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STRICTLY BARRELLED DISKS IN INDUCTIVE LIMITS OF QUASI-(LB)-SPACES

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ABSTRACT. A strictly barrelled disk B in a Hausdorff locally convex space E is a disk such that the linear span of B with the topology of the Minkowski functional of B is a strictly barrelled space. Valdivia's closed graph theorems are used to show that closed strictly barrelled disk in a quasi-(LB)-space is bounded. It is shown that a locally strictly barrelled quasi-(LB)-space is locally complete. Also, we show that a regular inductive limit of quasi-(LB)-spaces is locally complete if and only if each closed bounded disk is a strictly barrelled disk in one of the constituents.

KEY WORDS AND PHRASES. Quasi-(LB)-space, strictly barrelled space, inductive limit.

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1. INTRODUCTION.

Throughout this paper, we use the word *space* to denote a Hausdorff locally convex space. An absolutely convex set will be called a *disk*. If A is a disk in a space E, its linear span E_A may be endowed with the semi-normed topology given by the Minkowski functional of A. When distinction is needed, we will denote this topology by ρ_A. When A is a bounded disk, it is easy to see that E_A is normed and that $id: E_A \to E$ is continuous. If E_A is a Banach space (resp. Baire space), we call A a Banach (resp. Baire) disk. If every bounded subset of E is contained in a bounded Banach (resp. Baire) disk, we say that E is locally complete (resp. locally Baire). Locally complete spaces are also called fast complete, and according to [1; 5.1.6, pg. 152], a space is locally complete if and only if every closed bounded disk is already a Banach disk.

DEFINITION 1.1: Following [2], a space E is strictly barrelled if given any ordered absolutely convex web $\mathcal{W}$ on E there exists a strand $(W(k)) = \{W(k) : k \in N\}$ of $\mathcal{W}$ such that for each positive integer k, the closure $\overline{W(k)}$ is a zero neighborhood in E, where $W(k)$ denotes the kth member of a strand $(W(k))$.

DEFINITION 1.2: Let A be a disk. If E_A is a strictly barrelled space, we will say that A is a strictly barrelled disk. If every bounded set is contained in a strictly barrelled disk, we say that E is locally strictly barrelled.

REMARK 1.3: Using [1; chapt. 9] and [2; Prop. 6.17, pg. 160],

locally complete $\Rightarrow$ locally Baire $\Rightarrow$ locally strictly barrelled.

These implications cannot be reversed; the first by [1; 1.2.12 pg. 17], the second by [2; Prop. 17, pg. 160 & Note 4, pg. 162]. Valdivia defines quasi-(LB)-spaces in [2], and proves a webbed-space equivalence in [2; Th. 4.1, pg. 153]. We will use this equivalence as our definition below.

DEFINITION 1.4: A space with an ordered, absolutely convex strict web is called a quasi-(LB)-space.

2. QUASI-(LB)-SPACES AND STRICTLY BARRELLED DISKS.

The following generalizes [3; Th. 3, pg. 173] and [4; Th. 1, pg. 222].

THEOREM 2.1: *Let B be a closed strictly barrelled disk in a quasi- (LB)-space. Then B is bounded.*

PROOF: Let (E, τ) be the quasi- (LB)-space that contains B. Denote by η the topology induced on E_B by the following system of neighborhoods: $\{(n^{-1}B)\cap V : V$ is a τ-closed zero neighborhood, $n \in N\}$. Using the ordered strict web on (E, τ) and the construction in [4; Th. 1, pg. 222], we have that (E_B, η) is a quasi-(LB)-space. The map $id: (E_B, \eta) \to (E_B, p_B)$ is continuous and (E_B, p_B) is strictly barrelled. Therefore, by [2; Th. 6.5(a), pg. 163], this map is open, implying that for any τ-zero neighborhood V, $id(B \cap V)$ is a neighborhood of zero in (E_B, p_B). In particular, there exists $a > 0$ such that $aB \subset B \cap V \subset V$. We conclude that B is τ-bounded. ♦

The result that follows uses the closed graph theorem of Valdivia [2].

THEOREM 2.2: *Any locally strictly barrelled quasi-(LB)-space is locally complete.*

PROOF: Assume (E, t) is such a space and suppose A is bounded in E. There is a bounded disk $B \supset A$ such that (E_B, p_B) is strictly barrelled. Because $id: (E_B, p_B) \to (E_B, t)$ is continuous, [2; Th. 7.6 pg. 164] shows that there is a Fréchet space F for which $E_B = id(E_B) \subset F$ and the following injections are continuous: $(E_B, p_B) \to F \to (E_B, t)$. Hence, there is a bounded Banach disk D in F, with $A \subset B \subset D$, and D is a bounded Banach disk in E as well. ♦

3. INDUCTIVE LIMITS.

In this section we consider sequences (E_n, t_n), $n \in \mathbf{N}$ of spaces with $E_1 \subset E_2 \subset \ldots$, and for every positive integer n, E_n injects continuously into E_{n+1}. We put $E = ind_n E_n$ for the inductive limit. Recall that an inductive limit is called regular if for any of its bounded subsets, there is a constituent space such that the subset is contained in and bounded in that constituent.

THEOREM 3.1: *Let $E = ind_n E_n$ be an inductive limit of quasi-(LB)-spaces. Suppose B is a disk in (E_n, t_n). Then:*

(a) If there exists $m \geq n$ such that B is a closed strictly barrelled disk in (E_m, t_m), then B is a closed bounded strictly barrelled disk in both (E_n, t_n) and (E_m, t_m). Moreover, B is contained in a bounded Banach disk in (E_n, t_n) and (E_m, t_m).

(b) If (a) holds for every bounded disk in E_n, then E_n is locally complete.

(c) If E is regular and locally complete, then E_n is locally complete for every positive integer n.

PROOF: *(a)*: If the assumptions are satisfied, then from the continuity of $id: (E_n, t_n) \to (E_m, t_m)$, B is t_n- closed. As a strictly barrelled, closed disk in (E_m, t_m), B is t_n- bounded by Theorem 2.1. We use Theorem 2.2 in both (E_m, t_m) and (E_n, t_n) to conclude that B is contained in a bounded Banach disk in both spaces.

(b): Obvious consequence of *(a)*.

(c): Let E be any fixed natural number and let $A \subset E_n$ be bounded. By the assumptions and topology on E, A is bounded in E, and contained in an E- closed, bounded Banach disk D, where D itself is contained in and bounded in some (E_m, t_m); clearly $m \geq n$. As $id: (E_m, t_m) \to E$ is continuous, D is t_m- closed and of course is a bounded Banach disk there. We apply part *(a)* to the disk $D \cap E_n$ and we are done. ♦

In [5] we have that if each (E_n, t_n) is webbed and locally complete, then is $E = ind_n E_n$ regular if and only if it is locally complete. One can ask what happens if the inductive limit is regular but the spaces (E_n, t_n) are not locally complete; see for example [6] and [7]. It is not difficult to prove a similar type of result using quasi-(LB)-spaces; the details follow. Compare also [4; Th. 3, pg 223] and [3; Th.5, pg. 174].

<u>THEOREM</u> 3.2: *Suppose each (E_n, t_n) is a quasi-(LB)-space and $E = ind_n E_n$ is regular. Then E is locally complete if and only if for each closed, bounded disk $B \subset E_n$, there is an $m \in N$ such that B is a strictly barrelled disk in (E_m, t_m).*

PROOF: If E is locally complete, the conclusion follows directly from from 3.1 *(c)*. Conversely, take a closed, bounded disk B in E. There is an $n \in \mathbf{N}$ such that $B \subset E_n$ and is t_n- bounded, and there is an $m \in \mathbf{N}$ with $B \subset E_m$ and B is a strictly barrelled disk. If $m > n$, we use 3.1 *(a)*. On the other hand, if $n \geq m$, then 2.1 tells us that $n = m$ and *(a)* of 3.1 applies. In either case, E is locally complete.

♦

We want to construct a regular inductive limit of non-locally complete quasi-(LB)-spaces, but first we need:

<u>LEMMA</u> 3.3: *A finite product of locally convex spaces is locally complete if and only if each space is locally complete.*

PROOF: One may use bornologies, [8; 3.2(3), pg 43], to prove that any product of locally complete spaces is locally complete. Conversely, let $E = F \times G$, and assume that E is locally complete. Suppose, without loss of generality, that F is *not* locally complete. This means there is a disk B, closed and bounded in F, and B is not a Banach disk in F. Then $B' = B \times \{0\}$ is an E- closed and bounded disk that is not a Banach disk, a contradiction. Hence, F is locally complete. The proof for general finite products can is done by induction. ♦

EXAMPLE 3.4: Let E_0 be an non-regular (LB)-space. Then E_0 is a quasi-(LB)-space by [2; Prop 3.5, pg 152]. For each positive integer n, put $E_n = \oplus\{E_0 : i = 1,2,...\} \cong \prod\{E_0 : i = 1,2,...\}$. The lemma, the non-regularity of E_0 and [2; Prop 3.3, pg 151] imply that each E_n is a non-locally complete quasi-(LB)-space.

Set $E = ind_n E_n = \oplus\{E_0 : n \in \mathbf{N}\}$. As a direct sum, if $A \subset E$ is bounded, then there is a finite subset I of $\mathbf{N}$ such that A is bounded in $\oplus\{E_0 : i \in I\}$. If $n = max\{i : i \in I\}$, then A is bounded in E_n, and E is therefore regular. Next, we use 3.2. Let $B \subset E_I = E_0$ be a closed, bounded disk that is not a Banach disk. Using the defintion of the direct sum topology t of E and the fact that t induces on E_0 its own topology, we have that B is a closed bounded disk in E, also. The disk B cannot be a Banach disk in E, so E is not locally complete. From 3.2, we see that B is in fact a really bad disk; not only is it a non-Banach disk in E, it cannot be a strictly barrelled disk in any E_n. ♦

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