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CHARACTERIZATIONS OF MULTINOMIAL DISTRIBUTIONS BASED ON CONDITIONAL DISTRIBUTIONS

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ABSTRACT. Several characterizations of the joint multinomial distribution of two discrete random vectors are derived assuming conditional multinomial distributions.

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1. INTRODUCTION.

Suppose that the distributions of $X \mid Y = y$ and $Y \mid X = x$ both are given for every real values $x, y,$ then the joint distribution of X and Y is tried to be reconstructed. Brucker [4], then Fraser and Streit [6], Castillo and Galambos [5] characterized a bivariate normal distribution given that $X \mid Y = y$ and $Y \mid X = x$ both have normal distribution under some given conditions. Bischoff and Fieger [3], then Hamedani [8] gave characterizations of multivariate normal distribution, Dinh and Nguyen [9] gave a characterization of matrix variate normal distribution. In the case X and Y are identically distributed, and $Y \mid X = x$ has a normal distribution of mean $ax + b$ and variance σ^2, Ahsanullah [1] showed that $|a| < 1$ and X and Y have a joint bivariate normal distribution. In his paper Ahsanullah also proposed a conjecture for a multidimensional version of his result. Hamedani [7], then Arnold and Pourahmadi [2] gave counterexamples to this conjecture, and they also gave different characterizations for multivariate normal distribution based on conditional multivariate normality. Nguyen [10] gave a characterization for matrix variate normal distribution having identically distributed row vectors. In this note we consider the problem of characterization of multinomial distribution based on conditional multinomial distributions. In Section 2 a characterization of multinomial distribution is given based on two conditional multinomial distributions. In Section 3 a characterization of the joint multinomial of two identically distributed random vectors based on one conditional multinomial distribution is given. A conjecture similar to a conjecture proposed by Ahsanullah [1] is also raised and answered by a counterexample. Some supplementary conditions are added to this conjecture making it to be sufficient to characterize a multinomial distribution.

2. THE FIRST CHARACTERIZATION.

In this section we go to characterize a joint multinomial distribution of two discrete random vectors based on the conditional multinomial distribution of one vector given the other vector. A discrete random vector $X = (X_1,...,X_k)'$ is defined to have a multinomial $(n,p_1,...,p_k)$ distribution if its density is given by

$$p(x_1,...,x_k) = \frac{n!}{\prod_{i=1}^{k} x_i! \left(n - \sum_{i=1}^{k} x_i\right)!} \prod_{i=1}^{k} p_i^{x_i} (1 - P)^{n - \sum_{i=1}^{k} x_i}, \tag{2.1}$$

where $p_i > 0$, $i = 1,...,k$, $\sum_{i=1}^{k} p_i < 1$, $0 \le \sum_{i=1}^{k} x_i \le n$, x_i nonnegative integer, $i = 1,...,k$, $P = \sum_{i=1}^{k} p_i$. Its moment generating function (m.g.f.) is given by

$$M(s_1,...,s_k) = \left(\sum_{i=1}^{k} p_i e^{s_i} + 1 - P\right)^n, \tag{2.2}$$

for all real numbers $s_1,...,s_k$. It is clear that M is a continuous function in $(s_1,...,s_k)$.

THEOREM 2.1. Let $X = (X_1,...,X_m)'$ and $Y = (Y_1,...,Y_k)'$ be two discrete random vectors, whose components taking values on the set of nonnegative integers. Suppose $X \mid Y = y = (y_1,...,y_k)$ has a multinomial $\left(n - \sum_{j=1}^{k} y_j, p_1,...,p_m\right)$ distribution for all nonnegative integers $y_1,...,y_k$, $\sum_{j=1}^{k} y_j \le n$, and $Y \mid X = x = (x_1,...,x_m)$ has a multinomial $\left(n - \sum_{i=1}^{m} x_i, q_1,...,q_k\right)$ distribution for all nonnegative integers $x_1,...,x_m$, $\sum_{i=1}^{m} x_i \le n$. Then X and Y have a joint multinomial distribution.

PROOF. From the conditional distribution of $X \mid Y = y$,

$$P_{X,Y}(x,y) = \frac{\left(n - \sum_{j=1}^{k} y_j\right)!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j\right)!} \prod_{i=1}^{k} p_i^{x_i} (1 - P)^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j} P_Y(y), \tag{2.3}$$

and from the conditional distribution of $Y \mid X = x$,

$$p_{X,Y}(x,y) = \frac{\left(n - \sum_{i=1}^{m} x_i\right)!}{\prod_{j=1}^{k} y_j! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j\right)!} \prod_{j=1}^{k} q_j^{y_j} (1-Q)^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j} p_X(x), \tag{2.4}$$

where $Q = \sum_{i=1}^{k} q_i$. Equating (2.3) and (2.4), then simplifying,

$$\frac{p_X(x) \prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i\right)!}{\prod_{i=1}^{m} p_i^{x_i} (1-Q)^{\sum_{i=1}^{m} x_i} (1-P)^{n - \sum_{i=1}^{m} x_i}} = \frac{p_Y(y) \prod_{j=1}^{k} y_j! \left(n - \sum_{j=1}^{k} y_j\right)!}{\prod_{j=1}^{k} q_j^{y_j} (1-P)^{\sum_{j=1}^{k} y_j} (1-Q)^{n - \sum_{j=1}^{k} y_j}}, \tag{2.5}$$

for all nonnegative integers $x_1,\dots,x_m,\ y_1,\dots,y_k,\ \sum_{i=1}^{m} x_i + \sum_{j=1}^{k} y_j \le n$. The left side of (2.5) depends only on $x_1,\dots,x_m,$ meanwhile the right side of (2.5) depends only on $y_1,\dots,y_k,$ therefore their common value K does not depend on x and y. Hence,

$$p_X(x) = \frac{K \prod_{i=1}^{m} p_i^{x_i} (1-Q)^{\sum_{i=1}^{m} x_i} (1-P)^{n - \sum_{i=1}^{m} x_i}}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i\right)!}, \tag{2.6}$$

for all nonnegative integers $x_1,\dots,x_m,\ \sum_{i=1}^{m} x_i \le n,$ and

$$p_Y(y) = \frac{K \prod_{j=1}^{k} q_j^{y_j} (1-P)^{\sum_{j=1}^{k} y_j} (1-Q)^{n - \sum_{j=1}^{k} y_j}}{\prod_{j=1}^{k} y_j! \left(n - \sum_{j=1}^{k} y_j\right)!}, \tag{2.7}$$

for all nonnegative integers $y_1,\dots,y_k,\ \sum_{j=1}^{k} y_j \le n.$ To find $K,$ sum up (2.6) on all possible values of x or sum up (2.7) on all possible values of y, and using the fact that they are density functions,

$$\sum_{x} p_X(x) = \frac{K}{n!} \sum_{x_1+\dots+x_m \le n} \frac{n!}{\left(\prod_{i=1}^{k} x_i!\right)\left(n - \sum_{i=1}^{k} x_i\right)!} \prod_{i=1}^{m} [p_i(1-Q)]^{x_i} (1-P)^{n - \sum_{i=1}^{k} x_i}$$

$$= \frac{K}{n!}\left[\sum_{i=1}^{m} p_i\left(1-Q\right)+1-P\right]^n$$

$$= \frac{K}{n!}\,[1-PQ]^n = 1.$$

Hence,

$$K = \frac{n!}{[1-PQ]^n}. \tag{2.8}$$

Substitute K in (2.6),

$$P_X(x) = \frac{n!}{\prod_{i=1}^{m} x_i!\left(n-\sum_{i=1}^{m} x_i\right)!}\prod_{i=1}^{m}\left[\frac{p_i(1-Q)}{1-PQ}\right]^{x_i}\left[\frac{1-P}{1-P(1-Q)}\right]^{n-\sum_{i=1}^{m} x_i} \tag{2.9}$$

and then substitute (2.9) in (2.4),

$$P_{X,Y}(x,y) = \frac{n!}{\prod_{i=1}^{m} x_i!\prod_{j=1}^{k} y_j!\left(n-\sum_{i=1}^{m} x_i-\sum_{j=1}^{k} y_j\right)!}\prod_{i=1}^{m}\left[\frac{p_i(1-Q)}{1-PQ}\right]^{x_i}$$

$$\cdot\prod_{j=1}^{k}\left[\frac{q_j(1-P)}{1-PQ}\right]^{y_j}\left[\frac{(1-P)(1-Q)}{1-PQ}\right]^{n-\sum_{i=1}^{m} x_i-\sum_{j=1}^{k} y_j}, \tag{2.10}$$

for all nonnegative integers $x_1,\dots,x_m$, $y_1,\dots,y_k$, $\sum_{i=1}^{m} x_i+\sum_{j=1}^{k} y_j \le n$. Then X and Y have a joint multinomial distribution.

3. THE SECOND CHARACTERIZATION.

A characterization of the joint multinomial distribution of two identically distributed random vectors in this section is based on only one conditional multinomial distribution. It is trivial that if X and Y are random vectors whose components have values on the set of nonnegative integers and if Y has a multinomial $(n,q_1,\dots,q_k)$ distribution and $X\mid Y=y=(y_1,\dots,y_k)$ has a multinomial $\left(n-\sum_{j=1}^{k} y_j,p_2,\dots,p_m\right)$ distribution for all $y_1,\dots,y_k$, $\sum_{j=1}^{k} y_j \le n$, then the joint distribution of X and Y is multinomial with density given by

$$P_{X,Y}(x,y) = \frac{n!}{\prod_{i=1}^{m} x_i!\prod_{j=1}^{k} y_j!\left(n-\sum_{i=1}^{m} x_i-\sum_{j=1}^{k} y_j\right)}\left[\prod_{i=1}^{m}(p_i(1-Q))^{x_i}\right]\left[\prod_{j=1}^{k} q_j^{y_j}\right]$$

$$\cdot\,[(1-P)(1-Q)]^{n-\sum_{i=1}^{m} x_i-\sum_{j=1}^{k} y_j}, \tag{3.1}$$

for all nonnegative integers $x_1,\dots,x_m$, $y_1,\dots,y_n$ such that $\sum_{i=1}^{m} x_i+\sum_{j=1}^{k} y_j \le n$.

If X and Y are identically distributed and if their joint distribution is a multinomial $(n,p_1,...,p_m,p_1,...,p_m)$ distribution, then $\sum_{i=1}^{m} p_i < \frac{1}{2}$ and the marginal distribution of X and Y is a multinomial $(n,p_1,...,p_m)$ distribution. The conditional distribution of X given $Y = y = (y_1,...,y_m)$ is a multinomial distribution with density given by

$$P_{X\mid Y=y}(x\mid y) = \frac{\left(n - \sum_{j=1}^{m} y_j\right)!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j\right)!} \left[\prod_{i=1}^{m} \left(\frac{p_i}{1-P}\right)^{x_i}\right] \left[\frac{1-2P}{1-P}\right]^{n - \sum_{j=1}^{m} y_j - \sum_{i=1}^{m} x_i}, \tag{3.2}$$

for all nonnegative integers $x_1,...,x_m$ such that $\sum_{i=1}^{m} x_i \le n - \sum_{j=1}^{m} y_j$.

THEOREM 3.1. Let $X = (X_1,...,X_m)'$ and $Y = (Y_1,...,Y_m)'$ be two identically distributed discrete random vectors whose components have values on the set of nonnegative integers. Suppose $X \mid Y = y = (y_1,...,y_m)'$ has a multinomial $\left(n - \sum_{j=1}^{m} y_j, p_1,...,p_k\right)$ distribution for all nonnegative integers $y_1,...,y_m$, $\sum_{j=1}^{m} y_j \le n$, then X and Y have a joint multinomial distribution.

PROOF. Since $X \mid Y = y = (y_1,...,y_m)$ has a multinomial $\left(n - \sum_{j=1}^{m} y_j, p_1,...,p_k\right)$ distribution, the joint distribution of X and Y is given by

$$P_{X,Y}(x,y) = \frac{\left(n - \sum_{j=1}^{m} y_j\right)!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j\right)!} \left(\prod_{i=1}^{m} p_i^{x_i}\right)(1-P)^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j} P(y),$$

where P is the marginal density function of X and Y, $x_1,...,x_m$, $y_1,...,y_m$ are nonnegative integers, $\sum_{i=1}^{m} x_i + \sum_{j=1}^{m} y_j \le n$. Hence, the range of each component is from 0 to n, and the m.g.f. $M(s_1,...,s_m)$ of X and Y is a continuous function in R^m, $M > 0$ for all $(s_1,...,s_m)$ of R^m. The joint m.g.f. of X and Y is given by

$$M_{X,Y}(s,t) = E[e^{s'X+t'Y}] = E[e^{t'Y} E[e^{s'X} \mid Y]]$$

$$= E\left[e^{t'Y}\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^{n - \sum_{i=1}^{m} Y_i}\right]$$

$$= \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^{n} E\left[e^{\left(t - 1\,\ell n\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)\right)' Y}\right]$$

$$= \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^n M\left(t - \mathbf{1}\ \ell n\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)\right), \tag{3.3}$$

where $\mathbf{1} = (1,...,1)'$ of R^m, for all $s = (s_1,...,s_m)'$, $t = (t_1,...,t_m)'$ of R^m. Hence,

$$M(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^n M\left(-\mathbf{1}\ \ell n\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)\right), \tag{3.4}$$

for all $s \in R^m$.

Let M_1 and M_2 be two m.g.f. solutions of (3.4). Set $\frac{M_1(s)}{M_2(s)} = h(s)$. Then $h(s)$ is continuous on R^m and $h(0) = 1$. From

$$M_1(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i\right)^n M\left(-\mathbf{1}\ \ell n\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i\right)\right), \tag{3.5}$$

and

$$M_2(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i\right)^n M_2\left(-\mathbf{1}\ \ell n\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i\right)\right), \tag{3.6}$$

$$h(s) = h\left(\mathbf{1}\ \ell n\left(\frac{1}{\sum_{i=1}^{m} p_i e^{s_i} + 1 - P}\right)\right), \tag{3.7}$$

for all s of R^m.

Iterating the right side of (3.7) using (3.7) itself,

$$h(s) = h(\mathbf{1}\ \ell n\ A_1) = h(\mathbf{1}\ \ell n\ A_2) = ... = h(\mathbf{1}\ \ell n\ A_\ell) = ..., \tag{3.8}$$

where the sequence $\{A_\ell\}$, $\ell = 1,2,3,...,$ is defined recursively by

$$A_1 = \frac{1}{\sum_{i=1}^{m} p_i e^{s_i} + 1 - P}, \quad A_\ell = \frac{1}{PA_{\ell-1} + 1 - P} \text{ for all } \ell = 2,3,....$$

It is trivial to show that if $A_1 \ge 1$, then $A_1 \ge A_3 \ge A_5 \ge ... \ge 1$ and $A_2 \le A_4 \le ... \le 1$, and if $A_1 \le 1$, $A_1 \le A_3 \le A_5 \le ... \le 1$ and $A_2 \ge A_4 \ge ... \ge 1$. The two sequences $\{A_{2\ell-1}\}$ and $\{A_{2\ell}\}$ are convergent and both of them converge to 1. Hence, by the fact that the function h is continuous on R^m and the function ℓn is continuous on $(0,\infty)$,

$$h(s) = h(\mathbf{1}\ \ell n(\lim_{n\to\infty} A_n)) = h(\mathbf{1}.0) = h(\mathbf{0}) = 1.$$

Therefore the equation (3.4) has a unique solution. By (3.2) this solution is the m.g.f. of a multinomial distribution, and Y has a multinomial distribution. By the result (3.1), the joint distribution of X and Y is multinomial.

The following question will be studied regarding multinomial distribution. If $X_1,...,X_k$ are identically distributed discrete random variables having values on the set of nonnegative integers, where $k \ge 3$ and if $X_1 \mid X_2 = x_2,...,X_k = x_k$ has a binomial $\left(n - \sum_{i=2}^{k} x_i, p\right)$ distribution $0 < p < 1$ for all $x_2,...,x_k$ nonnegative integers, $\sum_{i=2}^{k} x_i \le n$, then does it imply that $X_1,....X_k$ have a joint multinomial distribution? The answer for this question is given by the following counterexample.

EXAMPLE 3.1. Let X_1, X_2, X_3 be three discrete random variables having a joint density function

$$P(0,0,0) = P(2,0,0) = P(0,2,0) = P(0,0,2) = 1/9$$
$$P(1,0,0) = P(0,1,1) = 2/9$$
$$P(0,0,1) = P(1,0,1) = P(0,1,0) = P(1,1,0) = 1/36.$$

Then it is trivial that X_1, X_2, X_3 are identically distributed with density function

$$P(0) = 11/18,\ \ P(1) = 5/18,\ \ P(2) = 2/18,$$

and this is not a binomial $(2,p)$ distribution, since there does not exist any p for a binomial (2,p) distribution to fit to this distribution. This fact shows that the joint distribution of X_1, X_2, X_3 is not a trinomial distribution, meanwhile, it is easy to check that $X_1 \mid X_2 = x_2, X_3 = x_3$ has a binomial $\left(2 - x_2 - x_3, \frac{1}{2}\right)$ distribution for all $x_2, x_3, x_2 + x_3 \leq 2$.

In addition to the given conditions of the above question, if $X_1,\ldots,X_k$ are supposed to have an exchangeable distribution in $x_1,\ldots,x_k$, we go to show that the joint distribution of $X_1,\ldots,X_k$ is a multinomial distribution.

If $k = 2$, since X_1 and X_2 are identically distributed and $X_1 \mid X_2 = x_2$ has a binomial $(n - x_2, p)$ distribution, by Theorem 3.1, X_1 and X_2 have a joint trinomial distribution. For $k > 2$, the proof follows by mathematical induction. For some positive integer $k \geq 2$, suppose that for any $m = 2,\ldots,k$, if $X_1 \mid X_2 = x_2,\ldots,X_m = x_m$ has a binomial $\left(n - \sum_{i=2}^{m} x_i, p\right)$ distribution, then the joint distribution of $(X_1,\ldots,X_m)$ is a multinomial distribution, we go to show that it is also true if the number of random variables is $k + 1$.

From $X_1 \mid X_2 = x_2,\ldots,X_k = x_k, X_{k+1} = x_{k+1}$ has a binomial $\left(n - \sum_{i=2}^{k+1} x_i, p\right)$ distribution, then

$$P(x_1 \mid x_2,\ldots,x_k,x_{k+1}) = \frac{P(X_1 = x_1, X_2 = x_2,\ldots,X_k = x_k, X_{k+1} = x_{k+1})}{P(X_2 = x_2,\ldots,X_{k+1} = x_{k+1})}$$
$$= \frac{P(X_1 = x_1,\ldots,X_k = x_k, X_{k+1} - x_{k+1})/P(X_3 = x_3,\ldots,X_{k+1} = x_{k+1})}{P(X_2 = x_2,\ldots,X_k = x_k, X_{k+1} = x_{k+1})/P(X_3 = x_3,\ldots,X_{k+1} = x_{k+1})}.$$

Set $Y_1 = X_1 \mid X_3 = x_3,\ldots,X_{k+1} = x_{k+1}$, and $Y_2 = X_2 \mid X_3 = x_3,\ldots,X_{k+1} = x_{k+1}$. Then $P(Y_1 = x_1 \mid Y_2 = x_2) = P(x_1 \mid x_2,\ldots,x_{k+1})$ and $Y_1 \mid Y_2 = x_2$ has a binomial $\left(n - \sum_{i=2}^{k} x_i, p\right)$ distribution, and Y_1 and Y_2 are identically distributed since $X_1,\ldots,X_{k+1}$ have a joint exchangeable distribution in $x_1,\ldots,x_{k+1}$. By Theorem 3.1, Y_1 and Y_2 have a binomial $\left(n - \sum_{i=3}^{k+1} x_i, q\right)$ distribution for some $0 < q < 1$. From $Y_2 = X_2 \mid X_3 = x_3,\ldots,X_{k+1} = x_{k+1}$ has a binomial $\left(n - \sum_{i=3}^{k+1} x_i, q\right)$ distribution and by induction hypothesis, $X_2,\ldots,X_{k+1}$ have a joint multinomial distribution. Hence, $X_1,\ldots,X_{k+1}$ have a joint multinomial distribution. By mathematical induction principle, this result of joint multinomial of $X_1,\ldots,X_k$ is true for all $k \geq 2$. Therefore the following result is proved.

THEOREM 3.2. Let $X_1,\ldots,X_k$ be identically distributed random variables having values on the set of nonnegative integers. Suppose that their joint density is exchangeable in $x_1,\dots,x_k$, and $X_1 \mid X_2 = x_2,\dots,X_k = x_k$ has a binomial $\left(n - \sum_{i=2}^{k} x_i, p\right)$ distribution, then $X_1,\dots,X_k$ have a joint multinomial distribution.

Theorem 3.2 can be generalized to Theorem 3.3. below in the case of identically distributed random vectors by using the result of Theorem 3.1 and a similar proof for Theorem 3.2.

THEOREM 3.3. Let $X_1,\dots,X_k$ be identically distributed $m \times 1$ random vectors whose components have values on the set of nonnegative integers. Suppose that the joint density of $X_1,\dots,X_k$ is exchangeable in $x_1,\dots,x_k$ and $X_1 \mid X_2 = x_2,\dots,X_k = x_k$ has a multinomial $\left(n - \sum_{i=2}^{k} \sum_{j=1}^{m} x_{ij}, p_1\right)$ distribution, where $x_i = (x_{i,1},\dots,x_{i,m})'$, $i = 2,\dots,k$, $p_1 = (p_{1,1},\dots,p_{1,m})'$, then $X_1,\dots,X_k$ have a joint multinomial distribution.

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