $S' = \{\{Y_j X_{nj}, j=1,2,\ldots,k_n\}\}$

converge, say to Z. In case S' does converge, we wish to determine the exact relationship between X and Z.

Here we take an index of convergence β_S of the system S and under the weak hypothesis $E|Y_1|^{\beta_S+\delta} < \infty$ for some $\delta > 0$, we obtain a necessary and sufficient condition for convergence of the system S'. When S' converges we then obtain the exact relationship between the limit distributions of X and Z. These results are then applied to a most commonly occurring system, namely that consisting of normed random variables whose distribution is attracted to stable laws with exponent α. In this case we find $\beta_S = \alpha$. In the classical central limit theorem ($\alpha = 2$) we find that if $EY^2 < \infty$, then X is attracted to the normal distribution if and only if XY is attracted to the normal distribution, and moreover the same norming constants work. For $\alpha < 2$ we find that if $E|Y_1|^{\alpha+\delta} < \infty$ and X is attracted to a stable of index α, then XY is attracted to the same stable law, the same norming constants work, and the exact scale change between the two resulting stable laws in calculated. We are also interested in when the limit laws of S and S' are of the same type. A necessary and sufficient condition for this to happen is that the limit law be either of purely stable type, or a mixture of stable and normal type.

2. PRELIMINARIES -- AN INDEX OF CONVERGENCE

Let us recall several important facts from [3].

If $S = \{\{X_{nj}, j = 1, 2, \ldots, k_n\}\}$ is an infinitesimal system of random variables, the functions M_n are defined by

$$M_n(x) = \begin{cases} \sum_{j=1}^{k_n} F_{X_{n,j}}(x) & \text{for } x < 0 \\ \\ \sum_{j=1}^{k_n} \{F_{X_{n,j}}(x)-1\} & \text{for } x > 0 \end{cases} \qquad (2.1)$$

where $F_{X_{n,j}}$ is the distribution function of X_{nj}.

We shall say that the system S is convergent to X if there exists a sequence of real numbers $\{C_n \quad n = 1, 2, \ldots\}$ such that $\sum_{j=1}^{k_n} X_{n,j} - C_n$ converges in distribution to X as $n \to \infty$. In this case X is infinitely divisible with characteristic function whose logarithm is of the form

$$\log f_X(u) = i u \alpha - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \{ e^{iux} - 1 - \frac{iux}{1 + x^2} \} dM(x).$$

We shall simply write $X \sim (\alpha, \sigma^2, M)$ to express the fact that X has such a characteristic function. The symbol $\int_{-c}^{d}$ represents $\lim\{\int_{a}^{d} + \int_{-c}^{b} : a \downarrow 0$ and $b \uparrow 0\}$. All integrals are taken in the Lebesgue-Stieltjes sense. The set of points of continuity of a function g will be denoted Cont(g). Discont(g) is defined to be the set of points at which g is not continuous. We shall hence forth use Σ to denote $\Sigma_{j=1}^{k_n}$ and F_{nj} to denote $F_{X_{nj}}$.

DEFINITION. <u>Let S be an infinitesimal system</u>. <u>The index β_S for the system S is defined by</u>

$$\beta_S = \inf\{\delta > 0 : \lim_{(x,n) \to (0,\infty)} |x|^{\delta} \int_{|t| > |x|} dM_n(t) = 0\}$$

<u>with $\beta_S = \infty$ if no such δ above exists</u>.

One should remark here that the index β_S as defined above has some relation to an index defined in 1961 by Blumenthal and Getoor [2] and later generalized by Berman (1965). They define the index β^M for the Levy spectral function M by

$$\beta^M = \inf\{c > 0 : \int_{-1}^{1} |x|^c dM(x) < \infty\}, \quad \text{and proved}$$

$$\beta^M = \inf\{c > 0 : \lim_{x \to 0} \{ |x|^c (M(-|x|) - M(|x|)) \} = 0\}.$$

As a similar result in [4] it was shown that then S converges to $X \sim (\gamma, \sigma^2, M)$

we have for a ϵ Cont(M)

$$\beta_S = \inf\{\varepsilon \geq \beta^M : \int_{|x|<a} |x|^\varepsilon dM_n(x) \to \int_{|x|<a} |x|^\varepsilon dM(x)\} \qquad (2.2)$$

with $\beta_S = \infty$ if the above set is empty. An easy way of calculating β_S is

then given by noting that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{|x|<\varepsilon} |x|^\gamma dM_n(x) = 0 \quad \text{if} \quad \gamma > \beta_S \qquad (2.3)$$

$$\liminf_{n \to \infty} \int_{|x|<\varepsilon} |x|^\gamma dM_n(x) = \infty \quad \text{for any} \quad \varepsilon > 0 \quad \text{if} \quad \gamma < \beta_S .$$

We next recall a variational sum result proved in [4].

THEOREM 1. <u>Suppose S is an infinitesimal system converging to</u>

$X \sim (\alpha, \sigma^2, M)$, <u>and suppose that g is a bounded function which satisfies</u>

$g(x) = 0(|x|^\gamma)$ <u>as</u> $x \to 0$ <u>for some</u> $\gamma > \beta_S$ <u>and which is either continuous or</u>

<u>is of bounded variation over</u> $(-\infty, 0)$ <u>and over</u> $(0, \infty)$ <u>with</u> discont(g) $\cap$

discont(M) = $\emptyset$. <u>Then we have</u>

$$\lim_{n \to \infty} \Sigma E(g(X_{nj})) = \lim_{n \to \infty} \int_{(-\infty,\infty)} g(x) dM_n(x) = \int_{(-\infty,\infty)} g(x) dM(x) .$$

REMARK 1. Some comments about the index β_S are in order. If the

system S is convergent to $X \sim (\alpha, \sigma^2, M)$, then $\beta^M \leq \beta_S \leq \infty$ and either of the

two equalities is possible. For an example where $\beta_S = \beta^M$, see [4] where

it is shown that if $\{X_1, X_2, \ldots\}$ is a sequence of independent identically

distributed random variables belonging to the domain of attraction of a

stable distribution with exponent α, and if $S = \{\{X_j/B_n , j = 1, 2, \ldots, n\}\}$

then $\beta_S = \alpha$. It is clear that in this case $\beta^M = \begin{cases} \alpha & \text{if } \alpha < 2 \\ 0 & \text{if } \alpha = 2 \end{cases}$. In [3] an

example is given showing β_S can be anything. Namely for $\gamma \leq \infty$ a conver-

gent system S is found such that $\beta_S = \lambda$. To show that β_S measures "how"

the system converges rather than to what it converges, an example in [3]

is given showing that for any Levy spectral function M and for any $\lambda \geq \beta^M$

there exists a system S converging to (0,0,M) with $\beta_S = \lambda$. The index β_S

has proved to be the appropriate index for studying variational sums of in-

finitesimal systems [4], and has been shown to be an extension of the Blum-

enthal-Getoor index for stochastic processes with independent increments

[5] which allows a unified treatment of variational sums of such processes.

Let us also state for reference the general limit theorem as found in

Gnedenko and Kolmogorov (1968) or Tucker (1967).

THEOREM 2. <u>Suppose</u> $S = \{\{X_{nj} : j = 1, 2, \ldots, k_n\}\}$ <u>is an infinitesimal</u>

<u>system</u>. <u>A necessary and sufficient condition that</u> $\Sigma X_{nj} - c_n \to X \sim (\alpha, \sigma^2, M)$

<u>is that the following three conditions all hold</u>

A) $M_n(x) \to M(x)$ for all $x \in \text{Cont}(M)$

B) $\lim\limits_{\varepsilon \to 0} \begin{Bmatrix} \lim \sup \\ \lim \inf \end{Bmatrix}_{n \to \infty} \left\{ \int_{|x|<\varepsilon} x^2 dM_n(x) - \Sigma (\int_{|x|<\varepsilon} x dF_{X_{nj}}(x))^2 \right\} = \sigma^2$

C) $\int_{|x|<\tau} x dM_n(x) - c_n \to \alpha + \int_{|x|<\tau} x^3/(1+x^2) dM(x) - \int_{|x|<\tau} x/(1+x^2) dM(x)$

for any $\tau > 0$.

3. A LIMIT THEOREM FOR SCALE MIXED SYSTEMS AND APPLICATIONS

We shall now proceed to answer the questions posed in the introduction.

Namely when S is convergent, the index β_S yields the needed tool for obtain-

ing conditions insuring S' is convergent. The precise formulation, and the

exact relationship between the limit laws of S and of S' is given in the following theorem. Also to be noted is that the convergence properties of S' are the same as that of S (i.e., $\beta_S = \beta_{S'}$) and the behavior of the limiting Levy functions are the same ($\beta^M = \beta^\Lambda$). In particular since an infinitely divisible distribution is continuous if and only if the corresponding Levy spectral function is unbounded, it follows that the limit distributions of S and S' are simultaneous continuous or not continuous.

THEOREM 3. <u>Suppose</u> <u>that</u> $S = \{\{X_{nj}, j = 1, 2, \ldots, k_n\}\}$ <u>is</u> <u>convergent</u> <u>to</u> $X \sim (\gamma, \sigma^2, M)$ <u>i.e.</u>, $\Sigma X_{nj} - c_n \overset{d}{\longrightarrow} X$. <u>Let</u> $\{Y_j, j = 1, 2, \ldots\}$ <u>be</u> <u>a</u> <u>sequence</u> <u>of</u> <u>independent</u> <u>identically</u> <u>distributed</u> <u>random</u> <u>variables</u> <u>with</u> <u>non-degenerate</u> <u>common</u> <u>distribution</u> <u>function</u> F_Y <u>and</u> <u>with</u> $E|Y|^{\beta_S + \kappa} < \infty$ <u>for</u> <u>some</u> $\kappa > 0$. <u>Assume</u> <u>also</u> <u>that</u> $\{Y_j, j = 1, 2, \ldots\}$ <u>is</u> <u>independent</u> <u>of</u> <u>the</u> <u>system</u> S. <u>Then</u> <u>a</u> <u>necessary</u> <u>and</u> <u>sufficient</u> <u>condition</u> <u>that</u> $S' = \{\{Y_j X_{nj}, j = 1, 2, \ldots, k_n\}$ $n = 1, 2, \ldots\}$ <u>be</u> <u>convergent</u> <u>is</u>

$$C_0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \int_{|x| < \epsilon} x^2 dM_n(x)$$
$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \int_{|x| < \epsilon} x^2 dM_n(x) \tag{3.1}$$

<u>exists</u> <u>and</u> <u>be</u> <u>finite</u> (<u>so</u> <u>that</u> <u>by</u> <u>Remark</u> 1 <u>necessarily</u> $\beta_S \leq 2$). <u>If</u> S' <u>con-</u> <u>verges</u> <u>to</u> $Z \sim (\gamma', (\sigma')^2, \Lambda)$, <u>then</u> <u>the</u> <u>following</u> <u>are</u> <u>true:</u>

1^0 <u>The</u> <u>centering</u> <u>constants</u> <u>for</u> S' <u>can</u> <u>be</u> <u>chosen</u> <u>as</u>

$$d_n = \Sigma \int_{|x| < \tau} x dF_{X_{nj} Y_j}(x) - \gamma' - \Sigma \int_{|x| < \tau} x^3/(1 + x^2) dF_{X_{nj} Y_j}(x)$$
$$+ \Sigma \int_{|x| > \tau} x/(1 + x^2) dF_{X_{nj} Y_j}(x)$$

2^0

$$\Lambda(x) = \begin{cases} \int_{(-\infty,0)} \{1-F_Y(x/t)\}dM(t) + \int_{(0,\infty)} F_Y(x/t)dM(t) & \text{for } x < 0 \\[2em] -\int_{(-\infty,0)} F_Y(x/t)dM(t) + \int_{(0,\infty)} \{F_Y(x/t)-1\}dM(t) & \text{for } x > 0 \end{cases}$$

3^0
$$(\sigma')^2 = C_0 \, \text{Var}(Y) + \sigma^2 (E(Y))^2$$
where C_0 is as defined in (3.1)

4^0
$$\beta_S = \beta_{S'}$$

5^0
$$\beta^\Lambda = \beta^M.$$

NOTE: 1. If $\beta_S < 2$ then (3.1) is automatically satisfied with $C_0 = 0$ by (2.3) and necessarily $\sigma^2 = 0$ (otherwise $\beta_S \geq 2$), so 3^0 says $(\sigma')^2 = 0$ in this case.   Also when $\beta_S = 2$ and (3.1) holds we need only assume $E(Y^2) < \infty$ to obtain the conclusion.

2. It should be noted that the mere meaningfulness of formula 2^0 may not be sufficient for the conclusion of the theorem.   Examples of this are given in [3].

PROOF.   1) We first note that S' is indeed an infinitesimal system. Now, an easy calculation yields

$$F_{X_{nj}Y_j}(x) = \int_{(-\infty,0)} \{1-F_Y(x/t)dF_{nj}(t) + \int_{(0,\infty)} F_Y(x/t)dF_{nj}(t) \qquad (3.2)$$
$$+ P[X_{nj}=0]\chi_{[0,\infty)}(x).$$

Here $\chi_A(x) = 1$ if $x \in A$ and 0 if $x \notin A$. Let

$x \in A$ and 0 if $x \notin A$. Let

$$\Lambda_n(x) = \begin{cases} \sum F_{X_{nj}Y_j}(x) & \text{for } x < 0 \\ \sum \{F_{X_{nj}Y_j}(x) - 1\} & \text{for } x > 0 \end{cases} \tag{3.3}$$

and let

$$g(x) = \begin{cases} 1 - F_Y(x) & x > 0 \\ F_Y(x) & x < 0. \end{cases}$$

Then using (3.2) in (3.3) we have for $x < 0$

$$\Lambda_n(x) = \int_{(-\infty,0)} \{1 - F_Y(x/t)\} dM_n(t) + \int_{(0,\infty)} F_Y(x/t) dM_n(t)$$

$$= \int_{(-\infty,\infty)} g(x/t) dM_n(t).$$

while for $x > 0$ we have

$$\Lambda_n(x) = \sum \{-\int_{(-\infty,0)} g(x/t) dF_{nj}(t) - \int_{(0,\infty)} g(x/t) dF_{nj}(t)\}$$

$$= -\int_{(-\infty,\infty)} g(x/t) dM_n(t).$$

Thus we have

$$\Lambda_n(x) = \begin{cases} \int_{(-\infty,\infty)} g(x/t) dM_n(t) & \text{for } x < 0 \\ -\int_{(-\infty,\infty)} g(x/t) dM_n(t) & \text{for } x > 0. \end{cases} \tag{3.4}$$

2) In this part of the proof we shall show that if $E|Y|^{\beta_s + \delta} < \infty$, then we always have $\lim_{n \to \infty} \Lambda_n(x) = \Lambda(x)$ for all $x \in \text{Cont}(\Lambda)$. Indeed since $E|Y|^{\beta_s + \delta} < \infty$ we have $\{1 - F_Y(t) + F_Y(-t)\} = 0(t^{-\beta_s - \delta})$ as $t \to \infty$. By the defi-

nition of g, we can thus see that for fixed x, $g(x/t) = 0(|t|^{\beta_s+\delta})$ as

$t \to 0$. Let us write $g_x(t) = g(x/t)$. By Theorem 1 we have

$$\lim_{n \to \infty} \int_{(-\infty,\infty)} g_x(t) dM_n(t) = \int_{(-\infty,\infty)} g_x(t) dM(t) \qquad (3.5)$$

for every x such that $\text{discont}(g_x) \cap \text{discont}(M) = \emptyset$. Thus using (3.5),

(3.4) and 2^0 we have $\lim_{n \to \infty} \Lambda_n(x) = \Lambda(x)$ for all x such that $\text{discont}(g_x) \cap$

$\text{discont}(M) = \emptyset$. Let $\mathcal{U} = \{x: \text{discont}(g_x) \cap \text{discont}(M) = \emptyset\}$. It remains to

show $\text{Cont}(\Lambda) \subseteq \mathcal{U}$. Equivalently we show $\mathcal{U}^c \subseteq \text{discont}(\Lambda)$. If $y_0 \in \mathcal{U}^c$ then

there exists a t_0 such that g_{y_0} and M are both discontinuous at t_0. For

simplicity let us assume that $y_0 > 0$ and $t_0 > 0$. A similar argument can

be used in any other case. Thus we have $g_{y_0}(t_0+0) - g_{y_0}(t_0-0) > 0$ and

$dM(\{t_0\}) > 0$, so that for any small $h > 0$ we have

$$\Lambda(y_0+h) - \Lambda(y_0-h) \geq \{F_Y(\frac{y_0+h}{t_0}) - F_Y(\frac{y_0-h}{t_0})\} dM(\{t_0\})$$

$$= \{F_Y(\frac{y_0}{t_0-\eta}) - F_Y(\frac{y_0}{t_0+\beta})\} dM(\{t_0\}) = \{g_{y_0}(t_0+\beta) - g_{y_0}(t_0-\eta)\} dM(\{t_0\})$$

$$\geq \{g_{y_0}(t_0+0) - g_{y_0}(t_0-0)\} dM(\{t_0\}) = b > 0$$

where $\beta = t_0 h/(y_0-h)$ and $\eta = t_0 h/y_0 + h$. Since $\Lambda(y_0+h) - \Lambda(y_0-h) \geq b > 0$

for all h, we must have $y_0 \in \text{discont}(\Lambda)$. Thus $\Lambda_n(x) \to \Lambda(x)$ for all

$x \in \text{Cont}(\Lambda)$. To see that Λ is a Levy spectral function, noted that Λ is

monotone increasing on $(-\infty,0)$ and on $(0,\infty)$ and $\Lambda(+\infty) = \Lambda(-\infty) = 0$. We

must also show that $\int_{(-1,1)} x^2 d\Lambda(x) < \infty$. The proof of this is accomplished

in part 5) below.

3) In this part of the proof we shall determine how to calculate

integrals of the form $\int_{(-\infty,\infty)} f(x) d\Lambda_n(x)$ and $\int_{(-\infty,\infty)} f(x) d\Lambda(x)$ where Λ_n

is given by (3.4) and Λ is given by 2^0. Now for $0 < a < b < \infty$, let $\chi_{(a,b]}$ denote the indicator function of the interval $(a,b]$ and calculate

$$\Lambda_n(b) - \Lambda_n(a) = \int_{(-\infty,\infty)} \int_{(-\infty,\infty)} \chi_{\frac{1}{t}(a,b]}(u) dF_Y(u) dM_n(t)$$

$$= \int_{(-\infty,\infty)} \int_{(-\infty,\infty)} \chi_{(a,b]}(tu) dF_Y(u) dM_n(t)$$

so that

$$\int_{(-\infty,\infty)} f(x) d\Lambda_n(x) = \int_{(-\infty,\infty)} \int_{(-\infty,\infty)} f(ux) dF_Y(u) dM_n(x) \qquad (3.6)$$

holds when f is the indicator function of an interval in $(-\infty,\infty)$. Standard approximation techniques now allow us to conclude (3.6) holds for the function in question. Also (3.6) holds if we replace Λ_n by Λ and M_n by M.

   4) In this part of the proof we show that 4^0 holds. We shall use (2.4) to show $\beta_S = \beta_{S'}$. Now for any $\delta > 0$ we have

$$\int_{|x|<\epsilon} |x|^\delta d\Lambda_n(x) = \int_{(-\infty,\infty)} |x|^\delta \int_{(-\infty,\infty)} |u|^\delta \chi_{(-\epsilon,\epsilon)}(ux) dF_Y(u) dM_n(x)$$

$$= \int_{|x|<1} |x|^\delta \int_{|ux|<\epsilon} |u|^\delta dF_Y(u) dM_n(x) + \int_{|x|\geq 1} |x|^\delta \int_{|ux|<\epsilon} |u|^\delta dF_Y(u) dM_n(x).$$

However concerning the second term above we see that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{|x|\geq 1} |x|^\delta \int_{|ux|<\epsilon} |u|^\delta dF_Y(u) dM_n(x) = 0.$$

Thus to show $\beta_S = \beta_{S'}$, we must only show

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \int_{|x|<1} |x|^\delta \int_{|ux|<\varepsilon} |u|^\delta dF_Y(u) dM_n(x) = 0 \text{ if } \delta > \beta_S \text{ and}$$

$$\lim_{n \to \infty} \inf \int_{|x|<1} |x|^\delta \int_{|ux|<\varepsilon} |u|^\delta dF_Y(u) dM_n(x) = \infty \text{ for any } \varepsilon > 0 \text{ if } \delta < \beta_S .$$

Suppose then that $\delta < \beta_S$ and choose ε_1 so that $\int_{|x|<\varepsilon_1} |x|^\delta dF_Y(x) > 0$.

Now $\int_{|x|<1} \int_{|ux|<\varepsilon} |ux|^\delta dF_Y(u) dM_n(x) \geq \int_{|x|<1} |x|^\delta \int_{|u|<\varepsilon} |u|^\delta dF_Y(u) dM_n(x)$

so that $\lim_{n \to \infty} \inf \int_{|x|<1} \int_{|xu|<\varepsilon_1} |ux|^\delta dF_Y(u) dM_n(x) \geq$

$\int_{|u|<\varepsilon_1} |u|^\delta dF_Y(u) \lim_{n \to \infty} \inf \int_{|x|<1} |x|^\delta dM_n(x) = \infty$, and thus $\beta_S \leq \beta_{S'}$. To see

that $\beta_S \geq \beta_{S'}$, let $\gamma > \beta_S$ be such that $E|Y|^\gamma < \infty$ and pick $\alpha > 0$ such that

$\delta - \gamma < \alpha < \delta - \beta_S$ i.e., such that $\beta_S < \delta - \alpha < \gamma$. Then $E|Y|^{\delta - \alpha} < \infty$ and also

$\int_{|x|<1} |x|^{\delta - \alpha} dM_n(x)$ is bounded in n. Now $\int_{|x|<1} \int_{|ux|<\varepsilon} |ux|^\delta dF_Y(u) dM_n(x) \leq$

$\varepsilon^\alpha E|Y|^{\delta - \alpha} \int_{|x|<1} |x|^{\delta - \alpha} dM_n(x)$. Thus

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \int_{|x|<1} \int_{|ux|<\varepsilon} |ux|^\delta dF_Y(u) dM_n(x) = 0. \text{ I.e. } \beta_{S'} \leq \beta_S \text{ and hence}$$

$\beta_{S'} = \beta_S$.

5) In this part of the proof we shall show 5^0 holds by showing

$$\int_{|x|<1} |x|^\delta d\Lambda(x) = \begin{cases} < \infty & \text{if } \delta > \beta^M \\ \infty & \text{if } \delta < \beta^M. \end{cases}$$

Suppose $\delta \geq \beta^M$ and that $\int_{(-1,1)} |x|^\delta dM(x) < \infty$. Now we know by hypothesis

that there is some $\gamma \geq \beta_S \geq \beta^M$ such that $E|Y|^\gamma < \infty$. Without loss of

generality we may assume $\delta \leq \gamma$ so that we have

$$\int_{|x|<1} |x|^{\delta} d\Lambda(x) = \int_{|x|<1} |x|^{\delta} \int_{|ux|<1} |u|^{\delta} dF_Y(u) dM(x)$$

$$+ \int_{|x|\geq 1} |x|^{\delta} \int_{|ux|<1} |u|^{\delta} dF_Y(u) dM(x) \leq E|Y|^{\delta} \int_{|x|<1} |x|^{\delta} dM(x) + M(-1) - M(1) < \infty$$

i.e. $\beta^{\Lambda} \leq \beta^{M}$. On the other hand, if $\delta < \beta^{M}$ so that $\int_{|x|<1} |x|^{\delta} dM(x) = \infty$,

then choosing $1 \geq c > 0$ such that $P[Y < \frac{1}{c}] > 0$ we have

$$\int_{|x|<1} |x|^{\delta} d\Lambda(x) \geq \int_{|x|<c} |x|^{\delta} \int_{|ux|<1} |u|^{\delta} dF_Y(u) dM(x)$$

$$\geq \{\int_{|x|<c} |x|^{\delta} dM(x)\}\{\int_{|u|<\frac{1}{c}} |u|^{\delta} dF_Y(u)\} = \infty.$$

Thus $\beta^{\Lambda} \geq \beta^{M}$ and hence $\beta^{\Lambda} = \beta^{M}$. In particular, choosing $\delta = 2$ we see Λ is

indeed a Levy spectral function.

6) An elementary calculation using the Helly Bray theorem and Theorem

2 shows that the centering can be taken as in 1^0 when (3.1) holds.

7) In this part of the proof we shall show that the finiteness of

the limit in (3.1) is necessary. We shall show that if

$$\lim_{n \to \infty} \sup \int_{|x|<\epsilon} x^2 dM_n(x) = \infty \qquad (3.7)$$

then

$$\lim_{n \to \infty} \sup\{\int_{|x|<\epsilon} x^2 d\Lambda_n(x) - \Sigma(\int_{|x|<\epsilon} x dF_{X_{nj}Y_j}(x))^2\} = \infty. \qquad (3.8)$$

Hence by the general limit theorem S' cannot be convergent. Let $\epsilon > 1$ and for simplicity of notation let us write

$$\sigma_n^2(\epsilon) = \int_{|x|<\epsilon} x^2 d\Lambda_n(x) - \Sigma(\int_{|x|<\epsilon} x dF_{X_{nj}Y_j}(x))^2.$$

Then

$$\sigma_n^2(\epsilon) \geq \int_{|x|<\epsilon^2} x^2 \int_{|ux|<\epsilon} u^2 dF_Y(u) dM_n(x) \tag{3.9}$$

$$- \Sigma\{\int_{|x|<\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}^2$$

$$- 2\Sigma\{\int_{|x|<\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}\{\int_{|x|\geq\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}$$

$$- \Sigma\{\int_{|x|\geq\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}^2.$$

CLAIM 1. $\lim\sup\limits_{n\to\infty} \Sigma\{\int_{|x|\geq\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}^2 = 0.$

PROOF. By the Schwartz inequality,

$$(\int_{|x|\geq\epsilon^2} \{x \int_{|ux|<\epsilon} u dF_Y(u)\} dF_{nj}(x))^2$$

$$\leq (\int_{|x|\geq\epsilon^2} \{x \int_{|ux|<\epsilon} u dF_Y(u)\}^2 dF_{nj}(x)) P[|X_{nj}| \geq \epsilon^2]$$

$$\leq \max_{j\leq j\leq k_n} P[|X_{nj}| \geq \epsilon^2] \int_{|x|\geq\epsilon^2} x^2 \int_{|ux|<\epsilon} u^2 dF_Y(u) dF_{nj}(x).$$

Using the infinitesimality of the system, we have the first term converging
to zero. The sum of the second terms is bounded, since by the Helly-Bray
theorem

$$\lim_{n \to \infty} \sup \int_{|x| \geq \varepsilon^2} x^2 \int_{|ux| < \varepsilon} u^2 dF_Y(u) dM_n(x)$$

$$\leq \{E(Y^2) \int_{\varepsilon^2 < |x| < \varepsilon} x^2 dM(x) + \varepsilon^2 \int_{|x| \geq \varepsilon} dM(x)\}.$$

This completes the proof of Claim 1. Note that using the above we actually
have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup \int_{|x| \geq \varepsilon^2} x^2 \int_{|ux| < \varepsilon} u^2 dF_Y(u) dM_n(x) = 0. \qquad (3.10)$$

CLAIM 2.

$$\left| \Sigma \{ \int_{|x| < \varepsilon^2} x \int_{|ux| < \varepsilon} u dF_Y(u) dF_{nj}(x) \} \{ \int_{|x| \geq \varepsilon^2} x \int_{|ux| < \varepsilon} u dF_Y(u) dF_{nj}(x) \} \right|$$

$$\leq \{E(Y^2)\}^{\frac{1}{2}} o(1) \{ \int_{|x| < \varepsilon^2} x^2 dM_n(x) \}^{\frac{1}{2}}.$$

PROOF. By the Schwarz inequality for sums, the sum in the claim is
no larger than

$$\{ \Sigma (\int_{|x| < \varepsilon^2} x \int_{|ux| < \varepsilon} u dF_Y(u) dF_{nj}(x))^2 \}^{\frac{1}{2}} \{ \Sigma (\int_{|x| \geq \varepsilon^2} x \int_{|ux| < \varepsilon} u dF_Y(u) dF_{nj}(x))^2 \}^{\frac{1}{2}}.$$

Applying Claim 1 to the second term and the Schwarz inequality to the

first term in the above product we have that the left hand side does not

exceed

$$\{\int_{|x|<\epsilon^2} x^2 (\int_{|ux|<\epsilon} u dF_Y(u))^2 dM_n(x)\}^{\frac{1}{2}} o(1) \leq \{E(Y^2)\}^{\frac{1}{2}} o(1) \{\int_{|x|<\epsilon} x^2 dM_n(x)\}^{\frac{1}{2}}$$

which completes the proof of Claim 2.

Now using the inequality

$$\Sigma\{\int_{|x|<\epsilon^2} x \int_{|ux|<\epsilon} u dF_Y(u) dF_{nj}(x)\}^2$$

$$\leq \int_{|x|<\epsilon^2} x^2 \{\int_{|ux|<\epsilon} u dF_Y(u)\}^2 dM_n(x)$$

and using Claims 1 and 2 in (3.9) yields

$$\sigma_n^2(\epsilon) \geq \int_{|x|<\epsilon^2} x^2 \{\int_{|ux|<\epsilon} u^2 dF_Y(u) - (\int_{|ux|<\epsilon} u dF_Y(u))^2\} dM_n(x)$$

$$\qquad\qquad (3.11)$$

$$- 2(E(Y^2))^{\frac{1}{2}} o(1) \{\int_{|x|<\epsilon^2} x^2 dM_n(x)\}^{\frac{1}{2}} + o(1).$$

Since (3.7) implies $\beta_S \geq 2$, we know $E(Y^2) < \infty$. Let $\delta > 0$ be chosen such

that $\text{Var}(Y) - \delta > 0$ and choose $\epsilon_0^2 > 0$ so small that if $|x| < \epsilon_0^2$, then

$$\text{Var}(Y) - \delta \leq \int_{|ux|<\epsilon_0} u^2 dF_Y(u) - (\int_{|ux|<\epsilon_0} u dF_Y(u))^2 \leq \text{Var}(Y) + \delta.$$

Then for $\varepsilon \leq \varepsilon_0$ in (3.11) we have

$$\sigma_n^2(\varepsilon) \geq \{Var(Y)-\delta\}\int_{|x|<\varepsilon^2} x^2 dM_n(x) - 2E(Y^2)o(1)\{\int_{|x|<\varepsilon^2} x^2 dM_n(x)\}^{\frac{1}{2}} + o(1)$$

$$(3.12)$$

$$= \{\int_{|x|<\varepsilon^2} x^2 dM_n(x)\}^{\frac{1}{2}}\{(Var(Y)-\delta)(\int_{|x|<\varepsilon} x^2 dM_n(x))^{\frac{1}{2}} - 2E(Y^2)o(1)\} + o(1).$$

By (3.7) the lim sup as $n \to \infty$ of the right hand side of (3.12) is $+\infty$,

i.e., $\limsup_{n \to \infty} \sigma_n^2(\varepsilon) = \infty$, and S' cannot be convergent.

8) In this part of the proof we show that if (3.1) holds, then S'

is convergent to $X \sim (\alpha', (\sigma')^2, \tilde{\Lambda})$. Since we have already shown that

$\lim_{n \to \infty} \Lambda_n(x) = \Lambda(x)$ for all $x \in Cont(\Lambda)$, to obtain this part of the proof,

we need only show that with $\sigma_n^2(\varepsilon)$ as given we have

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sigma_n^2(\varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \sigma_n^2(\varepsilon) = (\sigma')^2.$$

Now, if $\beta_S < 2$, then by (2.3) we must have $\sigma^2 = 0$, and $C_0 = 0$. However by

part 5 of the proof $\beta_{S'} = \beta_S$ so that

$$0 \leq (\sigma')^2 \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{|x|<\varepsilon} x^2 d\Lambda_n(x) = 0$$

so that 3^0 holds with $(\sigma')^2 = 0$, and consequently S' converges to

$X \sim (\alpha', 0, \Lambda)$. We are thus left to consider the case when $\beta_S = 2$ (recall

(3.1) implies $\beta_S \leq 2$). Then we assume $E(Y^2) < \infty$ and as before we have

$$\sigma_n^2(\varepsilon) = \int_{|x|<\varepsilon^2} x^2 \int_{|ux|<\varepsilon} u^2 dF_Y(u) dM_n(x) \tag{3.13}$$

$$+ \int_{|x|\geq\varepsilon^2} x^2 \int_{|ux|<\varepsilon} u^2 dF_Y(u) dM_n(x) - \Sigma \left(\int_{|x|<\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x) \right)^2$$

$$- \Sigma \left(\int_{|x|\geq\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x) \right)^2$$

$$- 2\Sigma \left(\int_{|x|<\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x) \right) \left(\int_{|x|\geq\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x) \right).$$

Applying (3.10) to the second term, Claim 2 to the last term and Claim 1 to the 4th term on the right hand side of the last equality in (3.13) and upon adding and subtracting

$$\Sigma \left(\int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x) \right)^2$$

we have

$$\sigma_n^2(\varepsilon) = \int_{|x|<\varepsilon^2} x^2 \int_{|ux|<\varepsilon} u^2 dF_Y(u) dM_n(x) \tag{3.14}$$

$$- \left(\int_{|u|<1/\varepsilon} u dF_Y(u) \right)^2 \Sigma \left(\int_{|x|<\varepsilon^2} x dF_{nj}(x) \right)^2$$

$$+ \Sigma \left\{ \left(\int_{|x|<\varepsilon^2} \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x) \right)^2 \right.$$

$$\left. - \left(\int_{|x|<\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x) \right)^2 \right\} + g_1(n,\varepsilon)$$

where $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup g_1(n, \varepsilon) = 0.$

CLAIM 3. $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup |\Sigma\{(\int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x))^2$

$\qquad - (\int_{|x|<\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x))^2\}| = 0.$

PROOF.   Using the equality $a^2 - b^2 = (a-b)(a+b)$ we have

$|(\int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x))^2$ (3.15)

$\quad - (\int_{|x|<\varepsilon^2} x \int_{|ux|<\varepsilon} u dF_Y(u) dF_{nj}(x))^2|$

$\quad = \{2\int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x)$

$\quad + \int_{|x|<\varepsilon^2} x \int_{1/\varepsilon<|u|<\varepsilon/|x|} u dF_Y(u) dF_{nj}(x)\}\{\int_{1/\varepsilon<|u|<\varepsilon/|x|} u dF_Y(u) dF_{nj}(x)\}.$

For simplicity we denote

$$f_{nj}(\varepsilon) = \int_{|x|<\varepsilon^2} x \int_{1/\varepsilon<|u|<\varepsilon/|x|} u dF_Y(u) dF_{nj}(x).$$

Then the right hand side of (3.15) becomes equal to

$$2 f_{nj}(\varepsilon) \int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x) + f_{nj}^2(\varepsilon).$$

Thus to complete the proof of Claim 3 we must show

$$\lim_{\varepsilon \to 0} \ \limsup_{n \to \infty} \ \Sigma \ f^2_{nj}(\varepsilon) = 0 \tag{3.16}$$

and

$$\lim_{\varepsilon \to 0} \ \limsup_{n \to \infty} \{ \Sigma \ f_{nj}(\varepsilon) \int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x) \} = 0. \tag{3.17}$$

Concerning (3.16), note that by the Schwarz inequality

$$f^2_{nj}(\varepsilon) \leq \int_{|x|<\varepsilon^2} x^2 \int_{|u|>1/\varepsilon} u^2 dF_Y(u) dF_{nj}(x)$$

and letting $a(\varepsilon^2) = \lim\sup_{n \to \infty} \int_{|x|<\varepsilon^2} x^2 dM_n(x)$ we have $a(\varepsilon^2) \downarrow C_0$ as $\varepsilon \downarrow 0$ and hence

$$\lim_{\varepsilon \to 0} \ \limsup_{n \to \infty} \ \Sigma \ f^2_{nj}(\varepsilon) \leq \lim_{\varepsilon \to 0} \{ \int_{|u|>1/\varepsilon} u^2 dF_Y(u) a(\varepsilon^2) \} = 0$$

so that (3.16) holds. In a similar manner, concerning (3.17) we have

$$|f_{nj}(\varepsilon) \int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x) |$$

$$\leq \{ \int_{|x|<\varepsilon^2} |x| \int_{|u|>1/\varepsilon} |u| dF_Y(u) dF_{nj}(x) \} \{ E|Y| \int_{|x|<\varepsilon^2} |x| dF_{nj}(x) \}$$

$$\leq E(Y) \{ \int_{|u|>1/\varepsilon} |u| dF_Y(u) \} \{ \int_{|x|<\varepsilon^2} x^2 dF_{nj}(x) \}.$$

Thus $|\Sigma\{f_{nj}(\varepsilon) \int_{|x|<\varepsilon^2} x \int_{|u|<1/\varepsilon} u dF_Y(u) dF_{nj}(x)\}|$

$\leq E|Y|\{\int_{|u|>1/\varepsilon} |u| dF_Y(u) \int_{|x|<\varepsilon^2} x^2 dM_n(x)\}$, from which (3.17) follows

easily.

Now using Claim 3 in (3.14) and upon adding and subtracting

$(\int_{|u|<1/\varepsilon} u dFY(u))^2 \int_{|x|<\varepsilon^2} x^2 dM_n(x)$, we obtain from (3.14),

$$\sigma_n^2(\varepsilon) = \int_{|x|<\varepsilon^2} x^2\{\int_{|u|<\varepsilon/|x|} u^2 dF_Y(u) - (\int_{|u|<1/\varepsilon} u dF_Y(u))^2\} dM_n(x) \qquad (3.18)$$

$$+ (\int_{|u|<1/\varepsilon} u dF_Y(u))^2\{\int_{|x|<\varepsilon^2} x^2 dM_n(x) - \Sigma(\int_{|x|<\varepsilon} x dF_{nj}(x))^2\}$$

$$+ g_2(n,\varepsilon) \quad \text{where} \quad \lim_{\varepsilon \to 0} \limsup_{n \to \infty} g_2(n,\varepsilon) = 0.$$

Then, if we use the inequalities

$$\{\int_{|x|<\varepsilon^2} x^2 dM_n(x)\}\{\int_{|u|<1/\varepsilon} u^2 dF_Y(u)\}$$

$$\leq \int_{|x|<\varepsilon^2} x^2 \int_{|ux|<\varepsilon} u^2 dF_Y(u) dM_n(x) \leq E(Y^2) \int_{|x|<\varepsilon^2} x^2 dM_n(x)$$

in (3.18), we obtain the following two sided bound for $\sigma_n^2(\varepsilon)$:

$$\{\int_{|x|<\epsilon^2} x^2 dM_n(x)\}\{\int_{|u|<1/\epsilon} u^2 dF_Y(u) - (\int_{|u|<1/\epsilon} u dF_Y(u)^2\} + g_2(n, \epsilon) \qquad (3.19)$$

$$+ (\int_{|u|<1/\epsilon} u dF_Y(u))^2 \{\int_{|x|<\epsilon^2} x^2 dM_n(x) - \Sigma (\int_{|x|<\epsilon^2} x dF_{nj}(x))^2\}$$

$$\leq \sigma_n^2(\epsilon) \leq \{\int_{|x|<\epsilon^2} x^2 dM_n(x)\}\{E(Y^2) - (\int_{|u|<1/\epsilon} u dF_Y(u))^2\} + g_2(n, \epsilon)$$

$$+ (\int_{|u|<1/\epsilon} u dF_Y(u))^2 \{\int_{|x|<\epsilon^2} x^2 dM_n(x) - \Sigma (\int_{|x|<\epsilon^2} x dF_{nj}(x))^2\}.$$

Using the fact that $\lim_{\epsilon \to 0} \begin{Bmatrix} \lim \sup \\ \lim \inf \end{Bmatrix}_{n \to \infty}$ of the extremities in (3.19) are

equal when (3.1) holds, we obtain

$$C_0 \, \mathrm{Var}(Y) + (E(Y))^2 \sigma^2 \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \sigma_n^2(\epsilon)$$

$$\leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \sigma_n^2(\epsilon) \leq C_0 \, \mathrm{Var}(Y) + (E(Y))^2 \sigma^2$$

where C_0 is the constant given in (3.1), so that $(\sigma')^2$ exists and is given

by $(\sigma')^2 = C_0 \, \mathrm{Var}(Y) + (E(Y))^2 \sigma^2$.

9) In this part of the proof we show that if S' is convergent

(so that $\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \sigma_n^2(\epsilon) = (\sigma')^2 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \sigma_n^2(\epsilon))$ then necessarily

(3.1) holds. By part 7) we know that we may assume

$\lim_{n \to \infty} \sup \int_{|x|<\epsilon} x^2 dM_n(x) < \infty$ so that all of the previous equations leading

up to (3.19) still remain valid. Let us then subtract the quantity

$$(\int_{|x|<1/\varepsilon} u dF_Y(u))^2 \{\int_{|x|<\varepsilon^2} x^2 dM_n(x) - \Sigma (\int_{|x|<\varepsilon^2} x dF_{nj}(x))^2\} + g_2(n,\varepsilon)$$

throughout the inequality (3.19) to obtain

$$\{\int_{|x|<\varepsilon^2} x^2 dM_n(x)\}\{\int_{|x|<1/\varepsilon} u^2 dF_Y(u) - (\int_{|u|<1/\varepsilon} u dF_Y(u))^2\} \qquad (3.20)$$

$$\leq \sigma_n^2(\varepsilon) - (\int_{|x|<1/\varepsilon} x dF_Y(x))^2 \{\int_{|x|<\varepsilon^2} x^2 dM_n(x)$$

$$- \Sigma (\int_{|x|<\varepsilon^2} x dF_{nj}(x))^2\} - g_2(n,\varepsilon)$$

$$\leq \{\int_{|x|<\varepsilon^2} x^2 dM_n(x)\}\{E(Y^2) - (\int_{|u|<1/\varepsilon} u dF_Y(u))^2\}.$$

Now, $\lim\limits_{\varepsilon \to 0} \begin{Bmatrix} \lim \sup \\ \lim \inf \end{Bmatrix}_{n \to \infty}$ of the inside of (3.20) equals $(\sigma')^2 - (E(Y))^2 \sigma^2$

while on the outside of (3.20) the above limits yield

$\text{Var}(Y) \lim\limits_{\varepsilon \to 0} \begin{Bmatrix} \lim \sup \\ \lim \inf \end{Bmatrix}_{n \to \infty} \int_{|x|<\varepsilon^2} |x|^2 dM_n(x).$ We thus conclude (3.1)

holds with $\text{Var}(Y)C_0 = (\sigma')^2 - (E(Y))^2 \sigma^2$. This completes the proof of the

theorem.

Let us now turn to the framework of the more classical central

limit theorem. If $\{X_1, X_2, \ldots\}$ is a sequence of independent identically

distributed random variables with common distribution function F, we

shall write $X_1 \in \mathcal{D}(\alpha)$ to denote the fact that F (or X_1) is in the domain

of attraction of a stable law with characteristic exponent α. That is,

there exists norming constants $\{B_n , n = 1, 2, \ldots\}$ and centering constants

$\{A_n , n = 1, 2, \ldots\}$ such that $\Sigma_{j=1}^{n} X_j / B_n - A_n$ converges in law to a distri-

bution which is stable with exponent $\alpha \leq 2$.

In this case the relation between the scale mixed system S' and

the original system S takes on a particularly appealing form. Our con-

dition for convergence only depends upon the moments of Y and the index

α.

THEOREM 4. <u>Let</u> $\{X_1, Y_1, X_2, Y_2, \ldots\}$ <u>be independent random variables</u>,

X_n <u>having distribution function</u> F_X <u>and</u> Y_n <u>having distribution function</u>

F_Y

1) <u>If</u> $E|Y|^2 < \infty$, <u>then</u> $X \in \mathcal{D}(2)$ <u>implies</u> $YX \in \mathcal{D}(2)$ <u>and the same</u>

<u>norming constants work</u>. <u>Conversely if</u> $YX \in \mathcal{D}(2)$ <u>then</u> $X \in \mathcal{D}(2)$ <u>and</u>

<u>the same norming constants work</u>.

2) <u>If</u> $E|Y|^{\alpha+\delta} < \infty$ <u>for some</u> $\delta > 0$, <u>then</u> $X \in \mathcal{D}(\alpha)$ <u>implies</u> $YX \in \mathcal{D}(\alpha)$

<u>and the same norming constants work</u>.

PROOF. The direct statement in 1) and 2) will follow from Theorem

3 once we show that (3.1) holds. Let $S = \{\{X_j/B_n , j = 1, 2, \ldots, n\}\}$ then

as proven in [4], $X \in \mathcal{D}(\alpha)$ implies $\beta_S = \alpha$. For $\alpha < 2$ we have (3.1)

holding with $C_0 = 0$ by (2.4). For $\alpha = 2$ we know (see Feller (1971), [6]

pg. 314) that necessarily

$$\frac{n}{B_n^2} \int_{|y| < B_n} y^2 dF_X(y) \to C_0 \quad \text{as} \quad n \to \infty . \tag{3.21}$$

Using the fact that $\dfrac{n}{B_n^2} \displaystyle\int_{|y|<B_n\varepsilon} y^2 dF_X(y) = \int_{|x|<\varepsilon} y^2 dM_n(y)$ and that

$\displaystyle\int_{|y|<t} y^2 dF_X(y)$ is a slowly varying function of t yields (3.1) in this

case also. Applying Theorem 3 we obtain the convergence of

$S' = \{\{X_j Y_j/B_n, \ j = 1, 2, \ldots, n\}\}$. If $\alpha = 2$ then $M \equiv 0$ so by 2^0 of Theorem 3,

$\Lambda \equiv 0$ and S' converges to a normal distribution. If $\alpha < 2$, then by 3^0 of

Theorem 3 $(\sigma')^2 = 0$ and with

$$M(x) = \begin{cases} c_1 |x|^{-\alpha} & \text{if } x < 0 \\[2mm] -c_2 x^{-\alpha} & \text{if } x > 0 \end{cases} \tag{3.22}$$

we have by 2^0 of Theorem 3

$$\Lambda(x) = \begin{cases} c_1 \displaystyle\int_{(-\infty,0)} \{1-F_Y(x/t)d(|t|^{-\alpha})-c_2 \int_{(0,\infty)} F_Y(x/t)d(t^{-\alpha}) & \text{if } x < 0 \\[3mm] -c_1 \displaystyle\int_{(-\infty,0)} F_Y(x/t)d(|t|^{-\alpha})-c_2 \int_{(0,\infty)} \{F_Y(x/t)-1\}d(t^{-\alpha}) & \text{if } x > 0, \end{cases}$$

or equivalently, upon integrating by parts

$$\Lambda(x) = \begin{cases} c_1 \displaystyle\int_{(-\infty,0)} |t|^{-\alpha} dF_Y(\tfrac{x}{t}) + c_2 \int_{(0,\infty)} t^{-\alpha} dF_Y(\tfrac{x}{t}) & \text{if } x < 0 \\[3mm] c_1 \displaystyle\int_{(-\infty,0)} |t|^{-\alpha} dF_Y(\tfrac{x}{t}) + c_2 \int_{(0,\infty)} t^{-\alpha} dF_Y(\tfrac{x}{t}) & \text{if } x > 0. \end{cases} \tag{3.23}$$

Upon simplifying (3.23) we obtain

$$\Lambda(x) = \begin{cases} a_1 |x|^{-\alpha} & \text{if } x < 0 \\[2mm] -a_2 x^{-\alpha} & \text{if } x > 0, \end{cases} \tag{3.24}$$

i.e., $X_n Y_n$ is in the domain of attraction of a stable law with character-istic exponent α, and the same norming constants work.

Let us suppose now that $YX \in \mathcal{D}(2)$ and $E|Y|^2 < \infty$. We shall use the general limit theorem to show $\Sigma_{j=1}^n X_j / B_n - A_n$ converges to a normal distribution. With M_n given by (2.1) and Λ_n given by (3.3) we shall show $\Lambda_n(x) \to 0$ for $x \neq 0$ implies $M_n(x) \to 0$ for $x \neq 0$. Indeed let t be such that $P[|Y| > t] > 0$. Then

$$0 \leftarrow \int_{|x|>t} d\Lambda_n(x) = \int \int_{|ux|>t^2} dF_Y(u) dM_n(x)$$

$$\geq \int_{|x|>t} \int_{|ux|>t^2} dF_Y(u) dM_n(x) \geq \int_{|x|>t} P[|Y| > t] dM_n(x).$$

Since $P[|Y| > t] > 0$, $\int_{|x|>t} dM_n(x) \to 0$, thus the limiting Levy function is 0.

Let us complete the proof by showing

$$\lim_{\epsilon \downarrow 0} \begin{Bmatrix} \lim\sup \\ \lim\inf \end{Bmatrix}_{n \to \infty} \int_{|x|<\epsilon} x^2 dM_n(x) - n(\int_{|x|<\epsilon} x dF_X(x))^2$$

exists and is finite. In view of part 7) of the proof of Theorem 3 we know we must have

$$C_0 = \lim_{\epsilon \to 0} \lim_{n \to \infty}\sup \int_{|x|<\epsilon} x^2 dM_n(x) < \infty.$$

Utilizing the inequality (3.20) of Theorem 3 we see that we must only show for

$$C_1 = \lim_{\varepsilon \downarrow 0} \liminf_{n \to \infty} \int_{|x|<\varepsilon} x^2 dM_n(x)$$

we have $C_0 = C_1$. Now by Fatou's lemma and by (3.21)

$$C_2 \equiv \lim_{n \to \infty} \int_{|x|<\varepsilon} x^2 d\Lambda_n(x) = \liminf_{n \to \infty} \int \int_{|ux|<\varepsilon} x^2 u^2 dM_n(x) dF_Y u$$

$$\leq \lim_{\varepsilon \downarrow 0} \int \liminf_{n \to \infty} u^2 \int_{|ux|<\varepsilon} x^2 dM_n(x) dF_Y(u) = C_1 E(Y^2).$$

On the other hand for $\eta > 0$ choose δ so small that $\int_{|u| \geq \delta} u^2 dF_Y(u) >$ $(1-\eta)E(Y^2)$. Then $C_2 \geq \limsup_{n \to \infty} \int_{|u| \geq \delta} u^2 \int_{|x|<\varepsilon/\delta} x^2 dM_n(x) dF_Y(u) \geq$ $(1-\eta)E(Y^2) \limsup_{n \to \infty} \int_{|x|<\varepsilon/\delta} x^2 dM_n(x)$. Thus

$$(1-\eta)E(Y^2)C_0 \leq C_2 \leq E(Y^2)C_1 \leq E(Y^2)C_0.$$

Since $\eta > 0$ is arbitrary $C_1 = C_0$ and $\sum_{j=1}^{n} X_j / B_n - A_n$ converges to a normal distribution.

REMARK 2. i) The first part of Theorem 4 should be compared to a result of H. Tucker (1968) who considered sums of random variables in the domain of attraction of the stable distributions instead of their products. He shows that if $EY^2 < \infty$ and $X \in \mathcal{D}(\alpha)$ then $X + Y \in \mathcal{D}(\alpha)$ and the same norming constants work. If $X + Y \in \mathcal{D}(\alpha)$ and $E(Y)^2 < \infty$, then a

slight modification of his methods yields $X \in \mathcal{D}(\alpha)$. Combining our result with his we find that if X, Y, ε are independent random variables, $EY^2 < \infty$ and $E\varepsilon^2 < \infty$, then $X \in \mathcal{D}(\alpha) \Rightarrow YX + \varepsilon \in \mathcal{D}(\alpha)$ and the same norming constants work. For $\alpha = 2$, $X \in \mathcal{D}(2) \Longleftrightarrow YX + \varepsilon \in \mathcal{D}(2)$ with the same norming constants.

ii) In Theorem 4 we can actually calculate the scale change involved in the distribution of the limit laws of S and S' when $\alpha < 2$. Namely, in going from (3.23) to (3.24) we have

$$\Lambda(x) = \begin{cases} a_1 |x|^{-\alpha} & x < 0 \\ -a_2 x^{-\alpha} & x > 0 \end{cases}$$

where $a_1 = C_1 \int_{(0,\infty)} t^\alpha dF_Y(t) + C_2 \int_{(-\infty,0)} |t|^\alpha dF_Y(t)$ and $a_2 = C_2 \int_{(0,\infty)} t^\alpha dF_Y(t) + C_1 \int_{(-\infty,0)} |t|^\alpha dF_Y(t)$. This follows immediately from (3.23) by the change of variables $z = xt^{-1}$.

iii) It can be shown that for $\alpha < 2$, $YX \in \mathcal{D}(\alpha)$ and $E|Y|^{\alpha+\eta} < \infty$ implies that $E|X|^{\alpha-\delta} < \infty$ and $E|X|^{\alpha+\delta} = \infty$ for all δ, hence $\beta_S = \alpha$. I suspect that in fact $X \in \mathcal{D}(\alpha)$ however, I have been unable to establish this for $\alpha \neq 2$.

iv) Suppose that

$$\lim_{\varepsilon \to 0} \begin{Bmatrix} \lim \sup \\ \lim \inf \end{Bmatrix}_{n \to \infty} \int_{|x| < \varepsilon} |x|^{\beta_S} dM_n(x) = C_0$$

then for $1 - F_Y(x) + F_Y(-x) \sim \dfrac{C_1}{|x|^{\beta_S}}$ we have $\Lambda_n(x) \to \dfrac{C_0 C_1}{|x|^{\beta_S}} + \Lambda(x)$ where $\Lambda(x)$

is given by 2^0 of Theorem 3. This is a Levy spectral function when

$\beta_S < 2$. Thus we see that some random scale changes introduce a stable

component into the limit law.

We can also use Theorem 4 to derive some interesting statements

about slowly varying functions which would be difficult to prove by

other means. The precise formulations are given in the following

corollary.

COROLLARY 1^0. Suppose $EY^2 < \infty$. Then $\displaystyle\int_{|x|<t} x^2 dF_X(x)$ is a slowly

varying function of t if and only if $\displaystyle\int\int_{|ux|<t} x^2 u^2 dF_X(x) dF_Y(u)$ is a

slowly varying function of t.

2^0. Suppose

a) $\displaystyle\int_{|x|<t} x^2 dF_X(x)$ varies regularly with exponent $2 - \alpha$ as a

function of t and also

$$\frac{P[X > x]}{P[|X| \geq x]} \xrightarrow[x \to \infty]{} p, \qquad \frac{P[X < -x]}{P[|X| \geq x]} \xrightarrow[x \to \infty]{} q$$

b) and $E|Y|^{\alpha+\delta} < \infty$ for some $\delta > 0$, then

$$\int_{(-\infty,\infty)} \int_{|ux|<t} x^2 u^2 dF_Y(u) dF_X(x)$$

varies regularly with exponent $2 - \alpha$ as a function of t and also

$$\frac{\int_{(-\infty,\infty)} P[tY \geq x]dF_X(t)}{\int_{(-\infty,\infty)} P[|tY| \geq x]dF_X(t)} \xrightarrow[x \to \infty]{} p',$$

$$\frac{\int_{(-\infty,\infty)} P[tY \leq -x]dF_X(t)}{\int_{(-\infty,\infty)} P[|tY| \geq x]dF_X(t)} \xrightarrow[x \to \infty]{} q'$$

for some p', q' $\geq$ 0 with p'+q' > 0.

PROOF. This follows immediately from Theorem 4 by utilizing the necessary and sufficient conditions given in Feller (1971) for a distribution to belong to the domain of attraction of a stable law.

In the previous theorem we observed an interesting phenomenon. Namely we took an infinitesimal system S and subjected it to an arbitrary random scale change with $E|Y|^{\beta_S + \delta} < \infty$ and we obtained a new system S' which was convergent. Moreover the limit distributions of S and S' were of the same type. In problems where X_{ni} represents a "true" or theoretical measurement of some occurrence and Y_i the scale change in the measuring device used to measure the occurrence, we obtain as an observation the product $X_{ni}Y_i$. It is of interest to determine when limit distribution calculated from the empirical data $\{\{X_{nj}Y_j, j = 1, 2, \ldots, k_n\}\}$ is of the same type as that from $\{\{X_{nj}, j = 1, 2, \ldots, k_n\}\}$. For normed sums in the domain of attraction of a stable law, Theorem 4 answers the question and Remark 2ii) allows

us to calculate the scale change. In general the following theorem tells

us that in non-stable limits the empirical data may yield a different

type distribution than the theoretical data $\{\{X_{nj}, j = 1, 2, \ldots, k_n\}\}$.

THEOREM 5. <u>In</u> <u>order</u> <u>that</u> <u>a</u> <u>limit</u> <u>distribution</u> <u>be</u> <u>preserved</u> <u>in</u> <u>type</u>

<u>under</u> <u>all</u> <u>random</u> <u>scale</u> <u>mixtures</u> <u>with</u> $E|Y|^{\beta_s + \delta} < \infty$ <u>it</u> <u>is</u> <u>necessary</u> <u>and</u>

<u>sufficient</u> <u>that</u> <u>the</u> <u>limit</u> <u>distribution</u> <u>type</u> <u>be</u> <u>either</u> <u>purely</u> <u>stable</u> <u>or</u> <u>a</u>

<u>mixture</u> <u>of</u> <u>stable</u> <u>and</u> <u>normal</u>.

PROOF. The sufficiency follows from 2^0 and 3^0 of Theorem 3 of

Theorem 3 as we calculated in Theorem 4, and in fact the scale change

involved is given in Remark 2ii).

Suppose now that the limit distribution is preserved in type when

subject to random scale change. Then with S and S' as defined in Theorem

3 we know $Z \sim (\gamma', (\sigma')^2, \Lambda)$ is of the same type as $X \sim (\gamma, \sigma^2, M)$, thus

$(\sigma')^2 = a^2 \sigma^2$ and $\Lambda(x) = M(x/a)$ for some constant a. As in 2) of the proof

of Theorem 3 we know that

$$\Lambda(x) = \begin{cases} \int_{-\infty}^{\infty} |M(x/t)| \, dF_Y(t) & x < 0 \\ -\int_{-\infty}^{\infty} |M(x/t)| \, dF_Y(t) & x > 0 \end{cases} \qquad (3.25)$$

If $M(x) \equiv 0$, then both X and Z are normally distributed. If $M(x) \not\equiv 0$,

then we must show M is given by (3.22) of Theorem 4. Using the fact

that $\Lambda(x) = M(x/a)$ in (3.25) and letting

$$h(t) = \frac{|M(x/t)|}{M(x/a)} , \quad \text{we see that } h \text{ is a function of } t \qquad (3.26)$$

alone, the equality holding a.e. $[dF_Y]$. Since Y could be chosen to be absolutely continuous we have (3.26) holding on a dense set of points t. For simplicity in calculation we shall consider the case $x > 0$, $t > 0$, $a > 0$ and denote $N(x) = M(x)/M(1)$. The other cases may be considered similarly. Rewriting the equation (3.26) yields

$$N(x/a)h(t) = N(x/t)$$

and hence for $x = a$

$$h(a/y) = \frac{N(xy/a)}{N(x/a)} = N(y).$$

Thus

$$N(y^2) = N(y)h(a/y) = (N(y))^2$$

and by induction for any k

$$N(y^k) = N(y^{k-1})h(a/y) = N^k(y).$$

Letting $U(x) = -\ell n |N(e^x)|$, the above equation becomes

$$1/k \ U(k \ \ell n \ x) = U(\ell n \ x) \quad \text{or}$$

$1/k \ U(ky) = U(y)$. By Lemma 3, page 277 of Feller (1971), this implies $U(x) = x^\rho$ or equivalently $N(x) = x^\rho$. That $-2 < \rho < 0$ follows from M being

a Levy Spectral function. Thus $M(x) = \dfrac{M(1)}{x^{\alpha}}$ for $x > 0$. Similarly we can

show $M(x) = \dfrac{M(-1)}{|x|^{\beta}}$ for $x < 0$. We see that $\alpha = \beta$ since the limit distribution

is of the same type whether multiplied by $Y \leq 0$ or $Y \geq 0$. The result then

follows easily from the calculation of $\Lambda(x)$ in both cases. This completes

the proof of the theorem.

REMARK. The problem considered here could have been solved by de-

fining the class C_S of random variables by

$$C_S = \{Y: \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup\{\int_{|x| > \varepsilon} \{1 - F_Y(xt) + F_Y(-xt)\} dM_n(x) = 0\}\} \;,$$

and proving the main theorem for members of this class. Some such account

must be made of the complex interaction of $M_n(t)$ as $(t,n) \to (0,\infty)$ and

$1 - F_Y(t)$ as $t \to \infty$. Since moment conditions are perhaps a more natural

approach, we choose to introduce the index β_S instead. To avoid the

borderline case $1 - F_Y(y) \sim y^{-\beta_S}$, we choose to assume $E|Y|^{\beta_S + \delta} < \infty$.

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