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NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces E, a subset X is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.
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INTRODUCTION

Let E be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on X, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously X is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field K that is locally compact. However, if K is not locally compact then, under reasonable conditions, for a subset X of a normed space E over K compactness of $f(X)$ for all $f \in E'$ implies weak compactness of X (we point out that if such an X has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout K is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $|\ |$, and E is a normed K-vector space, where we assume $\|\ \|$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^\times| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0,r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0,1)$.

E' is the space of all linear continuous functions $E \to K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). E is called *normpolar* if the norm is polar i.e. if $\|x\| = \sup\{|f(x)| : f \in E', |f| \leq \|\ \|\}$ $(x \in E)$, in other words, if $\jmath : E \to E''$ is an isometry. E' is always normpolar. We assume throughout this note that E is normpolar.

A subset A of a (normed) space E is *absolutely convex* if it is a module over B_K. A set $X \subset E$ is *convex* if it is either empty or an additive coset of an absolutely convex set. A subset A of E is called *edged* if it is absolutely convex and, in case the valuation of K is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The *weak topology* $w = \sigma(E, E')$ is the weakest topology on E making all $f \in E'$ continuous. The *weak-star topology* $w' = \sigma(E', E)$ is the weakest topology on E' making all evaluation maps $f \mapsto f(a)$ $(a \in E)$ continuous. For $X \subset E'$ we denote its w'-closure by $\overline{X}^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF w'-PRECOMPACT SETS

LEMMA 1.1. *Let X be a bounded subset of E'. Then $\{x \in E : \inf_{f\in X} |f(x)| > 0\}$ is open in E.*

Proof. X is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{x \in E : |f(x)| > \frac{1}{n}$ for all $f \in X\}$ is open. Then so is $\bigcup_n U_n = \{x \in E : \inf_{f\in X} |f(x)| > 0\}$.

LEMMA 1.2. *Let K be not locally compact. Let $X \subset E'$ and $a \in E$ be such that $X(a) := \{f(a) : f \in X\}$ is precompact. Suppose $X \subset g + U$ where U is an edged zero neighbourhood in E', U w'-closed and where $g \in E' \setminus U$. Then for any $\varepsilon > 0$ there exists a $b \in E$ for which $\|a - b\| \leq \varepsilon$ and $\inf\{|f(b)| : f \in X\} > 0$.*

Proof. There exists an $r \in |K^\times|$ such that $B_{E'}(0, r) \subset U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^\times$ given by '$\alpha \sim \beta$ iff $|\alpha - \beta| < |\beta|$' yields an open partition of $C := \{\lambda \in K : |\delta| r\varepsilon \leq |\lambda| \leq r\varepsilon\}$ that is infinite because K is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\gamma \in C$ such that

$$(*) \qquad |f(a) - \gamma| \geq |\gamma| \qquad (f \in X).$$

U is w'-closed and edged, $g \notin U$, so by [6], 4.8 there exists a $c \in E$ such that $g(c) = \gamma$, $|f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_{E'}(0, r)$ so $\|a - b\| = \|c\| = \|\jmath(c)\| \leq |\gamma| r^{-1} \leq \varepsilon$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(c) - \gamma| = |f(c) - g(c)| = |u(c)| < |\gamma|$. This, combined with $(*)$, yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f\in X} |f(b)| > 0$.

COROLLARY 1.3. *Let K be not locally compact, let E be a Banach space. Let $X \subset E'$ be w'-precompact. Suppose $X \subset g + U$ where U is an edged zero neighbourhood in E', U w'-closed, $g \in E' \setminus U$. Then $\{x \in E : \inf_{f\in X} |f(x)| > 0\}$ is open and dense in E.*

Proof. Just combine Lemmas 1.1 (w'-precompactness implies w'-boundedness hence norm boundedness by completeness) and 1.2.

DEFINITION 1.4. Let us call $X \subset E'$ *σ-decomposable in E'* if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in E' such that each U_n is w'-closed and $X \subset \bigcup_n (f_n + U_n)$, $g \notin \bigcup_n (f_n + U_n)$.

THEOREM 1.5. (SEPARATION THEOREM) *Let K be not locally compact, let E be a Banach space, let $X \subset E'$ be w'-precompact and σ-decomposable in E'. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.*

Proof. Without loss, assume $g = 0$. Let $\{f_n + U_n : n \in \mathbb{N}\}$ be a covering of X like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{x \in E : \inf_{f\in X_n} |f(x)| > 0\}$ is open and dense in E, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{x \in E : f(x) \neq 0 \text{ for all } f \in X\} \supset \bigcap_n \{x \in E : \inf_{f\in X_n} |f(x)| > 0\} \neq \varnothing$.

REMARK. It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. *Let K be not locally compact, let $X \subset E$ be weakly precompact and σ-decomposable in E* (see below). *Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \notin f(X)$.* Here, X is called σ-decomposable in E if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in E such that each U_n is weakly closed and $X \subset \bigcup_n (x_n + U_n)$, $a \notin \bigcup_n (x_n + U_n)$.

COROLLARY 1.6. *Let K be not locally compact, let E be a Banach space, let $X \subset E'$ be σ-decomposable in E'. Suppose $X(a) := \{f(a) : f \in X\}$ is compact for all $a \in E$. Then X is w'-compact.*

Proof. The map $f \mapsto (f(a))_{a\in E}$ is a homeomorphism of (E', w') onto a subspace of K^E. The image of X lies in the compact subset $\prod_{a\in E} X(a)$ so X is w'-precompact. Since E' is w'-quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that X is w'-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \notin X(a)$. Now $X(a) \subset \overline{X}^{w'}(a) \subset \overline{X(a)} = X(a)$, so $g(a) \notin \overline{X}^{w'}(a)$ i.e. $g \notin \overline{X}^{w'}$.

To find examples of σ-decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. *Let $n \in \mathbb{N}$, let D be an n-dimensional subspace of E'. Then for each $t \in (0,1)$ there exist $a_1, a_2, \dots, a_n \in B_E$ such that $\max_{1\le i\le n} |f(a_i)| \ge t\|f\| \qquad (f \in D)$.*

Proof. First assume that the valuation of K is dense. The space $H := \{x \in E : f(x) = 0 \text{ for all } f \in D\}$ has codimension n in E. Choose $s \in (t,1)$ and let $g_1, \dots, g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} \le \|g_i\| \le t^{-1}$ for $i \in \{1, \dots, n\}$. There exist $b_1, \dots, b_n \in E/H$ such that $g_i(b_j) = \delta_{ij}$ $(i,j \in \{1,\dots,n\})$. Let $i \in \{1,\dots,n\}$, let $g = \Sigma\lambda_j g_j \in (E/H)'$. Then $\|g\| \ge \sqrt{s}\max|\lambda_j|\ \|g_j\|$ and $|g(b_i)| = |\lambda_i|$ so $|g(b_i)| \le \max_j |\lambda_j| \le s\max_j|\lambda_j|\ \|g_j\| \le \sqrt{s}\|g\|$. So $\|b_i\| < 1$. Thus, with $\pi : E \to E/H$ denoting the canonical quotient map, there exist $a_1, \dots, a_n \in B_E$ with $\pi(a_i) = b_i$ for each i. The adjoint π' of π maps $(E/H)'$ isometrically onto D. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $\|g\| = \|f\|$. We have, writing $g = \sum_{j=1}^n \lambda_j g_j$, $\max_{1\le i\le n}|f(a_i)| = \max_i |g(b_i)| = \max_i|\lambda_i| \ge t\max_i|\lambda_i|\,\|g_i\| \ge t\|\Sigma\lambda_i g_i\| = t\|g\| = t\|f\|$.

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the b_i have norm ≤ 1 (rather than < 1), but one can use that E/H is a strict quotient i.e. there exist $a_1, \dots, a_n \in E$ with $\|a_i\| = \|b_i\|$ and $\pi(a_i) = b_i$ for each i.

LEMMA 1.8. *Let D be a subspace of E', D of countable type. Then there is a sequence $a_1, a_2, \dots \in B_E$ such that $\|f\| = \sup_n |f(a_n)|$ for all $f \in D$.*

Proof. Let $D_1 \subset D_2 \subset \dots$ be finite-dimensional subspaces of D, $\bigcup_n D_n$ is dense in D. Let $t \in (0,1)$. By Lemma 1.7 there exists a finite set $F_n^t \subset B_E$ such that $\max_{x\in F_n^t}|f(x)| \ge t\|f\|$ for all $f \in D_n$.

So, for $F^t := \bigcup_{n\in\mathbb{N}} F_n^t$ we obtain

$$(*) \qquad \|f\| \ge \sup_{x\in F^t}|f(x)| \ge t\|f\| \qquad (f \in \bigcup_n D_n).$$

Now $F := \bigcup_{t\in\mathbb{Q}\cap(0,1)} F^t$ is countable and $(*)$ implies $\|f\| = \sup_{x\in F}|f(x)|$ for all $f \in \bigcup_n D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. *Let $X \subset E'$ be such that $X(a) := \{f(a) : f \in X\}$ is separable for each $a \in E$ and $[X]$ is of countable type. Then X is σ-decomposable in E'.*

Proof. Let $g \in E' \setminus X$. Then $D := [\{g\} \cup X]$ is of countable type so by Lemma 1.8 there exist $a_1, a_2, \dots \in B_E$ such that

$$(*) \qquad \|h\| = \sup_{n\in\mathbb{N}}|h(a_n)| \qquad (h \in D).$$

For each $m,n \in \mathbb{N}$ the set $U_{mn} := \{h \in E' : |h(a_n)| \le \frac{1}{m}\}$ is an edged w'-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering F_{mn} no member of which contains g. Then $\bigcup_{m,n} F_{mn}$ still avoids g; it remains to be shown that it covers X. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all

m, n so $|f(a_n) - g(a_n)| = 0$ for all n. Now $f - g \in D$, so by $(*)$ we obtain $\|f - g\| = 0$ i.e. $f = g$, a contradiction since $g \notin X$.

COROLLARY 1.10. *Let $X \subset E'$. If X is norm precompact, or X is w'-precompact and $[X]$ is of countable type, then X is σ-decomposable in E'.*

PROPOSITION 1.11. *Let $X \subset E'$ be such that $X(a)$ is separable for each $a \in E$. Suppose that for each $h \in \overline{X}^{w'}$ the set $X \cup \{h\}$ is w'-metrizable. Then X is σ-decomposable in E'.*

Proof. Let $g \in E' \setminus X$. If $g \notin \overline{X}^{w'}$ then there exists a w'-zero neighbourhood U such that $(g + U) \cap X = \varnothing$. We may assume that U is of the form $\{f \in E' : |f(a_1)| \leq \varepsilon, \dots, |f(a_n)| \leq \varepsilon\}$ for some $\varepsilon > 0$, $n \in \mathbb{N}$, $a_1, \dots, a_n \in E$. Then U is w'-closed and edged. By separability of $X(a_1) \times \dots \times X(a_n)$ only countably many of the cosets $f + U : f \in X$ cover X and none of them contains g. Now let $g \in \overline{X}^{w'}$. By w'-metrizability there exist w'-neighbourhoods of zero $U_1 \supset U_2 \supset \dots$ such that $X \cap \bigcap_n (g + U_n) = \varnothing$. We may suppose that the U_n are w'-closed and edged. By separability, like above, for each n the set $X \setminus (g + U_n)$ is covered by countably many additive cosets of U_n none of them containing g. Their union is a countable covering of X avoiding g.

2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of §1. Recall ([5], p. 57) that E is said to have property $(*)$ if for each subspace D of countable type, every $f \in D'$ has an extension $\overline{f} \in E'$. By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete K has $(*)$. For general K, spaces with a base, in particular spaces of countable type, have $(*)$ ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that E is assumed to be normpolar.

THEOREM 2.1. *Let K be not locally compact, let X be a subset of E such that $f(X)$ is compact for all $f \in E'$. Then each one of the following properties implies that X is weakly compact and weakly metrizable.*

(i) *E has property $(*)$.*
(ii) *E' is of countable type.*
(iii) *$[X]$ is of countable type.*

Moreover, in case (i) *X is norm compact and the weak and norm topology coincide on X.*

Proof. The natural isometry $j : E \to E''$ is easily seen to be a homeomorphism of E with the weak topology onto $j(E)$ with the restriction of the w'-topology $\sigma(E'', E')$. We show that $j(X)$ is σ-decomposable in E''. First note that the predual E' is normpolar. In case (i), from weak precompactness of X it follows that X is norm precompact by [7], Th. 3 (the assumption made throughout [7] that E is complete is easily seen to be superfluous here). So $j(X)$ is norm precompact in E'' and therefore σ-decomposable by Corollary 1.10. For case (ii) observe that every ($w'-$) bounded subset of E'' is w'-metrizable ([8], 6.1) which applies to $j(X) \cup \{\theta\}$ for any $\theta \in E''$. For each $f \in E'$ the set $j(X)(f) = f(X)$ is compact hence separable so $j(X)$ is σ-decomposable in E'' by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, $j(X)$ is σ-decomposable, and from Corollary 1.6 we conclude that $j(X)$ is w'-compact, so $X = j^{-1}(j(X))$ is w-compact. Observe that X is w-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that $j(X)$ is w'-metrizable in case (ii), so X is weakly metrizable. Now let X satisfy (iii). Then $[j(X)]$ is of countable type so by Lemma 1.8 there exist $f_1, f_2, \dots \in B_{E'}$ such that $\|j(x)\| = \sup_{n \in \mathbb{N}} |f_n(x)|$ for all $x \in X$. The formula $d(x, y) = \sup_n |f_n(x) - f_n(y)| 2^{-n}$ defines an ultrametric d on X (if $d(x, y) = 0$ then $|f_n(x) - f_n(y)| = 0$ for all n so $\|x - y\| = 0$). By boundedness of X the induced topology is weaker than the weak topology on X, but by weak compactness these topologies coincide and so X is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on X the weak and norm topology coincide, and that therefore X is norm compact and w-metrizable.

REMARKS.

1. If K is not spherically complete the space ℓ^∞ does not have property $(*)$ ([4], 4.15 $(\delta) \Rightarrow (\gamma)$) but since $(\ell^\infty)' \simeq c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $\ell^\infty \widehat{\otimes} \ell^\infty$ ([3], 2.3) and the space D of [4], 4.J.
2. Let K be not spherically complete, let $E := \ell^\infty$, let $X := \{0\} \cup \{e_1, e_2, \dots\} \subset \ell^\infty$, when $e_1, e_2, \dots$ are the unit vectors. Then (ii) and (iii) above hold. X is weakly compact (since $\lim_{n\to\infty} e_n = 0$ weakly) but is obviously not norm compact.
3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := K^{\mathbf{N}}$ with the product topology. Then $E' \cong \bigoplus_{\mathbf{N}} K$. Let $X := \{e_1, e_2, \dots\}$ where $e_1, e_2, \dots$ are the unit vectors of $K^{\mathbf{N}}$. Then E is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large n, so $f(X)$ is finite (hence compact) and contains 0. Yet, X is not (weakly) compact as $0 = w - \lim_{n\to\infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

COROLLARY 2.2. (p-adic Eberlein-Šmulian Theorem I) *Let K be not locally compact and let X, E satisfy one of the conditions* (i), (ii), (iii) *of Theorem* 2.1. *Then the following are equivalent.*

(α) *X is weakly compact.*

(β) *X is weakly sequentially compact.*

(γ) *X is weakly countably compact.*

Proof. Each one of the properties (α), (β), (γ) implies compactness of $f(X)$ for all $f \in E'$. By Theorem 2.1 X is weakly metrizable and from that the equivalence of (α), (β), (γ) follows easily.

NOTE. In Corollary 2.2, (α), (β), (γ) are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if K is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over K has $(*)$.

THEOREM 2.3. (p-adic Eberlein-Šmulian Theorem II) *Let K be locally compact, let $X \subset E$. Then each one of the above statements* (α), (β), (γ) *is equivalent to 'X is norm compact'.*

Proof. We have $(\alpha) \Rightarrow (\gamma)$, $(\beta) \Rightarrow (\gamma)$. It suffices to prove that (γ) implies that X is a norm compactoid (then X is weakly metrizable since the norm and weak topology coincide on X ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0,1]$ and a t-orthogonal sequence $e_1, e_2, \dots$ in X such that $\inf_n \|e_n\| > 0$. By (γ) there is a weak accumulation point a of $\{e_1, e_2, \dots\}$. This a is in the weak closure D of $[\![e_1, e_2, \dots]\!]$ which equals the norm closure, so $a = \sum_{i=1}^{\infty} \lambda_i e_i$ where $\|\lambda_i e_i\| \to 0$. If $\lambda_j \neq 0$ for some j, let $U := \{x \in E : |\delta_j(x)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the jth coordinate function $\Sigma \xi_i e_i \mapsto \xi_j$ on D. Then $a + U$ is a weak neighbourhood of a but for each $n \in \mathbb{N}$, $n \neq j$ we have $|\delta_j(a - e_n)| = |\lambda_j|$ so $e_n \notin a + U$, a contradiction. Hence, $a = 0$. But then $\{x \in E : |f(x)| < 1\}$ is a weak neighbourhood of a containing no e_n if $f \in E'$ is such that $f(e_n) = 1$ for all n. Contradiction, so X is a norm compactoid.

REMARK. Corollary 2.2 for strongly polar spaces E and Theorem 2.3 were first proved directly by the first author.

REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). *Let* $X \subset E$. *Suppose one of the conditions* (i), (ii), (iii) *of Theorem* 2.1 *is satisfied. Then the following are equivalent.* (α) X *is weakly relatively compact.* (β) X *is weakly relatively sequentially compact.* (γ) X *is weakly relatively countably compact.* We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets X of $\ell^\infty(I)$ where I has at least the cardinality of the continuum, but is non-measurable, and where K is not spherically complete. The K-valued characteristic function of a subset $S \subset I$ is denoted ξ_S and is given by $\xi_S(x) := 1$ if $x \in S$, $\xi_S(x) := 0$ if $x \in I \setminus S$.

1. *Let* $X := \{\xi_S : S \subset I\}$. *Then* X *is a weakly compact but not weakly sequentially compact subset of* $\ell^\infty(I)$.

Proof. X is bounded and since $\ell^\infty(I)' \simeq c_0(I)$ ([4],4.21) the weak topology on X is the topology of pointwise convergence. Clearly the map $f \mapsto \left(f(i)\right)_{i\in I}$ is a homeomorphism $X \to \{0,1\}^I$, hence X is weakly compact. To prove that X is not weakly sequentially compact, let $\phi : I \to Y$ be a surjection where $Y := \{\xi_A : A \subset \mathbb{N}\} \subset \ell^\infty$. The formula $\phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \dots)$ ($x \in I$) defines subsets $S_1, S_2, \dots$ of I. If $\xi_{S_{n_1}}, \xi_{S_{n_2}}, \dots$ is a subsequence of $\xi_{S_1}, \xi_{S_2}, \dots$ then, by surjectivity of ϕ, there is an $x \in I$ for which $(\xi_{S_{n_1}}(x), \xi_{S_{n_2}}(x), \dots) = (1,0,1,0,1,\dots)$, so the subsequence is not weakly convergent.

2. *Let* $Z := \{\xi_S : S \subset I,\ S \text{ countable}\} \subset \ell^\infty(I)$. *Then* Z *is weakly sequentially compact but not weakly compact.*

Proof. Clearly the weak closure of Z equals X of above, so Z is not weakly compact. On the other hand, if $\xi_{S_1}, \xi_{S_2}, \dots$ is a sequence in Z then $S := \cup S_n$ is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of S, hence at all points of I, to an element of Z.

3. *Let* $T := \{\xi_{\{i\}} : i \in I\} \subset \ell^\infty(I)$. *Then* $f(T)$ *is compact for all* $f \in \ell^\infty(I)'$ *but* T *is not weakly countably compact.*

Proof. Let $f \in \ell^\infty(I)'$. As $\ell^\infty(I)' \simeq c_0(I)$ we have that $f(\xi_{\{i\}}) = 0$ except for $i \in \{i_1, i_2, \dots\}$ where we may assume the $i_n \in I$ to be distinct. Then $\xi_{\{i_n\}} \to 0$ weakly so $T_1 := \{0\} \cup \{\xi_{\{i_n\}} : n \in \mathbb{N}\}$ is weakly compact and $f(T) = f(T_1)$ is compact. However the only weak accumulation point of $\{\xi_{\{i_1\}}, \xi_{\{i_2\}}, \dots\}$ is $0 \notin T$ so that T is not weakly countably compact.

REFERENCES

[1] L.V. KANTOROVICH and G.P. AKILOV, *Functional Analysis. Pergamon Press* (1982).

[2] C. PEREZ-GARCIA and W.H. SCHIKHOF, Compact operators and the Orlicz-Pettis property in p-adic analysis, Report 9101, Department of Mathematics, University of Nijmegen, The Netherlands.

[3] C. PEREZ-GARCIA andW.H. SCHIKHOF, Non-reflexive and non-spherically complete subspaces of the p-adic space ℓ^∞, To appear in *Indag. Mathem. N.S.*

[4] A.C.M. van ROOIJ, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York (1978).

[5] A.C.M. van ROOIJ, Notes on p-adic Banach spaces, Report 7633, Department of Mathematics, University of Nijmegen, The Netherlands.

[6] W.H. SCHIKHOF, Locally convex spaces over non-spherically complete valued fields, *Bull. Soc. Math. Belg. Sér.* **B 38** (1986), 187-224.

[7] W.H. SCHIKHOF, p-Adic non-convex compactoids, *Proc. Kon. Ned. Akad. Wet.* **A92** (1989), 339-342.

[8] W.H. SCHIKHOF, A perfect duality between p-adic Banach spaces and compactoids, To appear in *Indag. Mathem. N.S.*