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THE SPACE OF ENTIRE FUNCTIONS OF TWO VARIABLES AS A METRIC SPACE

W. C. SISARCICK

Department of Mathematics
Marshall University
Huntington, West Virginia 25701

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Section 1. Introduction.

Let Γ^2 denote the space of entire functions of two variables. If $f(z,w) \in \Gamma^2$, $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$, the series converging absolutely for all (z,w) and uniformly in every bicylinder centered at $(0,0)$, [2]. Here, a metric is defined on Γ^2 and three classes of linear functionals on Γ^2 are characterized.

We use the following notation.

(1) $<a_{m,n}>_{m+n=k}^{\infty} \equiv <a_{k,0}, a_{k-1,1} \cdots a_{0,k}, a_{k+1,0} \cdots>.$

$$(2) \quad \sum_{m+n=0}^{\infty} a_{m,n} \equiv a_{0,0}+a_{1,0}+a_{0,1}+a_{2,0}+a_{1,1}+\cdots$$

$$\equiv \lim_{N\to\infty} \sum_{m+n=0}^{N} a_{m,n}.$$

Definition 1.1. The sequence $<a_{m,n}>_{m+n=k}^{\infty}$ is said to have limit a as $m+n \to \infty$, written $\lim_{m+n\to\infty} a_{m,n} = a$, if and only if for any $\varepsilon > 0$, there exists an $N = N(\varepsilon) \geq 0$ such that $|a_{m,n}-a| < \varepsilon$ if $m+n > N$.

Lemma 1.2. If $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$, then for each (z,w), the sequence $<a_{m,n} z^m w^n>_{m+n=0}^{\infty}$ is such that $\lim_{m+n\to\infty} a_{m,n} z^m w^n = 0$.

Proof. Given (z,w), let $S_N = \sum_{j+k=0}^{N} |a_{j,k} z^j w^k|$. Since $\lim_{N\to\infty} S_N$ exists, $0 = \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = \lim_{N\to\infty} (S_N - S_{N-1}) = \lim_{N\to\infty} \sum_{j+k=N}^{\infty} |a_{j,k} z^j w^k|$. Hence given $\varepsilon > 0$, there exists an $M = M(\varepsilon)$ such that if $N > M$, $|a_{N-j,j} z^{N-j} w^j| < \varepsilon$ for each j, $0 \leq j \leq N$. Let $m+n = N$. Then $0 \leq n \leq N$ and $|a_{m,n} z^m w^n| < \varepsilon$. Therefore given $\varepsilon > 0$, there exists an $M = M(\varepsilon)$ such that $m+n > M \Rightarrow |a_{m,n} z^m w^n| < \varepsilon$. Hence $\lim_{m+n\to\infty} a_{m,n} z^m w^n = 0$.

Lemma 1.3. A necessary and sufficient condition that $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$

is an entire function is that for the sequence $<|a_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$, one has $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$.

<u>Proof</u>. Let $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$ and $T = \overline{\lim}_{m+n\to\infty} |a_{m,n}|^{1/m+n}$. If $T > 0$, choose (z,w) such that $|z| \geq |w| > 1/T$. $(1/T = 0$ if $T = \infty)$. Then choose p such that $|w| > p > 1/T$. Then $|\frac{1}{z}| \leq |\frac{1}{w}| < \frac{1}{p} < T$. By definition of T, there exists a sequence $<(m_k, n_k)>^{\infty}\ k = 1$ such that $< m_k + n_k >_{k=1}^{\infty}$ increases monotonically to ∞ and $|a_{m_k,n_k}|^{\frac{1}{m_k+n_k}} > 1/p$ for all k. Hence $|a_{m_k,n_k} z^{m_k} w^{n_k}| > (\frac{z}{p})^{m_k} (\frac{w}{p})^{n_k} > 1$. This contradicts Lemma 1.2. Therefore $T = 0$. Hence $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$.

Conversely, let $\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n$ be a series such that for the sequence $<|a_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$, one has $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$. To show this series is an entire function, it sufficies to show [2] the series converges for each (z,w). Consider (z,w) fixed. Choose p such that $|z| < p$ and $|w| < p$. Let N be such that $m+n > N \Rightarrow |a_{m,n}|^{1/m+n} < 1/p$. Then for $m+n > N$, $|a_{m,n} z^m w^n| < (\frac{|z|}{p})^m (\frac{|w|}{p})^n$, $\sum_{m+n=N+1}^{\infty} |a_{m,n} z^m w^n| \leq \sum_{m+n=N+1}^{\infty} (\frac{|z|}{p})^m (\frac{|w|}{p})^n < \infty$. Therefore $\sum_{m+n=0}^{\infty} |a_{m,n} z^m w^n| < \infty$.

Let $s_{p,q} = \sum_{m=0}^{p} \sum_{n=0}^{q} a_{m,n} z^m w^n$. To show the series converges, it sufficies to show [1] that given $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $|s_{p,q} - s_{m,n}| < \epsilon$ if $P > m > N$ and $q > n > N$. Since $\sum_{m+n=0}^{\infty} |a_{m,n} z^m w^n| < \infty$, given $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that $N > \max\{M,1\} \Rightarrow \sum_{j+k=N+1}^{\infty} a_{j,k} z^j w^k < \epsilon$. Choose such an N. Then $N = N(\epsilon)$. For $p > m > N$ and $q > n > N$, $|s_{P,q} - s_{m,n}|$

$$= \left| \sum_{j=0}^{P} \sum_{k=0}^{q} a_{j,k} z^j w^k - \sum_{j=0}^{m} \sum_{k=0}^{n} a_{j,k} z^j w^k \right| \le \sum_{j+k=m+n}^{\infty} \left| a_{j,k} z^j w^k \right|$$

$$\le \sum_{j+k=N+1}^{\infty} \left| a_{j,k} z^j w^k \right| < \epsilon.$$

<u>Definition 1.4</u>. Given $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^m \in \Gamma^2$ and $g(z,w) = \sum_{m,n=0}^{\infty} b_{m,n} z^m w^n \in \Gamma^2$, define $d(f,g) =$ $\sup\{|a_{0,0} - b_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} : m+n \ge 1\}$.

<u>Theorem 1.5</u>. The space (Γ^2, d) is a metric space.

<u>Proof</u>. Given f, g as in Definition 1.4, the set $\{|a_{m,n} - b_{m,n}|^{1/m+n} : m+n \ge 1\}$ is a bounded set by Lemma 1.3, so d is well defined. It is clear that $d(f,g) = 0$ if and only if $f = g$ and that $d(f,g) = d(g,f)$. Let $h(z,w) = \sum_{m,n=0}^{\infty} c_{m,n} z^m w^n \in \Gamma^2$. Then $d(f,h) = \sup\{|a_{0,0} - c_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} : m+n \ge 1\} =$ $\sup\{|(a_{0,0} - b_{0,0}) + (b_{0,0} - c_{0,0})|, |(a_{m,n} - b_{m,n}) + (b_{m,n} - c_{m,n})|^{1/m+n} : m+n \ge 1\}$ $\le \sup\{|a_{0,0} - b_{0,0}| + |b_{0,0} - c_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} + |b_{m,n} - c_{m,n}|^{1/m+n} :$ $m+n \ge 1\} \le \sup\{|a_{0,0} - b_{0,0}|, |a_{m,n} - b_{m,n}|^{1/m+n} : m+n \ge 1\}$ + sup

$\{|b_{0,0}-c_{0,0}|, |b_{m,n}-c_{m,n}|^{1/m+n}: m+n \geq 1\} = d(f,g) + d(g,h)$. Hence d is a metric on Γ^2.

Section 2. The class of continuous linear functionals on Γ^2.

Definition 2.1. A function F from Γ^2 to $\mathbb{C}$ (complex plane) is a linear functional if and only if for all $f,g \in \Gamma^2, \alpha \in \mathbb{C}$, $F(f+g) = F(f)+F(g)$ and $F(\alpha f) = \alpha F(f)$.

Definition 2.2. A function F from Γ^2 to $\mathbb{C}$ is said to be continuous at $f \in \Gamma^2$ if and only if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g \in \Gamma^2$ and $d(f,g) < \delta$, then $|F(f)-F(g)| < \varepsilon$.

Definition 2.3. A function F from Γ^2 to $\mathbb{C}$ is said to be continuous if and only if it is continuous at each $f \in \Gamma^2$.

Lemma 2.4 The series $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ converges for all sequences $\langle a_{m,n}\rangle_{m+n=0}^{\infty}$ such that $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$ if and only if $\langle |b_{m,n}|^{1/m+n}\rangle_{m+n=1}^{\infty}$ is a bounded sequence.

Proof. Let $\langle |b_{m,n}|^{1/m+n}\rangle_{m+n=1}^{\infty}$ be a bounded sequence and $\langle a_{m,n}\rangle_{m+n=0}^{\infty}$ be such that $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$. Choose $M > 0$ such that $|b_{m,n}|^{1/m+n} \leq M$ if $m+n \geq 1$ and then $N \geq 0$ such that $m+n > N \Rightarrow |a_{m,n}|^{1/m+n} \leq \frac{1}{2M}$. Then if $m+n > N$, $|a_{m,n}b_{m,n}| \leq \frac{1}{(2M)^{m+n}} \cdot M^{m+n} = \frac{1}{2^{m+n}}$.

Therefore $\sum_{m+n=N+1}^{\infty} |a_{m,n}b_{m,n}| \leq \sum_{m+n=N+1}^{\infty} \frac{1}{2^{m+n}} < \infty$.

Hence the series $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ converges absolutely, hence it converges.

Conversely suppose for any sequence $<a_{m,n}>_{m+n=0}^{\infty}$ such that $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$, the series $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ converges. If $<|b_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$ is not bounded, for each $k \in Z^+$ there exists an (m_k,n_k) such that $|b_{m_k,n_k}|^{\frac{1}{m_k+n_k}} > k$ and $<m_k+n_k>_{k=1}^{\infty}$ is strictly increasing. Choose $a_{m,n} = 0$ if $(m,n) \neq (m_k,n_k)$, $a_{m_k,n_k} = k^{\frac{1}{m_k+n_k}}$. Then $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = \lim_{m_k+n_k\to\infty} |a_{m_k,n_k}|^{\frac{1}{m_k+n_k}} = \lim_{k\to\infty} \frac{1}{k} = 0$. But $|a_{m_k,n_k} b_{m_k,n_k}| > 1$ for each k so $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ does not converge. Therefore $<|b_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$ is bounded. The series $\Sigma a_{m,n}b_{m,n}$ does not converge since the only $\neq 0$ terms are > 1 and there are an infinite number of them.

We now characterize the class of continous linear functionals on Γ^2.

<u>Theorem 2.5</u>. Let F be a function from Γ^2 to the complex plane. Then F is a continous linear functional on Γ^2 if and only if there is a unique sequence $<b_{m,n}>_{m+n=0}^{\infty}$ such that $<|b_{0,0}|, |b_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$ is bounded and such that for all $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2$,

$$F(f) = \sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}.$$

Proof. Let $<|b_{0,0}|, |b_{m,n}|^{1/m+n}>_{m+n=1}^{\infty}$ be a bounded sequence, $M > 0$ be such that $|b_{0,0}| < M$, $|b_{m,n}|^{1/m+n} < M$, $m+n \geq 1$ and $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2$. Then $\lim_{m+n\to\infty} |a_{m,n}|^{1/m+n} = 0$ so $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ converges by Lemma 2.4. Hence we may define a function F from Γ^2 to the complex plane by $F(f) = \sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$. It is clear that F is a linear functional. Let $\varepsilon > 0$ and $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2$ be given. We show there exists a $\delta > 0$ such that if $g \in \Gamma^2$ and $d(f,g) < \delta$, then $|F(f)-F(g)| < \varepsilon$. Choose $\delta > 0$ such that $\delta M < 1$ and $\delta M + \left(\frac{\delta M}{1-\delta M}\right)^2 < \varepsilon$. Then if $g(z,w) = \sum_{m+n=0}^{\infty} C_{m,n}z^m w^n \in \Gamma^2$ and $d(f,g) < \varepsilon$, $|F(f)-F(g)| = |F(f-g)| = \left|\sum_{m+n=0}^{\infty} (a_{m,n}-C_{m,n})b_{m,n}\right|$

$$\leq |a_{0,0}-C_{0,0}|M + \sum_{m+n=1}^{\infty} |a_{m,n}-C_{m,n}|M^{m+n}$$

$$\leq \delta M + \sum_{m+n=1}^{\infty} (\delta M)^{m+n}$$

$$= \delta M + \sum_{m=1}^{\infty} (\delta M)^m \sum_{n=1}^{\infty} (\delta M)^n$$

$$= \delta M + \left(\frac{\delta M}{1-\delta M}\right)^2 < \varepsilon .$$

Conversely, let F be a continuous linear functional on Γ^2. Let $F(z^m w^n) = b_{m,n}$ for all $m+n \geq 0$. Given $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n}z^m w^n$, let $f_N(z,w) = \sum_{m+n=0}^{N} a_{m,n}z^m w^n$. Then $d(f_N,f) = \sup\{|a_{m,n}|^{1/m+n} : m+n > N\} \to 0$ as $N \to \infty$ so by the continuity of F, $F(f_N) \to F(f)$ as $N \to \infty$.

But $F(f_N) = \sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$. Therefore $\lim_{N\to\infty} \sum_{m+n=0}^{N} a_{m,n}b_{m,n} = F(f)$. Hence $\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$ converges and $F(f) = \sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}$. By Lemma 2.4, the sequence $<|b_{0,0}|^{1/m+n}>_{m+n=1}^{\infty}$ is bounded. Suppose $< C_{m,n} >_{m+n=0}^{\infty}$ is a sequence such that for all $f(z,w) = \sum_{m+n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2$, $F(f) = \sum_{m+n=0}^{\infty} a_{m,n}C_{m,n}$, then for $j, k \in Z_+$, $F(z^j w^k) = C_{j,k}$. But $F(z^j w^k) = b_{j,k}$.

Hence $C_{jk} = b_{j,k}$ and the sequence is unique.

Section 3. The class of continuous scalar homomorphisms on Γ^2.

Let $f,g \in \Gamma^2$, $\alpha \in \mathbb{C}$ (complex field). Define $(f+g)(z,w) = f(z,w) + g(z,w)$, $(f\cdot g)(z,w) = f(z,w)g(z,w)$, $(\alpha f)(z,w) = \alpha\cdot f(z,w)$. Then Γ^2 becomes a commutative algebra with a unit. In this section we characterize the continuous linear functionals on Γ^2 that preserve multiplication. That is the continuous scalar homomorphisms on Γ^2.

Lemma 3.1. Given $\epsilon > 0$ and $(b,c) \in \mathbb{C}\times\mathbb{C}$, there exists a $\delta > 0$ such that if $f, g \in \Gamma^2$ and $d(f,g) < \delta$, then $|f(b,c)-g(b,c)| < \epsilon$.

Proof. Given $\epsilon > 0$ and $(b,c) \in \mathbb{C}\times\mathbb{C}$, let $R = \max\{|b|,|c|\}$ choose $\delta > 0$ such that $\delta R < 1$ and $\delta + \left(\frac{\delta R}{1-\delta R}\right)^2 < \epsilon$. Then if $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n}z^m w^n$ and $g(z,w) = \sum_{m,n=0}^{\infty} b_{m,n}z^m w^n$ are in Γ^2 and $d(f,g) < \delta$, $|f(b,c)-g(b,c)| = |\sum_{m+n=0}^{\infty} (a_{m,n}-b_{m,n})b^m c^n|$ $\leq |a_{0,0}-b_{0,0}| + \sum_{m+n=1}^{\infty} |a_{m,n}-b_{m,n}|R^{m+n} < \delta + \sum_{m+n=1}^{\infty} (\delta R)^{m+n} =$

$$\delta + \sum_{m=1}^{\infty} (\delta R)^m \sum_{n=1}^{\infty} (\delta R)^n = \delta + \left(\frac{\delta R}{1-\delta R}\right)^2 < \varepsilon.$$

Theorem 3.2. Let F be a function from Γ^2 to $\mathbb{C}$, $F \not\equiv 0$. Then F is a continuous scalar homorphism on Γ^2 if and only if there exists a unique $(b,c) \in \mathbb{C}\times\mathbb{C}$ such that for all $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$,

$$F(f) = f(b,c).$$

Proof. Let F be a $\not\equiv 0$ continuous scalar homomorphism on Γ^2. By Theorem 2.5, there is a unique sequence $\langle b_{m,n} \rangle_{m+n=0}^{\infty}$ such that for all $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$, $F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$. For each m and n, $b_{m,n} = F(z^m w^n) = F(z)^m F(w)^n = b_{1,0}^m b_{0,1}^n$. Therefore

$$F(f) = \sum_{m+n=0}^{\infty} a_{m,n} b_{1,0}^m b_{0,1}^n = f(b_{1,0}, b_{0,1}).$$

Conversely, given $(b,c) \in \mathbb{C}\times\mathbb{C}$, define a function F from Γ^2 to $\mathbb{C}$ by $F(f) = f(b,c)$. Then F is clearly a $\not\equiv 0$ scalar homomorphism. Given $\varepsilon > 0$, let $\delta > 0$ be such that if $f,g \in \Gamma^2$ and $d(f,g) < \delta$, then $|f(b,c)-g(b,c)| < \varepsilon$. Then $|F(f)-F(g)| = |F(f-g)| = |(f-g)(b,c)| = |f(b,c)-g(b,c)| < \varepsilon$. Hence F is continuous.

Section 4. The class of bounded linear functionals on Γ^2.

Definition 4.1 . Let F be a linear functional on Γ^2. Then F is said to be bounded if and only if there exists an $M \geq 0$ such that for all $f \in \Gamma^2$, $|F(f)| \le Md(f,0)$. Here, 0 denotes the function identically zero on $\mathbb{C}x\mathbb{C}$.

Lemma 4.2. Let F be a linear functional on Γ^2. If F is bounded, F is continuous but not conversely.

Proof. Let F be a bounded linear functional on Γ^2. Let $f_o \in \Gamma^2$ $\epsilon > 0$ be given and let $M \ge 0$ be such that for all $f \in \Gamma^2$, $|F(f)| \le Md(f,0)$. Choose $\delta = \epsilon/M+1$. Then if $g \in \Gamma^2$ and $d(f_o,g) < \delta$, $|F(f_o)-F(g)| = |F(f_o-g)| \le Md(f_o-g,o) \le (m+1)d(f_o,g) < (M+1)\ \delta < \epsilon$. Therefore F is continuous at f_o. Hence F is continuous.

For an example of a continuous linear functional that is not bounded, let $b_{m,n} = n$. Then $<|n|^{1/m+n}>_{m+n=1}^{\infty}$ is a bounded sequence. Define a function F from Γ^2 to $\mathbb{C}$ by $F\ (\sum_{m,n=0}^{\infty} a_{m,n}z^m w^n) = \sum_{m+n=1}^{\infty} na_{m,n}$. By Theorem 2.5, F is a continuous linear functional on Γ^2. If F is bounded, there exists an $M \ge 0$ such that $|\sum_{m+n+1}^{\infty} na_{m,n}| \le M$ $\sup\{|a_{0,0}|, |a_{m,n}|^{1/m+n}: m+n \ge 1\}$ for all $<a_{m,n}>_{m+n=0}^{\infty}$ such that $|a_{m,n}|^{1/m+n} \to 0$ as $m+n \to \infty$. Let $k \in Z^+$, $k > \max\{M,2\}$. Let $a_{0,k} = k$, $a_{m,n} = 0$ if $(m,n) = (0,k)$. Then $|a_{m,n}|^{1/m+n} \to 0$ as $m+n \to \infty$ since the sequence has only one non-zero term. But

$|\sum_{m+n=1}^{\infty} na_{m,n}| = k^2$, $M \sup\{|a_{0,0}|, |a_{m,n}|^{1/m,n} : m+n \geq 1\} =$
$M \cdot k^{1/k} < k \cdot k^{1/k} < k \cdot k^{1/k} < k^2$, a contradiction. Hence F is not bounded.

Definition 4.3. Let B denote the class of bounded linear functionals on Γ^2. For $F, G \in B$, $\alpha \in \mathbb{C}$, $f \in \Gamma^2$, define $(F+G)(f) = F(f) = G(f)$, $(\alpha F)(f) = \alpha \cdot F(f)$, $||F|| = \inf\{M \geq 0 \mid \text{for all } f \in \Gamma^2, |F(f)| \leq Md(f,0)\}$.

Theorem 4.4. With respect to Definition 4.3, B is a normed linear space.

Proof. Let $F \in B$. Then $|F(f)| \leq ||F||\ d(f,0)$ for all $F \in \Gamma^2$. If not, for some $f_o \in \Gamma^2$, $|F(f_o)| > ||F||\ d(f_{0,0})$. Then $d(f_o,0) \neq 0$ so choose $\epsilon > 0$ such that $|F(f_o)| = ||F|| d(f_o,0) + \epsilon\ d(f_o,0)$. By Definition of $||F||$, there exists an $M \geq 0$ such that $|F(f)| \leq Md(f,0)$ for all $f \in \Gamma^2$ and $||F|| + \epsilon > M$. Then $d(f_o,0)(||F|| + \epsilon)$ $= |F(f_o)| \leq Md(f_o,0)$. Hence $||F|| + \epsilon \leq M$, a contradiction. Therefore $|F(f)| \leq ||F|| d(f,0)$ for all $f \in \Gamma^2$ and $||F||$ is the smallest number to satisfy this inequality for all $f \in \Gamma^2$.

For $F, G \in B$, $\alpha \in \mathbb{C}$, $F+G$ and αF are clearly linear functionals on Γ^2. For $f \in \Gamma^2$, $|(F+G)(f)| = |F(f)+G(f)| \leq |F(f)| + |G(f)| \leq ||F||\ d(f,0) + ||G||\ d(f,0) = (||F|| + ||G||)\ d\ (f,0)$. Hence $F+G \in B$ and $||F+G|| \leq ||F|| + ||G||$. Also $|(\alpha F)(f)| = |\alpha \cdot F(f)| = |\alpha|\ |F(f)| \leq |\alpha|\ ||F||\ d(f,0)$. Hence $\alpha F \in B$ and $||\alpha F|| \leq |\alpha|\ ||F||$. Suppose it is possible to have $||\alpha F|| < |\alpha|\ ||F||$. Choose

$\epsilon > 0$ such that $||\alpha F|| + \epsilon\,|\alpha|\;||F|| = |\alpha|\;||F||$. Then for all $f \in \Gamma^2$, $|\alpha|\;|F(f)| = |(\alpha F)(f)| \le ||\alpha F||\,d(f,0) = (1 - \epsilon)\,|\alpha|\;||F||\;d(f,0)$. Therefore $|F(f)| \le (1 - \epsilon)\;||F||\;d(f,0)$, a contradiction. Hence $||\alpha F|| = |\alpha|\;||F||$.

It is clear that $||\cdot||$ evaluated at the zero linear functional on Γ^2 is 0 and if $||F|| = 0$ then $|F(f)| = 0$ for all $f \in \Gamma^2$, hence $F \equiv 0$. Also the remaining properties required for B to be a normed linear space follow trivially. Hence B is a normed linear space with respect to Definition 4.3.

Theorem 4.5. Let F be a function from Γ^2 to $\mathbb{C}$. Then $F \in B$ if and only if there exists unique $(a,b,c,) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}$ such that for all $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \in \Gamma^2$,

$$F(f) = a_{0,0}a + a_{1,0}b + a_{0,1}c.$$

Also, $||F|| = |a| + |b| + |c|$.

Proof. Let $F \in B$. Then F is continuous so there exist a unique sequence $\langle b_{m,n}\rangle_{n+n=0}^{\infty}$ such that $F\left(\sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\right) = \sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}$

and $\left|\sum_{m+n=0}^{\infty} a_{m,n} b_{m,n}\right| \le ||F||\sup\{|a_{0,0}|, |a_{m,n}|^{1/m+n} : m+n \ge 1$ for all sequences $\langle a_{m,n}\rangle_{m+n=0}^{\infty}$ such that $|a_{m,n}|^{1/m+n} \to 0$ as $m+n \to \infty$. Suppose $b_{k,j} \ne 0$ for some (k,j) with $k+j \ge 2$. Choose $a_{m,n} = 0$ if $(m,n) \ne (k,j)$ and choose $a_{k,j}$ such that $||F|| >$ $|a_{k,j}|^{1-\frac{1}{k+j}} \cdot |b_{k,j}|$. Then $|a_{m,n}|^{1/m+n} \to 0$ as $m+n \to \infty$,

$|\sum_{m+n=0}^{\infty} a_{m,n}b_{m,n}| = |a_{k.j}b_{k,j}| \le ||F|| \, |a_{k,j}|^{\frac{1}{k+j}}$. Therefore $|a_{k,j}|^{1-\frac{1}{k+j}} \cdot |b_{k,j}| \le ||F||$, a contradiction. Hence $b_{k,j} = 0$ if $k+j \ge 2$. Hence $F(\sum_{m,n=0}^{\infty} a_{m,n}z^m w^n) = a_{0,0}b_{0,0} + a_{1,0}b_{1,0} + a_{0,1}b_{0,1}$. Also $|F(\sum_{m,n=0}^{\infty} a_{m,n}z^m w^n)| \le |a_{0,0}| \, |b_{0,0}| + |a_{1,0}| \, |b_{1,0}| + |a_{0,1}| \, |b_{0,1}| \le (|b_{0,0}| + |b_{1,0}| + |b_{0,1}|)d(f,0)$.

Therefore $||F|| \le |b_{0,0}| + |b_{1,0}| + |b_{0,1}|$. To show equality here, it sufficies to show there exists an $f_o \in \Gamma^2$ such that $|F(f_o)| = (|b_{0,0}| + |b_{1,0}| + |b_{o,1}|)d(f_o, 0)$. If $b_{o,o} = |b_{o,o}|e^{i\theta_1}$, $b_{1,0} = |b_{1,0}|e^{i\theta_2}$, $b_{0,1} = |b_{0,1}|e^{i\theta_3}$, choose $f_o(z,w) = e^{-i\theta_1} + e^{-i\theta_2}z + e^{-i\theta_3}w$. Then $|F(f_o)| = |b_{o,o}| + |b_{1,0}| + |b_{0,1}| = (|b_{0,0}| + |b_{1,0}| + |b_{0,1}|)d(f_o,0)$.

Conversely, given $(a,b,c) \in ¢x¢x¢$, define a function F from Γ^2 to ¢ by $F(f) = a_{0,0}a + a_{1,0}b + a_{0,1}c$, $f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2$. By Theorem 2.5, F is a continuous linear functional on Γ^2. Since $|F(f)| \le |a_{o,o}| \, |a| + |a_{1,0}| \, |b| + |a_{0,1}| \, |c| \le (|a| + |b| + |c|)d(f,0)$, $F \in B$.

Corollary 4.6. With respect to Definition 4.3, B is a Banach space.

Proof. Let $\langle F_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in B, F_n corresponding to (a_n,b_n,c_n). Then for any $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $m,n > N(\epsilon) \Rightarrow ||F_n-F_m|| < \epsilon$. That is $|a_n-a_m|+|b_n-b_m| + |c_n-c_m| < \epsilon$. Hence each of $\langle a_n \rangle$, $\langle b_n \rangle$, $\langle c_n \rangle$ is a Cauchy sequence. Let $a_n \to a$, $b_n \to b$, $c_n \to c$ as $n \to \infty$. Define a function F from Γ^2 to $\mathbb{C}$ by

$$F\left(\sum_{m,n=0}^{\infty} a_{m,n}z^m w^n\right) = a_{0,0}a + a_{1,0}b + a_{0,1}c.$$ By Theorem 4.5, $F \in B$.

Given $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $|a_m-a| + |b_m-b_n| + |c_m-c_n| < \epsilon/2$ if $m,n > N(\epsilon)$. Let $m \to \infty$ to get $|a-a_n| + |b-b_n| + |c-c_n| \le \epsilon/2$ if $n > N(\epsilon)$. Hence given $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that if $n > N(\epsilon)$, $|a-a_n| + |b-b_n| + |c-c_n| < \epsilon$. That is $||F_n-F|| < \epsilon$. Therefore B is a Banach Space.

If (a_1,b_1,c_1) and (a_2,b_2,c_2) are in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ and $\alpha \in \mathbb{C}$, it is clear that if addition and scalar multiplication are defined by $(a_1,b_1,c_1) + (a_2,b_2,c_2) = (a_1+a_2,\ b_1+b_2,\ c_1+c_2)$, $\alpha(a_1,b_1,c_1) = (\alpha a_1,\ \alpha b_1,\ \alpha c_1)$, that $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ is a vector space over $\mathbb{C}$. Also if $||\cdot||$ is defined by $||(a,b,c)|| = |a|+|b|+|c|$, $||\cdot||$ is a norm on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ making $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ into a Banach space. Define a function Ψ from B to $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ as follows: For $F \in B$, let (a,b,c) be the unique element of $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ such that $F(f) = a_{0,0}a + a_{1,0}b + a_{0,1}c$ for all

$$f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n}z^m w^n \in \Gamma^2.$$ Let $\Psi(F) = (a,b,c)$. The following theorem is straightforward to prove so the proof is omitted

Theorem 3.4.7. The spaces B and $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ are isometrically isomorphic Banach spaces.

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ABSTRACT. Three classes of linear functionals on the space of entire functions of two variables are characterized. Several results are proved.

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