Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 69-74

A NOTE ON RIEMANN INTEGRABILITY

G. A. BEER
Department of Mathematics
California State University
Los Angeles, California 90032

(Received November 21, 1977)

ABSTRACT. In this note we define Riemann integrability for real valued functions defined on a compact metric space accompanied by a finite Borel measure. If the measure of each open ball equals the measure of its corresponding closed ball, then a bounded function is Riemann integrable if and only if its set of points of discontinuity has measure zero.

Let $\mathscr{A}$ denote the algebra of sets generated by the open and closed subintervals of an interval $[a,b]$. A bounded real valued function f defined on $[a,b]$ is Riemann integrable if for each positive ε, there exist two functions ϕ and ψ that are linear combinations of characteristic functions of sets in $\mathscr{A}$ satisfying $\phi \le f \le \psi$ and

$$\int_a^b \psi\, dm - \int_a^b \phi\, dm < \varepsilon$$

where m denotes ordinary Lebesgue measure. Riemann integrability may be defined in an analagous way for real valued functions defined on a compact metric space K accompanied by a finite Borel measure. If we make a simple assumption about the balls of K, then the following famous theorem of Lebesgue extends: a bounded real valued function f defined on $[a,b]$ is Riemann integrable if and only if the set of points at which f is not continuous has Lebesgue measure zero.

Suppose that K is a compact metric space and μ is a finite Borel measure on K. Let $B_r(x) = \{y: d(x,y) < r\}$ and $\overline{B}_r(x) = \{y: d(x,y) \le r\}$ denote the open and closed balls of radius r about a point x in K. Let $\mathscr{A}$ denote the algebra generated by all such balls. Any element of $\mathscr{A}$ is of the form

$$\bigcup_{1 \le i \le m} \ \bigcap_{1 \le k \le n_i} A_{ik} \tag{1}$$

where A_{ik} is a ball or its complement and $\{m,n_1,\ldots,n_m\}$ are positive integers. A <u>step function</u> is a linear combination of characteristic functions determined by elements of $\mathscr{A}$. Hence a step function ϕ has the form $\Sigma\, d_i\chi_{A_i}$ where each d_i is real and $A_i \in \mathscr{A}$. Since $\mathscr{A}$ is an algebra, the $\{A_i\}$ can be taken to be pairwise disjoint. It is easy to see that if ϕ and ψ are step functions, then so are $\phi + \psi$, $\phi - \psi$, $\inf\{\phi,\psi\}$, and $\sup\{\phi,\psi\}$.

DEFINITION. A bounded real valued function f defined on K is <u>Riemann integrable</u> if for each positive ε there exist step functions ϕ and ψ such that $\phi \le f \le \psi$ and $\int \psi\, d\mu - \int \phi\, d\mu < \varepsilon$.

Given a bounded real valued function f defined on K, the <u>upper envelope</u> h of f is the function defined by

$$h(x) = \inf_{\delta>0} \sup_{y \in B_\delta(x)} f(y) \qquad x \in K$$

and the lower envelope g of f is defined by

$$g(x) = \sup_{\delta>0}\inf_{y \in B_\delta(x)} f(y) \qquad x \in K$$

It is well known that h is upper semicontinuous, g is lower semicontinuous, $g(x) \leq f(x) \leq h(x)$ for each x, and $g(x) = h(x)$ if and only if f is continuous at x (see Royden [1, p.49]).

THEOREM. Suppose $\mu(B_r(x)) = \mu(\overline{B}_r(x))$ for each x in K and for each positive r. A bounded real valued function f defined on K is Riemann integrable if and only if the set of points at which f is discontinuous has μ-measure zero.

Proof. Let h be the upper envelope of f and g its lower envelope. Let ψ be any step function that exceeds f. Since each member of $\mathscr{A}$ can be expressed in the form depicted in (1), the condition on the balls of K implies that each member of $\mathscr{A}$ is the union of an open set and a set of μ-measure zero. It follows that ψ can be represented as

$$\sum_{j=1}^{n} a_j \chi_{A_j}$$

where (i) A_j is an open set for $1 \leq j \leq m$ (ii) $\mu(A_j) = 0$ for $m < j \leq n$ (iii) $\{A_1, A_2, \ldots, A_n\}$ partition K.

Let $x \in \bigcup_{j=1}^{m} A_j$. Since ψ is constant near x, there exists $\delta > 0$ such that $\psi(x) \geq \sup_{y \in B_\delta(x)} f(y)$ so that $\psi(x) \geq h(x)$. Hence, $\mu\{x\colon \psi(x) < h(x)\} = 0$, and we have $\int \psi \, d\mu \geq \int h \, d\mu$. We now construct a decreasing sequence of step functions converging pointwise to h so that

$\inf \{\int \psi \, d\mu : \psi \geq f$ and ψ is a step function$\} = \int h \, d\mu$.

Let N be a fixed positive integer. Let $\{B_{r_1}(x_1), \ldots, B_{r_m}(x_m)\}$ be a cover of K by balls of radius at most $1/N$ such that if $y \in B_{r_i}(x_i)$, then $h(y) < h(x_i) + 1/N$. Now let $\theta_N : K \to R$ be the step function described by $\theta_N(x) = \inf \{h(x_i) + 1/N : x \in B_{r_i}(x_i)\}$. Define ψ_N to be θ_N. Given any positive integer p, define θ_{N+p} as above, and let ψ_{N+p} be $\inf \{\theta_{N+p}, \psi_{N+p-1}\}$. Clearly, for each p ψ_{N+p} is a step function, and $\psi_{N+p} \geq \psi_{N+p+1} \geq h$. To establish the pointwise convergence, suppose to the contrary that for some x_0 in K and $\varepsilon > 0$ we have for each p

$$\psi_{N+p}(x_0) > h(x_0) + 2\varepsilon$$

Pick n so large that $1/n < \varepsilon$. There exists a point x_n such that $d(x_0, x_n) < 1/n$ and $\psi_n(x_0) \leq h(x_n) + 1/n$. Clearly, $h(x_n) > h(x_0) + \varepsilon$ which violates the upper semicontinuity of h. Hence, $\{\psi_n\}$ is the desired sequence.

Using the above technique we can show in the same manner that $\int g \, d\mu = \sup \{\int \phi \, d\mu : \phi \leq f$ and ϕ is a step function$\}$. The proof is now completed by observing the equivalence of the following statements: (i) f is Riemann integrable (ii) $\int g \, d\mu = \int h \, d\mu$ (iii) f is continuous except at a set of points of μ-measure zero.

A simple example shows that the theorem need not hold if our condition on the balls of the metric space is omitted. Let K be the closed unit disc in the plane with the usual metric. If B is a Borel subset of K, define $\mu(B)$ to be $\mu_1(B \cap \{(x,y) : x^2 + y^2 = 1\}) + \mu_2(B \cap \{(x,y) : x^2 + y^2$

$< 1\}$ where μ_2 is two dimensional Lebesgue measure and μ_1 is one dimensional Lebesgue measure, considering the circle as having measure 2π. Then the characteristic function of the unit circle is Riemann integrable (being a step function), but its set of discontinuities has measure 2π.

REFERENCES

1. H. L. Royden. Real Analysis, Macmillan, New York, 1968.

KEY WORDS AND PHRASES. *Riemann integrable functions on a compact metric space, Compact metric space with Borel measure.*

AMS(MOS) SUBJECT CLASSIFICATIONS (1970). 28A25.