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EXPLICIT L_2 INEQUALITIES FOR PARABOLIC AND PSEUDOPARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

J. R. KUTTLER and V. G. SIGILLITO

Applied Physics Laboratory
The Johns Hopkins University
Laurel, Maryland 20810

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ABSTRACT. Explicit L_2 inequalities are derived for second and third order diffusion equations with Neumann boundary conditions. Such inequalities are useful in approximating solutions to partial differential equations by the method of a priori inequalities.

1. INTRODUCTION.

In this paper we derive explicit a priori inequalities which are useful in yielding approximate solutions, with norm or pointwise error bounds, of the Neumann initial-boundary value problem associated with the diffusion operator $Lu \equiv \Delta u - u_t$ and the related third order operator $L_1u \equiv \Delta(u + u_t) - u_t$. These inequalities complete a series of a priori inequalities which are applicable to the Dirichlet and Robin boundary value problems for parabolic and pseudoparabolic operators [5], [6], [7], [8]. A priori inequalities for second order elliptic operators with Dirichlet, Neumann or Robin boundary conditions have appeared in

[1], [2], [3].

A comprehensive treatment of explicit a priori inequalities and their applications is given in [11] (also see [9], [10]). Recently a priori inequalities have been shown to be useful in yielding upper and lower bounds in classical and Steklov eigenvalue problems [4].

Inequalities using more general parabolic and pseudoparabolic operators than L or L_1 can be derived by the method presented here but we have chosen these simpler cases to keep the derivations from becoming unnecessarily cluttered.

In the next section we introduce notation and then briefly describe the use of the a priori inequalities to approximate solutions of boundary value problems. The inequalities are then derived in the final section.

2. USE OF THE INEQUALITIES.

Our motivation for developing the a priori inequalities is the standard Neumann problem

$$\begin{aligned} \mathcal{L}u &= G \quad \text{in} \quad R, \\ u &= F \quad \text{on} \quad B, \\ \frac{\partial u}{\partial n} &= H \quad \text{on} \quad S, \end{aligned} \tag{2.1}$$

where $\mathcal{L}$ denotes either L or L_1, B is a region in n-dimensional Euclidean space with boundary ∂B, R is the time cylinder $B \times (0,T]$, and $\frac{\partial u}{\partial n} \equiv u_{,i}n_i$ is the normal derivative on $S = \partial B \times (0,T]$ where (n_i) is the unit outer normal vector on ∂S. A comma denotes differentiation and the summation convention is used so that repeated indices are to be summed over that index from 1 to n.

Suppose for definiteness that $\mathcal{L} = L$, then problem (2.1) can be solved approximately by the method of a priori inequalities providing the inequality

$$\int_R v^2 \, dxdt \le \alpha_1 \int_B v^2 \, dx + \alpha_2 \int_S \left(\frac{\partial v}{\partial n}\right)^2 dsdt + \alpha_3 \int_R (Lv)^2 dxdt, \tag{2.2}$$

can be obtained with explicit constants α_1, α_2, α_3 which depend on R but not on the function v which is an arbitrary smooth function.

The method of a priori inequalities puts

$$v = u - \sum_{i=1}^{N} a_i \phi_i \equiv u - u_a$$

into (2.2) for some set of test functions $\{\phi_i\}$. Here u denotes the solution of (2.1). This leads to

$$\int_R (u-u_a)^2 dxdt \leq \alpha_1 \int_B (F-u_a(x,0))^2 dx + \alpha_2 \int_S (H - \frac{\partial u_a}{\partial n})^2 dsdt + \alpha_3 \int_R (G-Lu_a)^2 dxdt \tag{2.3}$$

in which the right-hand side is in terms of known quantities and the undetermined coefficients a_i, $i=1,\cdots,N$. The right-hand side is now minimized with respect to the a_i yielding an L_2 bound on the error. Pointwise bounds are obtainable from this L_2 bound. The procedure for computing them is given in detail in [5], [10].

3. THE INEQUALITIES.

A. THE INEQUALITY FOR L. The desired inequality for the operator L is

$$\left(\int_R u^2\, dxdt\right)^{\frac{1}{2}} \leq c_1\left(\int_B u^2\, dx\right)^{\frac{1}{2}} + c_2\left(\int_S \left(\frac{\partial u}{\partial n}\right)^2 dsdt\right)^{\frac{1}{2}} + c_3\left(\int_R (Lu)^2 dxdt\right)^{\frac{1}{2}}, \tag{3.1}$$

for an arbitrary function $u \in R$, which is once continuously differentiable in t and twice continuously differentiable in x.

To obtain this inequality write $u = f + g + h$ where

$$\begin{array}{llllll} Lf = 0, & Lg = Lu, & Lh = 0, & \text{in} & R, \\ f = u, & g = 0, & h = 0, & \text{on} & B, \\ \frac{\partial f}{\partial n} = 0, & \frac{\partial g}{\partial n} = 0, & \frac{\partial h}{\partial n} = \frac{\partial u}{\partial n}, & \text{on} & S. \end{array}$$

Now substitute f, g, and h successively into the identity

$$\int_{B_t} \phi^2\, dx = \int_B \phi^2\, dx - 2\int_0^t \int_{B_\tau} \phi\, L\, \phi\, dxd\tau - 2\int_0^t \int_{B_\tau} \phi,_i\, \phi,_i\, dxd\tau$$

$$+ 2\int_0^t \int_{S_\tau} \phi \frac{\partial\phi}{\partial n}\, dsd\tau, \qquad (3.2)$$

where B_τ is the intersection of R with t = τ and $S_\tau = \partial B \times (0,T]$.

Putting f into (3.2) yields

$$\int_{B_t} f^2\, dx \leq \int_B u^2\, dx,$$

and integrating from t = 0 to t = T gives

$$\int_R f^2\, dxdt \leq T\int_B u^2\, dx. \qquad (3.3)$$

Putting g into (3.2) yields

$$\int_{B_t} g^2\, dx \leq -2\int_0^t \int_{B_\tau} g\, Lu\, dxd\tau \leq \alpha \int_0^t \int_{B_\tau} g^2\, dxd\tau + \alpha^{-1}\int_0^t \int_{B_\tau} (Lu)^2\, dxd\tau,$$

by the arithmetic-geometric mean inequality for arbitrary positive α. Multiplying by $e^{-\alpha t}$ and rearranging gives

$$\frac{d}{dt}\left(e^{-\alpha t}\int_0^t \int_{B_\tau} g^2\, dxd\tau\right) \leq \alpha^{-1} e^{-\alpha t}\int_0^t \int_{B_\tau} (Lu)^2\, dxd\tau.$$

Integration with respect to t from 0 to T and multiplication by $e^{\alpha T}$ then gives

$$\int_R g^2 \, dxdt \leq \alpha^{-1} e^{\alpha T} \int_0^T e^{-\alpha t} \int_0^t \int_{B_\tau} (Lu)^2 \, dxd\tau \, dt$$

$$= - \alpha^{-2} e^{\alpha T} \int_0^T \frac{d}{dt} e^{-\alpha t} \int_0^t \int_{B_\tau} (Lu)^2 \, dxd\tau \, dt$$

$$= - \alpha^{-2} e^{\alpha T} \left\{ e^{-\alpha T} \int_0^T \int_{B_\tau} (Lu)^2 \, dxd\tau \right.$$

$$\left. - \int_0^T e^{-\alpha t} \frac{d}{dt} \left[\int_0^t \int_{B_\tau} (Lu)^2 \, dxd\tau \right] dt \right\}$$

$$= \alpha^{-2} \int_0^T (e^{\alpha(T-t)} - 1) \int_{B_\tau} (Lu)^2 \, dxdt$$

$$\leq \alpha^{-2} (e^{\alpha T} - 1) \int_R (Lu)^2 \, dxdt,$$

where integration by parts was used in going from the first to the second line above. Setting $\alpha = \beta T^{-1}$ gives

$$\int_R g^2 \, dxdt \leq \beta^{-2}(e^{\beta} - 1) \, T^2 \int_R (Lu)^2 \, dxdt \leq 1.544138653 \, T^2 \int_R (Lu)^2 \, dxdt, \tag{3.4}$$

with the optimal choice of $\beta = 1.59362$.

Finally, putting h into (3.2) yields

$$\int_{B_t} h^2 \, dx = - 2 \int_0^t \int_{B_\tau} h_{,i} \, h_{,i} \, dxd\tau + 2 \int_0^t \int_{S_\tau} h \frac{\partial u}{\partial n} \, dsd\tau$$

$$\leq - 2 \int_0^t \int_{B_\tau} h_{,i} \, h_{,i} \, dxd\tau + \alpha^{-1} \int_0^t \int_{S_\tau} (\frac{\partial u}{\partial n})^2 \, dsd\tau$$

$$+ \alpha \int_0^t \int_{S_\tau} h^2 \, dsd\tau. \tag{3.5}$$

Now introduce a continuously differential vector field f_i which has the property that $\min_S f_i n_i = b > 0$ (see [1] and [2] for methods of constructing such vector fields). Then,

$$\begin{aligned}\int_{S_\tau} h^2 dx &\le b^{-1} \int_{S_\tau} h^2 f_i n_i ds = b^{-1} \int_{B_\tau} (h^2 f_i),_i \, dx \\ &= b^{-1} \int_{B_\tau} h^2 f_{i,i} + 2b^{-1} \int_{B_\tau} h\, h,_i \, f_i \, dx \\ &\le b^{-1} \, |f_{i,i}|_M \int_{B_\tau} h^2 dx + 2b^{-1} \, (|f_i f_i|_M \int_{B_\tau} h^2 dx \int_{B_\tau} h,_i h,_i dx)^{\frac{1}{2}} \\ &\le b^{-1} \, |f_{i,i}|_M \int_{B_\tau} h^2 dx + 2 \int_{B_\tau} h,_i h,_i dx + \frac{1}{2} \, b^{-2} \, |f_i f_i|_M \int_{B_\tau} h^2 dx.\end{aligned} \tag{3.6}$$

where the subscript M denotes the maximum value of the function.

Using this inequality in (3.5) gives

$$\int_{B_t} h^2 dx \le \int_0^t \int_{S_\tau} (\frac{\partial u}{\partial n})^2 \, ds d\tau + (|f_{i,i}|_M \, b^{-1} + \frac{1}{2} \, |f_i f_i|_M \, b^{-2}) \int_0^t \int_{B_\tau} h^2 dx d\tau, \tag{3.7}$$

where in (3.5) we have chosen $\alpha = 1$ to cancel out the term $-2 \int_0^t \int_{B_\tau} h,_i h,_i dx d\tau$ with that in (3.6). If we now define $K \equiv |f_{i,i}|_M \, b^{-1} + \frac{1}{2} \, |f_i f_i|_M \, b^{-2}$ and multiply (3.7) by e^{-Kt} we can write

$$\frac{d}{dt} \, (e^{-Kt} \int_0^t \int_{B_\tau} h^2 dx d\tau) \le e^{-Kt} \int_0^t \int_{S_\tau} (\frac{\partial u}{\partial n})^2 \, ds d\tau.$$

Integrating with respect to t from 0 to T and multiplying by e^T yields

$$\begin{aligned}
\int_R h^2\,dxdt &\le e^{K\cdot T}\int_0^T e^{-K\cdot t}\int_0^t\int_{S_\tau}\left(\frac{\partial u}{\partial n}\right)^2 ds d\tau\, dt \\
&= K^{-1}\int_0^T \left(e^{K\cdot(T-t)}-1\right)\int_{S_t}\left(\frac{\partial u}{\partial n}\right)^2 dsdt \\
&\le K^{-1}\left(e^{K\cdot T}-1\right)\int_S\left(\frac{\partial u}{\partial n}\right)^2 dsdt \\
&\equiv c_2^2\int_S\left(\frac{\partial u}{\partial n}\right)^2 dsdt,
\end{aligned}\tag{3.8}$$

where we have again used integration by parts in a similar manner to go from the first to the second line above.

As an example of a choice of the vector field f_i assume that B is star-shaped with respect to an interior point which we take as the origin. Then we could choose $f_i = x_i$, since then $n_i x_i > 0$ on the boundary of B. The quantity $x_i n_i$ is the distance from the origin to the tangent line at the point (x_i). In this case then $|f_i f_i|_M = |r^2|_M$ for r the distance to the origin and $|f_{i,i}|_M = n$. More specifically suppose B is a rectangle centered at the origin with sides of length 2a and 2b. Choose $f_1 = x/a$, $f_2 = y/b$. Then $|f_i f_i|_M = 2$, $t = 1$ and $f_{i,i} = a^{-1} + b^{-1}$. Thus

$$K = 1 + 1/a + 1/b.$$

Combining (3.3), (3.4) and (3.8) now yields the desired inequality (3.1) with

$$c_1 = T^{1/2}, \qquad c_2 = [K^{-1}(e^{KT} - 1)]^{1/2}, \qquad c_3 = 1.242633757\ T.$$

B. THE INEQUALITY FOR L_1. For the operator L_1 the a priori inequality

$$\left(\int_R u^2\,dxdt\right)^{\frac{1}{2}} \le c_1\left(\int_B (u^2 + u_{,i}u_{,i})dx\right)^{\frac{1}{2}} + c_2\left(\int_S \left|\frac{\partial}{\partial n}(u + u_t)\right|^2 dsdt\right)^{\frac{1}{2}}$$

$$+ c_3\left(\int_R (L_1 u)^2\,dxdt\right)^{\frac{1}{2}}$$

can be developed in an entirely analogous manner. The starting point is the identity

$$\int_{B_t} (\phi^2 + \phi_{,i}\phi_{,i})dx = \int_B (\phi^2 + \phi_{,i}\phi_{,i})dx - 2\int_0^t \int_{B_\tau} \phi\, L_1\, \phi\, dxd\tau$$

$$- 2\int_0^t \int_{B_\tau} \phi_{,i}\phi_{,i}dxd\tau + 2\int_0^t \int_{S_\tau} \phi\,\frac{\partial}{\partial n}(\phi + \phi_t)\,dsd\tau.$$

The derivation then exactly parallels that of (3.1) and we omit the details.

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REFERENCES

1. Bramble, J. and L. E. Payne. Bounds for Solutions of Second Order Elliptic Partial Differential Equations, Contributions to Differential Equations 1 (1963) 95-127.

2. Bramble, J. H. and L. E. Payne. Bounds in the Neumann Problem for Second Order Uniformly Elliptic Operators, Pacific J. Math. 12 (1962) 823-833.

3. Bramble, J. H. and L. E. Payne. Some Integral Inequalities for Uniformly Elliptic Operators, Contributions to Differential Equations 1 (1963) 129-135.

4. Kuttler, J. R. and V. G. Sigillito. Bounding Eigenvalues of Elliptic Operators, SIAM J. Math. Anal. (to appear).

5. Sigillito, V. G. Pointwise Bounds for Solutions of the First Initial-Boundary Value Problem for Parabolic Equations, SIAM J. Appl. Math. 14 (1966) 1038-1056.

6. Sigillito, V. G. Pointwise Bounds for Solutions of Semilinear Parabolic Equations, SIAM Rev. 9 (1967) 581-585.

7. Sigillito, V. G. A Priori Inequalities and Dirichlet Problem for a Pseudoparabolic Equation, SIAM J. Math. Anal. 7 (1976) 222-229.

8. Sigillito, V. G. and J. C. Pirkle. A Priori Inequalities and Norm Error Bounds for Solutions of a Third Order Diffusion-Like Equation, SIAM J. Appl. Math. 25 (1973) 69-71.

9. Sigillito, V. G. On a Continuous Method of Approximating Solutions of the Heat Equation, J. ACM. 14 (1967) 732-741.

10. Sigillito, V. G. A Priori Inequalities and Approximate Solutions of the First Boundary Value Problem for $\Delta^2 u = f$, SIAM J. Numer. Anal. 13 (1976) 251-260.

11. Sigillito, V. G. Explicit a Priori Inequalities with Applications to Boundary Value Problems, Research Notes in Mathematics Series No. 13, Pitman Publishing Limited, London, 1977.

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