HOMOMORPHISMS OF ERGODIC GROUP ACTIONS AND CONJUGACY OF SKEW PRODUCT ACTIONS

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ABSTRACT. Let G be a locally compact group acting ergodically on X. We discuss relationships between homomorphisms on the measured groupoid $X \times G$, conjugacy of skew product extensions, and similarity of measured groupoids. To do this, we describe the structure of homomorphisms on $X \times G$ whose restriction to an extension given by a skew product action is the trivial homomorphism.

KEY WORDS AND PHRASES. Homomorphism between ergodic group actions, groupoids, Mackey dense range, cocycles, skew products.
1991 AMS SUBJECT CLASSIFICATION CODES. 20L13, 22D40, 28D15.

1. INTRODUCTION. Ergodic group actions is one subfamily of the categorical generalization of locally compact groups as suggested by Mackey in [3]. This generalization include measured equivalence relations, measured foliations, and measured ergodic groupoids amongst others.

A concept for groups that extends to this category and in particular to ergodic group actions is the homomorphism. This paper discusses relationships between homomorphisms of ergodic group actions, their restrictions, conjugacy of skew product actions, and similarity of measured groupoids. These relationships provide an answer to the following problem about locally compact groups.

If $\phi : G \to H$ and $\psi : G \to K$ are continuous homomorphisms whose ranges are dense in H and K, respectively, does there exists an isomorphism $\eta : H \to K$ satisfying $\psi = \eta \circ \phi$? If so, H and K are conjugate G-spaces where the action of G on H is given by $h \cdot g = h\phi(g)$. We will investigate the extent to which this problem generalizes to ergodic group actions. This paper will extend some of the results obtained by Fabec in [1,2].

Before stating the main result, which is the theorem at the end of this section, we will briefly review notions of group actions and homomorphisms of measured groupoids. The proof of the main result will be given in the second section. In the third section, the theorem will be applied to find relationships between conjugacy of skew product extensions of an ergodic group action and similarity of measured groupoids. In particular, the above problem is answered there.

Let G be a second countable, locally compact group. Suppose μ is a σ-finite measure on a standard Borel space X. Then X is a G-space provided there exists a Borel mapping $(x, g) \mapsto x \cdot g$ from $X \times G$ to X satisfying

(1) $x \cdot e = x$ for all x,

(2) $x \cdot g_1 \cdot g_2 = x \cdot g_1 g_2$ for all x, g_1, g_2, and

(3) $\mu(E \cdot g) = 0$ if and only if $\mu(E) = 0$ for each g and Borel subset E.

Two G-spaces (X_1, μ_1) and (X_2, μ_2) are isomorphic if there exists a Borel isomorphism $\Phi : X_1 \to X_2$ such that $\Phi(x \cdot g) = \Phi(x) \cdot g$ μ_1-a.e. x for each g and $\Phi_* \mu_1 \equiv \mu_1 \circ \Phi^{-1} \sim \mu_2$ i.e. $\Phi_* \mu_1$ and μ_2 belong to the same measure class.

If (Y, ν) is a G-space then a Borel map $p : X \to Y$ is called an extension if $p(x \cdot g) = p(x) \cdot g$ for all (x, g) and $p_* \mu \sim \nu$. Two extensions $p_1 : (X_1, \mu_1) \to (Y, \nu)$ and $p_2 : (X_2, \mu_2) \to (Y, \nu)$ are isomorphic over Y if there exists a G-space isomorphism Φ from X_1 to X_2 such that $p_2 \circ \Phi(x) = p_1(x)$ a.e. x.

Let H be a complete separable metric group. A cocycle on the G-space (X, μ) with values in H is a Borel mapping $\phi : X \times G \to H$ satisfying $\phi(x, g_1 g_2) = \phi(x, g_1)\phi(x \cdot g_1, g_2)$ for all x, g_1, g_2. If one defines $(x, h) \cdot g = (x \cdot g, h\phi(x, g))$ then $(X \times H, \mu \times m_H)$ becomes a G-space where m_H is a Haar measure on H. This G space is denoted by $(X \times_\phi H, \mu \times m_H)$ and is called the skew product action defined by the cocycle ϕ.

The measure space $(X \times G, \mu \times m_G)$ is called the measured groupoid defined by the action of G on (X, μ). There is a partially defined multiplication on the groupoid, namely, $(x, g)(x \cdot g, g_1) = (x, gg_1)$. Also, $(x, g)^{-1} = (x \cdot g, g^{-1})$ defines an inverse operation in the groupoid. Given $z_1, z_2, z_3, z_4 \in X \times G$, one can directly verify that

$$\text{if } z_1 z_2 = z_3 \text{ then } z_1^{-1} z_3 \text{ is well-defined and } z_2 = z_1^{-1} z_3. \tag{1.1}$$

A homomorphism $h : X \times G \to Y \times H$ between measured groupoids or group actions is a pair of Borel maps $h = (p, \phi)$ where $p : X \to Y$ and $\phi : X \times G \to H$ satisfy

(4) ϕ is a cocycle,

(5) $p(x \cdot g) = p(x) \cdot \phi(x, g)$ for all (x, g), and

(6) $\mu(p^{-1}(E)) = 0$ if and only if $\nu(E) = 0$ whenever E is an invariant analytic subset of Y.

Statements (4) and (5) are necessary and sufficient conditions for $h : X \times G \to Y \times H$ to define a multiplicative mapping between groupoids. An action of G is ergodic if E is a null or conull Borel subset whenever $E \cdot g = E$ for all g. In case (X, μ) is an ergodic G-space and (Y, ν) is an ergodic H-space, one says h is a homomorphism between ergodic group actions. In this case, as pointed out by Ramsay in [5, page 394], statement (6) is equivalent to the condition: $p^{-1}(E)$ is null whenever E is an invariant analytic null subset of Y.

Two homomorphisms h and h' from $X \times G$ to $Y \times H$ are cohomologous or similar if there is a Borel mapping $A : X \to Y \times H$ satisfying $A(x)h(x, g) = h'(x, g)A(x \cdot g)$ a.e. (x, g). This will be denoted by $h \sim h'$. The idea of isomorphic groups extends to the notion of similar measured groupoids. Let id be the identity mapping. If there exist homomorphisms $J_1 : X \times G \to Y \times H$ and $J_2 : Y \times H \to X \times G$ for which $J_1 \circ J_2 \sim id$ and $J_2 \circ J_1 \sim id$, then $X \times G$ and $Y \times H$ are called similar measured groupoids. Also, a homomorphism $h : X \times G \to Y \times H$ is said to be trivial

if there is a Borel mapping $A : X \to Y \times H$ satisfying

$$A(x)h(x,g) = A(x \cdot g) \quad a.e. \ (x,g). \tag{1.2}$$

A homomorphism $h = (p, \phi) : X \times G \to Y \times H$ between ergodic group actions is said to have Mackey dense range if $(x,h) \mapsto p(x)h^{-1}$ provides an ergodic decomposition of the skew product G-space $X \times_\phi H$. In particular, a cocycle ϕ has Mackey dense range in the locally compact group H if the skew product action is ergodic. An ergodic decomposition satisfies a universal property. Namely, if f is a Borel function on $X \times_\phi H$ with values in a countably separated Borel space and f is essentially invariant, i.e. $f(x \cdot g, h\phi(x,g)) = f(x,h)$ a.e. (x,h,g), then there exists an essentially unique Borel function F defined on Y such that $F(p(x) \cdot h^{-1}) = f(x,h)$ a.e. (x,h).

One can also restrict homomorphisms between measured groupoids. Indeed, if $p_0 : X \to Y$ is an extension of G-spaces, the homomorphism given by $i(x,g) = (p_0(x), g)$ from $X \times G$ into $Y \times G$ is called an inclusion homomorphism. If $h = (p, \phi)$ is a homomorphism of $Y \times G$ into $W \times H$, the composition $h \circ i$ is called the restriction of h to $X \times G$ and will be denoted by $h|_{X \times G}$. So $h|_{X \times G}(x,g) = (p(p_0(x)), \phi(p_0(x), g))$ for all (x,g). The main result of this paper is the following theorem and it is an extension of Proposition 2.1 in Fabec [2].

THEOREM *Let $h = (p, \phi) : X \times G \to Y \times H$ be a homomorphism with Mackey dense range and let $b = (q, \psi) : X \times G \to Z \times K$ be a homomorphism between ergodic group actions. Then $b \sim a \circ h$ for some homomorphism $a : Y \times H \to Z \times K$ if and only if $b|_{X \times_\phi H \times G}$ is trivial.*

2. PROOF OF THEOREM.

To organize and divide the proof, we will begin with two lemmas. It will be shown that by changing a trivial homomorphism slightly one can change the 'almost everywhere' condition in (1.2) to one that will hold 'for all'.

LEMMA 2.1 *Let W be a G-space and let $\psi : W \times G \to K$ be a cocycle. Suppose $\eta : W \to K$ is a Borel mapping such that $\eta(w)\psi(w,g) = \eta(w \cdot g)$ a.e. (w,g). Then there exist a cocycle $\psi_0 : W \times G \to K$, a Borel mapping $A_2 : W \to K$, and a conull, Borel subset W_0 of W with $W_0 \cdot G$ Borel satisfying*

(a) $A_2(w)\psi_0(w,g) = A_2(w \cdot g)$ for all (w,g),

(b) $\psi_0(w,g) = \psi(w,g)$ for all $w \in W_0 \cdot G$ and $g \in G$, and

(c) $A_2(w) = \eta(w)$ a.e. w.

PROOF. Let $J = \{(w,g) : \eta(w) = \eta(w \cdot g)\psi(w,g)^{-1}\}$. If (w, g_1) and $(w \cdot g_1, g_2)$ are in J then $\eta(w \cdot g_1 g_2)\psi(w, g_1 g_2)^{-1} = \eta(w \cdot g_1 g_2)\psi(w \cdot g_1, g_2)^{-1}\psi(w, g_1)^{-1} = \eta(w \cdot g_1)\psi(w, g_1)^{-1} = \eta(w)$ and so $(w, g_1 g_2) \in J$. Since J is a conull, multiplicatively closed Borel subset of the groupoid $W \times G$, then by Lemma A.4 in Fabec [2] and Lemma 5.2 in Ramsay [4] there is a Borel, conull subset W_0 of W such that

(i) if $w, w \cdot g \in W_0$ then $(w,g) \in J$,

(ii) $W_0 \cdot G$ is a Borel subset, and

(iii) there exists a Borel mapping $\theta : W_0 \cdot G \to G$ such that $\theta(w) = e$ if $w \in W_0$,
    and $w \cdot \theta(w) \in W_0$ for all $w \in W_0 \cdot G$.

If $w \in W_0 \cdot G$ then define $A_2(w) = \eta(w \cdot \theta(w))\psi(w, \theta(w))^{-1}$, otherwise set $A_2(w) = e$.

Let $\psi_0 : W \times G \to K$ be the cocycle defined by $\psi_0(w, g) = \psi(w, g)$ if $w \in W_0 \cdot G$, and if $w \notin W_0 \cdot G$ set $\psi_0(w, g) = e$. If $w \in W_0 \cdot G, g \in G$ then $(w \cdot \theta(w), \theta(w)^{-1}g\theta(w \cdot g)) \in J$, and cosequently,

$A_2(w \cdot g)\psi_0(w, g)^{-1} = \eta(w \cdot g\theta(w \cdot g))\psi(w \cdot g, \theta(w \cdot g))^{-1}\psi(w, g)^{-1} =$

$\eta(w \cdot g\theta(w \cdot g))\psi(w, g\theta(w \cdot g))^{-1} = \eta(w\theta(w))\psi(w \cdot \theta(w), \theta(w)^{-1}g\theta(w \cdot g))\psi(w, g\theta(w \cdot g))^{-1} =$

$\eta(w\theta(w))\psi(w \cdot \theta(w), \theta(w)^{-1}) = A_2(w)$. Hence, $A_2(w \cdot g)\psi_0(w, g)^{-1} = A_2(w)$ for all (w, g). □

LEMMA 2.2 *Let* $b = (q, \psi) : W \times G \to Z \times K$ *be a trivial homomorphism between measured groupoids. Then there exist Borel mappings* $b_0 = (q_0, \psi_0) : W \times G \to Z \times K$, $A_0 = (A_1, A_2) : W \to Z \times K$, *and a conull,Borel subset* W_0 *of* W *with* $W_0 \cdot G$ *Borel satisfying*

(a) $\psi_0 : W \times G \to K$ *is a cocycle where* $\psi_0(w, g) = \psi(w, g)$ *for all* $w \in W_0 \cdot G$ *and* $g \in G$,

(b) $q(w) = q_0(w)$ *a.e* w, *and*

(c) $A_0(w)b_0(w, g) = A_0(w \cdot g)$ *for all* (w, g).

PROOF. Let $A = (\xi, \eta) : W \to Z \times K$ be a Borel mapping satisfying

$(\xi(w), \eta(w))(q(w), \psi(w, g)) = (\xi(w \cdot g), \eta(w \cdot g))$ a.e. (w, g). By Lemma 2.1 we can choose a cocycle $\psi_0 : W \times G \to K$, a Borel mapping $A_2 : W \to K$, and a conull, Borel subset W_0 with $W_0 \cdot G$ Borel such that (a) - (c) of Lemma 2.1 holds. Let $J = \{(w, g) : \xi(w) = \xi(w \cdot g)\}$. Since J is a conull, multiplicatively closed Borel subset of the groupoid $W \times G$ then by Lemma A.4 in Fabec [2] and Lemma 5.2 in Ramsay [4] there exists a conull, Borel subset W_1 of W such that

(i) if $w, w \cdot g \in W_1$ then $(w, g) \in J$,

(ii) $W_1 \cdot G$ is a Borel subset, and

(iii) there exists a Borel mapping $\theta : W_1 \cdot G \to G$ where $\theta(w) = e$ for $w \in W_1$, and $w \cdot \theta(w) \in W_1$ for all $w \in W_1 \cdot G$.

Let $A_1 : W \to Z$ be the Borel mapping given by $A_1(w) = \xi(w \cdot \theta(w))$ if $w \in W_1 \cdot G$ and if $w \notin W_1 \cdot G$ set $A_1(w) = z_0$ where z_0 is some fixed element of Z independent of w. If $w \in W_1 \cdot G$ then $(w\theta(w), \theta(w)^{-1}g\theta(w \cdot g)) \in J$ and so $A_1(w \cdot g) = \xi(w \cdot g\theta(w \cdot g)) = \xi(w\theta(w)) = A_1(w)$. Thus, $A_1(w \cdot g) = A_1(w)$ for all (w, g).

Define a Borel map $q_0 : W \to Z$ by letting $q_0(w) = A_1(w)A_2(w)$. Since $A_2(w) = \eta(w)$ a.e. w and $\xi(w) \cdot \eta(w) = q(w)$ a.e. w then $q_0(w) = q(w)$ a.e. w. Set $b_0 = (q_0, \psi_0)$ and $A_0 = (A_1, A_2)$. Then $(A_1(w), A_2(w))(q_0(w), \psi_0(w, g)) = (A_1(w \cdot g), A_2(w \cdot g))$ for all w, g. □

Turning to the proof of the Theorem, suppose $b \sim a \circ h$. Since 'cohomologous' defines an equivalence relation, one can assume $b = a \circ h$. If $A(x, h) = a(p(x)h^{-1}, h)$ then $A(x, h)b(x, g) = a(p(x)h^{-1}, h)a(p(x), \phi(x, g)) = a(p(x)h^{-1}, h\phi(x, g)) = a(p(xg)\phi(x, g)^{-1}h^{-1}, h\phi(x, g)) = A(xg, h\phi(x, g))$. Thus, $A(x, h)b(x, g) = A((x, h) \cdot g)$ for all $(x, h), g$ and $b|_{X \times H \times G}$ is trivial.

Conversely, suppose $b|_{X \times_\phi H \times G} : X \times_\phi H \times G \to Z \times K$ is trivial. According to Lemma 2.2, with $W = X \times_\phi H$, there are Borel mappings $b_0 = (q_0, \psi_0) : X \times_\phi H \times G \to Z \times K$ and $A_0 = (A_1, A_2) : X \times_\phi H \to Z \times K$ and conull Borel subsets $W_0, W_2 \subseteq X \times_\phi H$, with $W_0 \cdot G$ Borel for which

(a) ψ_0 is a cocycle and $\psi_0(x, h, g) = \psi(x, g)$ for all $(x, h) \in W_0 \cdot G, g \in G$,

(b) $q(x) = q_0(x, h)$ for all $(x, h) \in W_2$, and

(c) $A_0(x, h)b_0(x, h, g) = A_0(x \cdot g, h\phi(x, g))$ for all x, h, g.

Let $W_3 = W_2 \cap (W_0 \cdot G)$. Then $b_0(x, h, g) = b(x, g)$ for all $(x, h) \in W_3$, $g \in G$.

Set $E = \{(x, h_1, h_2) : (x, h_1), (x, h_2^{-1}h_1) \in W_3\}$. If $(x, h_1, h_2) \in E$, $g \in G$ then $A_0(x, h_1)b(x, g) = A_0(x \cdot g, h_1\phi(x, g))$ and $A_0(x, h_2^{-1}h_1)b(x, g) = A_0(x \cdot g, h_2^{-1}h_1\phi(x, g))$. In particular, by (1.1), $A_0(x, h_1)A_0(x, h_2^{-1}h_1)^{-1}$ is a well-defined element of $Z \times K$ for all $(x, h_1, h_2) \in E$. Fix a point z_0 in Z and define a Borel mapping $F : X \times_\phi H \times H \to Z \times K$ by

$$F(x, h_1, h_2) = \begin{cases} A_0(x, h_1)A_0\left(x, h_2^{-1}h_1\right)^{-1} & \text{if } (x, h_1, h_2) \in E \\ (z_0, e) & \text{otherwise.} \end{cases}$$

Given $g \in G, (x, h_1, h_2) \in E$ such that $(x \cdot g, h_1\phi(x, g), h_2) \in E$, then $F(x \cdot g, h_1\phi(x, g), h_2) = A_0(x \cdot g, h_1\phi(x, g)) A_0\left(x \cdot g, h_2^{-1}h_1\phi(x, g)\right)^{-1} = [A_0(x, h_1)b(x, g)]\left[A_0(x, h_2^{-1}h_1)b(x, g)\right]^{-1} = A_0(x, h_1)A_0(x, h_2^{-1}h_1)^{-1} = F(x, h_1, h_2)$. Since E is a conull, Borel subset and $(x, h_1) \mapsto p(x)h_1^{-1}$ is an ergodic decomposition, there exists a Borel mapping $a : Y \times H \to Z \times K$ satisfying $a\left(p(x)h_1^{-1}, h_2\right) = A_0(x, h_1)A_0(x, h_2^{-1}h_1)^{-1}$ a.e. (x, h_1, h_2). Moreover, $a\left(p(x)h^{-1}, h_1h_2\right) = A_0(x, h)A_0\left(x, h_2^{-1}h_1^{-1}h\right)^{-1}$ $= A_0(x, h)A_0\left(x, h_1^{-1}h\right)^{-1} A_0\left(x, h_1^{-1}h\right) A_0\left(x, h_2^{-1}h_1^{-1}h\right)^{-1} = a\left(p(x)h^{-1}, h_1\right) a\left(p(x)h^{-1}h_1, h_2\right)$ a.e. (x, h, h_1, h_2), and hence, a is almost multiplicative. By Proposition A.7 in Fabec [2], one can assume a is strictly multiplicative. Since a is given as a pair of Borel maps, there exist a Borel function $s : Y \to Z$ and a cocycle $\eta : Y \times H \to K$ which by definition satisfies

$$\left(s\left(p(x)h_1^{-1}\right), \eta\left(p(x)h_1^{-1}, h_2\right)\right) =$$

$$= \left(A_1(x, h_1), A_2(x, h_1)\right)\left(A_1(x, h_2^{-1}h_1) \cdot A_2(x, h_2^{-1}h_1), A_2(x, h_2^{-1}h_1)^{-1}\right) \tag{2.1}$$

for a.e. (x, h_1, h_2). So $\eta\left(p(x)h_1^{-1}, h_2\right) = A_2(x, h_1)A_2(x, h_2^{-1}h_1)^{-1}$ a.e. (x, h_1, h_2). Set $L(x, h) = A_2(x, h)^{-1}\eta(p(x)h^{-1}, h)$. Then $L(x, h_2^{-1}h_1) = A_2(x, h_2^{-1}h_1)^{-1}\eta\left(p(x)h_1^{-1}h_2, h_2^{-1}h_1\right) = A_2(x, h_1)^{-1}\eta\left(p(x)h_1^{-1}, h_2\right) \eta\left(p(x)h_1^{-1}h_2, h_2^{-1}h_1\right) = A_2(x, h_1)^{-1}\eta\left(p(x)h_1^{-1}, h_1\right) = L(x, h_1)$ a.e. (x, h_1, h_2). Since G is the only conull mutiplicatively closed subset of a group, $L(x, h_2^{-1}h_1) = L(x, h_1)$ a.e. h_1, for each h_2, a.e. x. Thus, by Lemma A.3 in Fabec [2], there exists a Borel mapping $B : X \to K$ such that $B(x) = L(x, h)^{-1}$ a.e. (x, h). It follows from the definition of $L(x, h)$ that

$$A_2(x, h) = \eta\left(p(x)h^{-1}, h\right) B(x) \quad a.e. \quad (x, h) \tag{2.2}$$

and so

$$\begin{aligned} A_2\left(x \cdot g, h\phi(x, g)\right) &= \eta\left(p(x \cdot g)\phi(x, g)^{-1}h^{-1}, h\phi(x, g)\right) B(x \cdot g) \\ &= \eta\left(p(x)h^{-1}, h\right) \eta(p(x), \phi(x, g)) B(x \cdot g) \\ &= A_2(x, h)B(x)^{-1}\eta(p(x), \phi(x, g)) B(x \cdot g) \end{aligned} \tag{2.3}$$

a.e. (x, h) for each g. On the other hand, $A_2(x \cdot g, h\phi(x, g)) = A_2(x, h)\psi_0(x, h, g) = A_2(x, h)\psi(x, g)$ a.e. (x, h, g) by (a) and (c). Together with (2.3), one obtains

$$\psi(x, g) = B(x)^{-1}\eta(p(x), \phi(x, g)) B(x \cdot g) \quad a.e. \quad (x, g). \tag{2.4}$$

Since $s\left(p(x)h^{-1}\right) = A_1(x, h)$ a.e. (x, h) by (2.1) and $A_1(x, h)A_2(x, h) = q_0(x, h)$ for all (x, h) by (c), then $q(x) = s\left(p(x)h^{-1}\right) A_2(x, h)$ a.e. (x, h) by (b). Combining this with (2.2), one has $q(x) = s\left(p(x)h^{-1}\right) \eta\left(p(x)h^{-1}, h\right) B(x) = s(p(x)) B(x)$ a.e. x.

Let $r : X \to Z$ be the Borel function given by $r(x) = s\,(p(x))$. So $q(x) = r(x)B(x)$ a.e. x and $s\,(p(x))\,\eta\,(p(x), \phi(x,g)) = s\,(p(x)\phi(x,g)) = s\,(p(x \cdot g)) = r(x \cdot g)$. Thus, by (2.4), one obtains $\Big(r(x), B(x)\Big)\Big(q(x), \psi(x,g)\Big) = \Big(s\,(p(x)), \eta\,(p(x), \phi(x,g))\Big)\Big(r(x \cdot g), B(x \cdot g)\Big)$ a.e. (x,g).

Finally, we have to show $a = (s, \eta)$ satisfies (6) of the definition of a homomorphism. Let Z_0 be an invariant analytic null subset of Z. So $\{x : s(p(x)) \in Z_0\}$ is null iff $\{x : q(x)B(x)^{-1} \in Z_0\}$ is null iff $\{x : q(x) \in Z_0 \cdot B(x)\}$ is null iff $\{x : q(x) \in Z_0\}$ is null. Since b is a homomorphism, $(p^{-1} \circ s^{-1})\,(Z_0)$ is null. Since h is a homomorphism and $s^{-1}(Z_0)$ is an invariant analytic subset of Y, $s^{-1}(Z_0)$ is null. Thus, a is a homomorphism and b is cohomologous to $a \circ h$. $\square$

3. SKEW PRODUCT ACTIONS.

In this section, the Theorem is applied to study the relationship between conjugacy of skew product actions and similarity of groupoids. The following proposition is itself interesting and will be useful.

PROPOSITION 3.1 *Let $h = (p, \phi) : X \times G \to Y \times H$ be a homomorphism with Mackey dense range and suppose $J = (q, \psi) : Y \times H \to Y \times H$ is a homomorphism such that $J \circ h \sim h$. Then $J \sim id$ where $id(y, h) = (y, h)$.*

PROOF. Choose a Borel mapping $B = (B_1, B_2) : X \to Y \times H$ satisfying

$$\Big(B_1(x), B_2(x)\Big)\Big(q(p(x)), \psi(p(x), \phi(x,g))\Big) = \Big(p(x), \phi(x,g)\Big)\Big(B_1(x \cdot g), B_2(x \cdot g)\Big) \qquad (3.1)$$

a.e. (x,g). By Fubini's Theorem, equation (3.1) holds a.e. x for a.e. g. But the set of all g for which there exists a conull Borel subset X_g of X, depending on g, so that (3.1) holds on $X_g \times \{g\}$ is a multiplicatively closed subset of G. Since G is the only conull multiplicatively closed subset of G, then (3.1) holds a.e. x for each g.

Consider the mapping $F : X \times_\phi H \times H \to H$ given by $F(x, h_1, h_2) = h_2^{-1} h_1 B_2(x)\psi(p(x), h_1^{-1}h_2)$. Then $F\,(x \cdot g, h_1\phi(x,g), h_2) = h_2^{-1} h_1 \phi(x,g) B_2(x \cdot g)\psi\Big(p(x \cdot g), \phi(x,g)^{-1}h_1^{-1}h_2\Big) = h_2^{-1} h_1 B_2(x)\psi\,(p(x), \phi(x,g))\,\psi\Big(p(x \cdot g), \phi(x,g)^{-1}h_1^{-1}h_2\Big) = h_2^{-1} h_1 B_2(x)\psi(p(x), h_1^{-1}h_2) = F(x, h_1, h_2)$ a.e. x, for each g, h_1, and h_2. Since F is invariant in its first two variables and $(x, h) \to p(x)h^{-1}$ is an ergodic decomposition of $X \times_\phi H$, there exists a Borel mapping $C : Y \times H \to H$ for which

$$C(p(x)h_1^{-1}, h_2) = h_2^{-1} h_1 B_2(x)\psi(p(x), h_1^{-1}h_2) \quad \text{a.e. } (x, h_1, h_2). \qquad (3.2)$$

Define $\eta : X \times H \to H$ by $\eta(x, h_1) = h_1 B_2(x)\psi(p(x), h_1^{-1})$. By using (3.1), $\eta\,(x \cdot g, h_1\phi(x,g)) = h_1\phi(x,g)B_2(x \cdot g)\psi\Big(p(x \cdot g), \phi(x,g)^{-1}h_1^{-1}\Big) = h_1 B_2(x)\psi\,(p(x), \phi(x,g))\,\psi\Big(p(x \cdot g), \phi(x,g)^{-1}h_1^{-1}\Big) = h_1 B_2(x)\psi(p(x), h_1^{-1}) = \eta(x, h_1)$ a.e. x for each g, h_1. Since J has Mackey dense range, $(x, h) \mapsto p(x)h^{-1}$ is an ergodic decomposition of the the skew product G-space $X \times_\phi H$ and thus there exists a Borel mapping $\xi : Y \to H$ such that $\xi(p(x)h_1^{-1}) = h_1 B_2(x)\psi(p(x), h_1^{-1})$ a.e. (x, h_1).

From (3.2), one finds $C(p(x)h_1^{-1}, h_2) = h_2^{-1} h_1 B_2(x)\psi(p(x), h_1^{-1})\psi(p(x)h_1^{-1}, h_2) = h_2^{-1}\xi(p(x)h_1^{-1})\psi(p(x)h_1^{-1}, h_2)$ a.e. (x, h_1, h_2). Also, by the definition of ξ and (3.2), one obtains $\xi(p(x)h_1^{-1}h_2) = h_2^{-1} h_1 B_2(x)\psi(p(x), h_1^{-1}h_2) = C(p(x)h_1^{-1}, h_2)$ a.e. (x, h_1, h_2). Consequently, $h_2\xi(p(x)h_1^{-1}h_2) = \xi\Big(p(x)h_1^{-1}\Big)\,\psi(p(x)h_1^{-1}, h_2)$ a.e. (x, h_1, h_2). Thus,

$$\xi(y)\psi(y, h_2) = h_2\xi(y \cdot h_2) \quad \text{a.e. } (y, h_2). \qquad (3.3)$$

On the other hand, as a consequence of (3.1) and the fact that J is a homomorphism, one finds

$p(x)h_1^{-1}\xi(p(x)h_1^{-1}) = p(x)h_1^{-1}h_1 B_2(x)\psi(p(x), h_1^{-1}) = p(x)B_2(x)\psi(p(x), h_1^{-1}) =$

$B_1(x)B_2(x)\psi(p(x), h_1^{-1}) = q(p(x))\psi(p(x), h_1^{-1}) = q(p(x)h_1^{-1})$ a.e. (x, h_1). Thus, $y \cdot \xi(y) = q(y)$ a.e.

y and by (3.3), $(y, \xi(y))(q(y), \psi(y, h_2)) = (y, h_2)(y \cdot h_2, \xi(y \cdot h_2))$ a.e. (y, h_2). Hence, $J \sim id$.   □

COROLLARY 3.2 *Let $\phi : X \times G \to H$ be a cocycle with Mackey dense range.*
Suppose J is an endomorphism of H such that $J \circ \phi \sim \phi$. Then J is an inner conjugation.

PROPOSITION 3.3 *Suppose $h = (p, \phi) : X \times G \to Y \times H$ and*
$h' = (p', \phi') : X \times G \to Y' \times H'$ are homomorphisms with Mackey dense range. If the skew product
G spaces $X \times_\phi H$ and $X \times_{\phi'} H'$ are conjugate extensions of X then $Y \times H$ and $Y' \times H'$ are similar
groupoids.

PROOF.   If $X \times_\phi H$ and $X \times_{\phi'} H'$ are conjugate extensions of X then there is a Borel

mapping $\xi : X \times_\phi H \to H'$ satisfying $\xi(x \cdot g, h\phi(x, g)) = \xi(x, h) \cdot \phi'(x, g)$ a.e. (x, h) for each g.

Let $\alpha : X \times_\phi H \to Y'$ be the Borel mapping given by $\alpha(x, h) = p'(x) \cdot \xi(x, h)^{-1}$. Then α is G-invariant

since $\alpha((x, h) \cdot g) = p'(x \cdot g)\xi(x \cdot g, h\phi(x, g))^{-1} = p'(x \cdot g)\phi'(x, g)^{-1}\xi(x, h)^{-1} = p'(x)\xi(x, h)^{-1} = \alpha(x, h)$

a.e. (x, h) for all g. Consequently, $(\alpha(x, h), \xi(x, h))(p'(x), \phi'(x, g)) = (\alpha((x, h) \cdot g), \xi((x, h) \cdot g))$

a.e. (x, h) for all g and so $h'|X \times_\phi H \times G$ is trivial. Similarly, $h|X \times_{\phi'} H' \times G$ is trivial.

By the Theorem, there must exist homomorphisms J and J' such that $J' \circ h' \sim h$ and $J \circ h \sim h'$.

Thus, $J' \circ J \circ h \sim h$ and $J' \circ J \sim id$ follows by Proposition 3.1 . Similarly, $J \circ J' \sim id$.

This shows $Y \times H$ and $Y' \times H'$ are similar groupoids.   □

In Proposition 3.3, if Y and Y' are points one obtains an interesting corollary that extends

Theorem 1.3 in Fabec [1]. Also, if X is a point, the problem stated in the introduction is answered.

COROLLARY 3.4 *Suppose $\phi : X \times G \to H$ and $\psi : X \times G \to K$ are cocycles with Mackey*
dense range. Then the following are equivalent

(a) *ϕ restricted to $X \times_\psi K \times G$ and ψ restricted to $X \times_\phi H \times G$ are both trivial*

(b) *there is an isomorphism J from H onto K such that $J \circ \phi$ is cohomologous to ψ*

(c) *the skew product actions of G on $X \times_\phi H$ and on $X \times_\psi K$ define*
   conjugate extensions of X.

PROOF. If (a) holds then by the Theorem, there are homomorphisms $J_1 : H \to K$ and

$J_2 : K \to H$ such that $J_1 \circ \phi \sim \psi$ and $J_2 \circ \psi \sim \phi$. So $J_1 \circ J_2 \circ \psi \sim \psi$ and $J_2 \circ J_1 \circ \phi \sim \phi$.

By Corollary 3.2, J_1 must be an isomorphism and (b) is proved.

If (b) holds then there is a Borel map $A : X \to K$ satisfying $J(\phi(x, g)) = A(x)\psi(x, g)A(x \cdot g)^{-1}$

a.e. x for each g. Let $\xi : X \times_\phi H \to X \times_\psi K$ be the function given by $\xi(x, h) = (x, J(h)A(x))$. Then

$\xi(x \cdot g, h\phi(x, g)) = (x \cdot g, J(h)J\phi(x, g)A(x \cdot g)) = (x \cdot g, J(h)A(x)\psi(x, g)) = \xi(x, h) \cdot g$ a.e. (x, h)

for each g. Note, ξ is a Borel isomorphism whose inverse is given by $\eta(x, k) = (x, J^{-1}(kA(x)^{-1}))$.

Moreover, a direct calculation shows $\xi_*(\mu \times m_H) \sim \mu \times m_K$ where m_H and m_K are Haar measures

on H and K, respectively. Thus, $X \times_\phi H$ and $X \times_\psi K$ are conjugate extensions of X.

Finally, in the proof of Proposition 3.3 it is shown (c) implies (a).   □

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