A STONE-WEIERSTRASS THEOREM FOR GROUP REPRESENTATIONS

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ABSTRACT.   It is well known that if G is a compact group and π a faithful

(unitary) representation, then each irreducible representation of G occurs in the

tensor product of some number of copies of π and its contragredient. We gener-

alize this result to a separable type I locally compact group G as follows: let

π be a faithful unitary representation whose matrix coefficient functions vanish

at infinity and satisfy an appropriate integrability condition.   Then, up to

isomorphism, the regular representation of G is contained in the direct sum of

all tensor products of finitely many copies of π and its contragredient.

   We apply this result to a symplectic group and the Weil representation associ-

ated to a quadratic form. As the tensor products of such a representation are

also Weil representations (associated to different forms), we see that any discrete

series representation can be realized as a subrepresentation of a Weil represen-

tation.

1. INTRODUCTION.

Let G be a separable locally compact group, and π a representation of G

("representation" will always mean a strongly continuous unitary representation

on a separable Hilbert space). For every pair of non-negative integers m , n ,

not both zero, let

$$\pi_{m,n} = (\pi \otimes \cdots \otimes \pi) \otimes (\bar\pi \otimes \cdots \otimes \bar\pi) ,$$

where π occurs m times, and its contragredient $\bar\pi$ occurs n times.

Our starting point is the following easy fact.

THEOREM 1. Let G be a compact group and π a faithful representation.

Then every irreducible representation of G occurs in

$$\sigma = \underset{m,n \geq 0}{\oplus'} \pi_{m,n} ,$$

where the prime means we omit (m , n) = (0 , 0) from the sum.

PROOF. Consider the set of linear combinations of the matrix coefficient

functions of the representation σ . These functions form an algebra, are

closed under complex conjugation, separate points of G , and do not all

vanish at any point of G . By the Stone-Weierstrass Theorem, they are sup-

norm dense in the continuous functions on G , and hence dense in $L^2(G)$.

But, by the Peter-Weyl Theorem, a representation whose coefficient functions

are dense in $L^2(G)$ must contain every irreducible representation.   ∎

REMARK. Since, as is well known, $\pi \otimes \bar\pi$ contains the trivial represent-

tation, it is easy to see that every irreducible representation occurs not

just once, but countably infinitely many times in σ .

This theorem, in a slightly different form, is originally due to Burnside,

in the case of a finite group G (see, e.g., Curtis and Reiner [2], Theorem

32.9); namely:

THEOREM 2. (Burnside). Let G be a finite group, π a faithful represen-
tation. Then every irreducible representation of G occurs in the tensor
product of some number of copies of π .

Rather than prove Burnside's theorem directly, we obtain it as an obvious
special case of the following.

THEOREM 3. Let n be a positive integer. Let G be a compact group, all
of whose elements have order dividing n . If π is a faithful representation
of G , then every irreducible representation of G occurs in the tensor
product of some number of copies of π .

PROOF. The argument is the same as that for Theorem 1; the only thing
that is not obvious is that the algebra spanned by coefficient functions is
closed under complex conjugation. But the complex conjugates of the matrix
coefficients of $\pi(g)$ are just the matrix coefficients of $\pi(g)^{-1}$ rearranged
(since π is unitary), and the matrix coefficients of $\pi(g)^{-1} = \pi(g)^{n-1}$ are
just sums of products of matrix coefficients of $\pi(g)$, so we are done.   ▌

We generalize Theorem 1 to a non-compact group.

2. PRELIMINARIES.

In this section we establish two results needed for the proof of the main
theorem. As above, G is a separable locally compact group.

Let ψ be a continuous square-integrable function on G . Let U be the
open set where ψ is not zero. We may restrict Haar measure to U and
speak of the closed subspace $L^2(U) \subseteq L^2(G)$, consisting of functions which
vanish off U .

LEMMA 4.   Let A be an algebra of continuous functions on G which vanish

at infinity and separate points, and suppose A is closed under complex con-

jugation. Let ψ , U be as above.

Then $\psi \cdot A = \{\psi \cdot f : f \in A\}$ is dense in $L^2(U)$.

PROOF.   Let $\phi \in C_c(U)$, and fix $\epsilon > 0$. Then $\phi / \psi \in C_c(U)$, and by

the Stone-Weierstrass Theorem there exists $\phi' \in A$ such that $\|\phi' - \phi/\psi\|_\infty < \epsilon$.

Consequently, $\|\psi \cdot \phi' - \phi\|_2 = \|\psi(\phi' - \phi/\psi)\|_2 \leq \|\psi\|_2 \cdot \epsilon$, and we have approximated

ϕ by a function in $\psi \cdot A$. Since $C_c(U)$ is dense in $L^2(U)$, the proof is

complete.   ▌

Now let σ be a representation of G on a Hilbert space H .

LEMMA 5.   Suppose the intersection of $L^2(G)$ with the set of coefficient

functions of σ spans a dense subspace of $L^2(G)$. Then (up to isomorphism)

the regular representation of G is quasi-contained in σ (i.e. contained

in a direct sum of copies of σ).

PROOF.   If u , v $\in H$ are such that the coefficient function $\phi_{u,v}(g)$

$= \langle \sigma(g)u,v \rangle$ is in $L^2(G)$, we define a linear map T_v from some subspace

of H to $L^2(G)$ by $T_v(w) = \phi_{w,v}$.

The domain of T_v will be all vectors w $\in H$ such that $\phi_{w,v}$ is square-

integrable; it is a G-invariant subspace, containing u . It is easy to check

that T_v is a closed G-map; hence we may apply Schur's Lemma to conclude that

there are closed G-subspaces $V_v \subseteq H$, $W_v \subseteq L^2(G)$ and a surjective unitary

G-isomorphism $U_v : V_v \to W_v$. Moreover, W_v contains the function $\phi_{u,v}$.

By hypothesis we may choose (countably many) coefficient functions $\phi_{u,v}$

whose span is dense in $L^2(G)$. The corresponding subspaces W_v thus span a

dense subspace of $L^2(G)$, and, sticking together the maps U_v^{-1} and

applying Schur's Lemma again, we get a unitary G-isomorphism of $L^2(G)$ into

$\oplus_\nu V_\nu \subseteq \oplus_\nu H$, a countable direct sum of copies of H . ∎

3. <u>THE MAIN RESULT</u>.

   We shall prove two weaker versions of our main result as a preliminary step.

   PROPOSITION 6. Let G be a separable locally compact group, and π a faithful representation of G . Suppose that a non-trivial coefficient function of π is in $L^p(G)$, for some finite p , and that all the coefficient functions vanish at infinity.

   Then the regular representation of G is quasi-contained in

$$\sigma = \bigoplus_{m,n \geq 0}' \pi_{m,n} .$$

   PROOF. By Lemma 5, it suffices to find coefficient functions of σ whose span is dense in $L^2(G)$. Suppose ϕ is a coefficient function of π which is in $L^p(G)$. Then some power ϕ^n of ϕ is in $L^2(G)$. Since ϕ^n is a coefficient function of $\pi_{n,0}$, we see that the algebra A spanned by the coefficient functions of σ contains $\phi^n \cdot A$ (note σ contains $\pi_{n,0} \otimes \sigma$). By Lemma 4, $A \cap L^2(G)$ contains a dense set of $L^2(U)$, where $U = \{g \in G : \phi(g) \neq 0\}$.

   Since $A \cap L^2(G)$ is invariant under translations by G , we see that it contains a dense subset of $L^2(g \cdot U)$, for any $g \in G$. But since U is open, any element of $L^2(G)$ can be approximated by a finite sum of elements belonging to spaces $L^2(g \cdot U)$, for appropriate choices of $g \in G$. Such a sum can be approximated by something in $A \cap L^2(G)$.

   We have approximated an arbitrary element of $L^2(G)$ by a sum of elements of $A \cap L^2(G)$, as desired. ∎

In fact, we can strengthen this result slightly, as follows:

PROPOSITION 7. Let π , σ be as above, and let τ be any representation of G . Then the regular representation of G is quasi-contained in $\tau \otimes \sigma$.

PROOF. The proof is analogous to that of Proposition 6; we just replace ϕ^n by $\phi_0 \cdot \phi^n$, where ϕ_0 is any coefficient function of τ , and let $U = \{g \in G : \phi_0 \cdot \phi(g) \neq 0\}$.   ∎

We are now ready to prove the main result, which is essentially Proposition 7 with "quasi-contained" changed to "contained".

THEOREM 8. Let G be a separable type I locally compact group, π a faithful representation whose coefficient functions vanish at infinity and one of whose non-trivial coefficient functions is in $L^p(G)$ for some finite p . Let τ be any representation.

Then the regular representation of G is (up to unitary isomorphism) contained countably infinitely many times in $\tau \otimes \sigma = \tau \otimes \left(\displaystyle\bigoplus_{m,n \geq 0}' \pi_{m,n} \right)$.

PROOF. We shall show that every subrepresentation of the regular representation occurs not just once but infinitely many times in $\tau \otimes \sigma$. It will suffice to consider multiplicity-free subrepresentations, i.e. those of the form

$$\eta = \int_{\hat{G}}^{\oplus} \nu \, d\mu(\nu) \ ,$$

where $\hat{G}$ is the set of equivalence classes of unitary irreducible representations of G , and μ is some <u>finite</u> Radon measure on $\hat{G}$ which is absolutely continuous with respect to the Plancherel measure. Fix such a μ , and fix $\varepsilon > 0$. By Proposition 7, we know that η is contained in $\tau \otimes \sigma$, so there

is a positive integer $M(1)$ such that $\tau \otimes \sigma_1 = \tau \otimes \left(\bigoplus_{m,n<M(1)}^{\oplus'} \pi_{m,n} \right)$

contains most of η , namely, $\tau \otimes \sigma_1$ contains $\int_{X_1}^{\oplus} \nu \, d\mu(\nu)$, for some

measurable set $X_1 \subseteq \hat{G}$ such that $\mu(\hat{G} \setminus X_1) < \epsilon/2$.

Now consider the representation $\tau \otimes \pi_{M(1),M(1)} \otimes \sigma = \tau \otimes \left(\bigoplus_{M(1)\leq m,n}^{\oplus'} \pi_{m,n} \right)$

$\Big($ the prime means omit $(m,n) = \big(M(1),M(1)\big)$ $\Big)$. By Proposition 4, this

representation contains η , so we may find a measurable set $X_2 \subseteq \hat{G}$ with

$\mu(\hat{G} \setminus X_2) < \epsilon/4$ and a positive integer $M(2)$ such that $\tau \otimes \sigma_2$

$= \tau \otimes \left(\bigoplus_{M(1)\leq m,n<M(2)}^{\oplus'} \pi_{m,n} \right)$ contains $\int_{X_2}^{\oplus} \nu \, d\mu(\nu)$.

Continuing inductively, we find measurable sets $X_1 , X_2 , X_3 , \ldots \subseteq \hat{G}$

such that $\mu(\hat{G} \setminus X_k) < 2^{-k} \cdot \epsilon$, and integers $M(1) < M(2) < M(3) < \ldots$ such

that $\tau \otimes \sigma_k = \tau \otimes \left(\bigoplus_{M(k-1)\leq m,n<M(k)}^{\oplus'} \pi_{m,n} \right)$ contains $\int_{X_k}^{\oplus} \nu \, d\mu(\nu)$.

But σ contains $\sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \ldots$, so $\tau \otimes \sigma$ contains countably

infinitely many copies of $\int_{X}^{\oplus} \nu \, d\mu(\nu)$, where $X = \bigcap X_k$. Now

$\mu(\hat{G} \setminus X) < \epsilon$, and since ϵ was arbitrary, the theorem is proved.   ∎

4.   WEIL REPRESENTATIONS AND THE DISCRETE SERIES.

The Weil representation has long been known as a fruitful place for

realizing various irreducible representations (see, e.g., Gelbart, [3] and

Howe, [5]). We shall apply Theorem 8 to show that any discrete series

representation of a symplectic group can be so realized.

In (Weil, [7]) there is constructed a representation of the symplectic

group associated to a quadratic form Q . The tensor products of this

representation with itself and/or its contragredient are again Weil represen-

tations, those associated to forms constructed by taking direct sums of

copies of $\pm Q$. Since the discrete series representations of G are sub-

representations of the regular representation, we have as an immediate conse-
quence of Theorem 8:

THEOREM 9. With G as above, any representation of the discrete series
of G is a subrepresentation of some Weil representation of G (associated
to some quadratic form).

5. COMPLEMENTS.

In this section we make some remarks and give some examples.

The conditions on π — that its matrix coefficient functions must
vanish at infinity, and that at least one of them must be in some $L^p(G)$ —
seem rather peculiar at first. We remark that they are necessary, for
example, to exclude finite dimensional representations of a non-compact group
(indeed, if π were finite dimensional, then σ would be a direct sum of
finite dimensional representations, and could not possibly contain the
regular representation). On the other hand, though something of this sort
is needed, the conditions we have given can be altered or weakened in
various ways; we give two examples. Firstly, it is not necessary to know
that the coefficient functions all vanish at infinity. It would suffice to
know that the ones belonging to some dense set of vectors did so (e.g.
K-finite vectors), or even just that one coefficient function achieves its
maximum absolute value on a compact set. Suitable modifications of Lemma 4
hold in these cases. Secondly, we note that in Theorem 8 the L^p condition
on π could without difficulty be replaced with an L^2 condition on τ .

Finally, we observe that neither of the conditions is actually as
unnatural as it may first seem. They are both satisfied in a great many
cases, including all connected semi-simple Lie groups with finite centre
(see Cowling, [1] and Howe, [4]).

EXAMPLE 1. Let G be the circle group; let χ_n be the character $e^{in\theta}$.
Then χ_1 , χ_{-1} are the only faithful irreducible representations (though of
course there are many other faithful representations), and if, for example,
we let $\pi = \chi_1$, then $\bar{\pi} = \chi_{-1}$, and $\pi_{m,n} = \chi_{m-n}$. Every χ_k equals some
$\pi_{m,n}$ (in fact infinitely many).

EXAMPLE 2. Let $G = SL_2(R)$, or $PSL_2(R)$. Then the results of Repka,
[6] show that if π is any non-trivial irreducible unitary representation,
then the regular representation is contained in $\underset{m,n\leq 3}{\oplus'} \pi_{m,n}$. This example
also shows that $\sigma = \oplus' \pi_{m,n}$ may contain subrepresentations which are not
contained in the regular representation (this happens if π is some "comple-
mentary series" representations).

REFERENCES

1. Cowling, M. Sur l'Algèbre de Fourier-Stieltjes d'un Groupe Semi-simple,
       preprint (1976).

2. Curtis, C.W. and Reiner, I. Representation Theory of Finite Groups and
       Associative Algebras, Interscience (1966).

3. Gelbart, S. Holomorphic Discrete Series for the Real Symplectic Group,
       Inv. Math. 19 (1973) 49-58.

4. Howe, R. On the Asymptotic Behaviour of Matrix Coefficients, preprint
       (1975).

5. Howe, R. Invariant Theory and Duality for Classical Groups over Finite
       Fields, preprint (1976).

6. Repka, J. Tensor Products of Unitary Representations of $SL_2(R)$,
       Bull. Amer. Math. Soc. 82 (1976) 930-932; also to appear, Amer. J.
       Math.

7. Weil, A. Sur Certains Groupes d'Opérateurs Unitaires, Acta Math. 111
       (1964) 143-211.