ANISOTROPIC NONLINEAR DIFFUSION WITH ABSORPTION: EXISTENCE AND EXTINCTION

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ABSTRACT. The authors prove that the nonlinear parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \varphi_{ij}(u) - f(u)$$

with homogeneous Dirichlet boundary conditions and a nonnegative initial condition has a nonnegative generalized solution u. They also give necessary and sufficient conditions on the constitutive functions φ_{ij} and f which ensure the existence of a time $t_0 > 0$ for which u vanishes for all $t \geq t_0$.

KEY WORDS AND PHRASES. Anisotropic nonlinear diffusion, finite extinction time, generalized solution.
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1 INTRODUCTION.

We consider the nonlinear initial-boundary value problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \varphi_{ij}(u) - f(u) && \text{in } Q_\infty \equiv \Omega \times (0, \infty) \\
\Phi(u) &= 0 && \text{on } \partial\Omega \times [0, \infty) \\
u(x,0) &= u_0(x) \geq 0 && \text{on } \overline{\Omega}
\end{aligned}
\tag{1.1}
$$

in which $\Phi = (\varphi_{ij})$ is an $n \times n$ symmetric matrix and the domain $\Omega \subseteq \mathbf{R}^n$ is bounded. We assume that the functions φ_{ij} and the nonnegative, nondecreasing function f are in $C([0, \infty)) \cap C^1((0, \infty))$ and satisfy $f(0) = \varphi_{ij}(0) = 0$. Furthermore, we assume that the matrix $\Phi'(s) = \frac{d}{ds}\Phi(s)$ is positive definite on $(0, \infty)$ and there exists a positive function $\lambda \in C((0, \infty))$ such that

$$\lambda(s)|\xi|^2 \leq \sum_{i,j=1}^{n} \varphi_{ij}'(s)\xi_i\xi_j \leq \Lambda'(s)|\xi|^2 \tag{1.2}$$

for $s > 0$, $\xi \in \mathbf{R}^n$ where $\Lambda(s) \equiv tr(\Phi(s))$, the trace of the matrix $\Phi(s)$, so that $\Lambda'(s) = tr(\Phi'(s))$. The main purpose of this article is to prove that the initial- boundary value problem (1.1) has a solution and to give necessary and sufficient conditions on the constitutive functions φ_{ij} and f to

ensure the existence of a finite extinction time (i.e., a time $t_0 > 0$ such that any solution u satisfies $u(x,t) = 0$ for all $(x,t) \in \overline{\Omega} \times [t_0, \infty)$).

Such problems have been considered for over two decades for the isotropic problem in which the matrix Φ is a scalar multiple of the identity matrix (i.e., $\Phi = \varphi I$) . (See [1],[2],[3],[4],[5],[6],[7] and their references.) The main thrust of these studies has been the determination of conditions on the functions φ and f which ensure the existence or nonexistence of a finite extinction time. For a single equation (see [2] for systems), the principal results can be summarized by those contained in [6] which contains all of the other results when homogeneous Dirichlet boundary conditions are specified. There it is shown that a sufficient condition to ensure the existence of a finite extinction time is that ($\varepsilon > 0$)

$$\int_0^\varepsilon \frac{ds}{\varphi(s)} < \infty \quad \text{or} \quad \int_0^\varepsilon \frac{ds}{f(s)} < \infty$$

holds. Conversely, the authors prove that if it is known that a solution has a finite extinction time, then

$$\int_0^\varepsilon \frac{ds}{\varphi(s) + f(s)} < \infty$$

must hold.

For anisotropic diffusion, such results are virtually nonexistent. Indeed the authors know of no results in which either existence of a solution or existence of a finite extinction time for problem (1.1) has been treated. Other problems for special cases of (1.1) have been studied. For example, Kersner [8] demonstrates some properties of the solution to the Cauchy problem for the equation $u_t = (u^m)_{xx} + (u^p)_{yy}$.

It is well-known that classical solutions to (1.1) do not, in general, exist and hence weak solutions must be considered. We prove the existence of such a solution. Our definition of a weak, or generalized, solution is quite similar to that of Benilan et al [9]. In addition to existence of a solution, we give necessary and sufficient conditions for the existence of a finite extinction time. In particular, we show that if

$$\int_0^\varepsilon \frac{ds}{f(s)} < \infty \tag{1.3}$$

or if there exists some $m \in \{1, 2, ..., n\}$ such that

$$\int_0^\varepsilon \frac{ds}{\varphi_{mm}(s)} < \infty, \tag{1.4}$$

then any solution to our problem has a finite extinction time. On the other hand, we show that a necessary condition for the existence of a finite extinction time is

$$\int_0^\varepsilon \frac{ds}{f(s) + tr(\Phi(s))} < \infty . \tag{1.5}$$

These results contain all of those for the isotropic problem for which a bounded domain and Dirichlet boundary conditions are specified.

2   STATEMENT OF MAIN RESULTS.

Since problem (1.1) does not, in general, have a classical solution even in the isotropic case (see [4]), it is necessary to work with a weaker formulation of a solution. In the following, we give the definition of a generalized solution. It should be noted that our definition is similar to that of Benilan et al (see p. 218 of [9]).

DEFINITION. The sequence of problems (Φ_k is the matrix (φ_{ij}^k))

$$\begin{aligned}
\frac{\partial v_k}{\partial t} &= \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \varphi_{ij}^k(v_k) - f_k(v_k) && \text{in } Q_\infty \\
\Phi_k(v_k) &= 0 && \text{on } \partial\Omega \times [0,\infty) \\
v_k(x,0) &= u_{0,k}(x) \geq 0 && \text{on } \overline{\Omega}
\end{aligned} \tag{2.1}$$

is called an *approximating sequence of problems* for the problem if the sequences of $C^2([0,\infty))$ functions $\{\varphi_{ij}^k\}_{k=1}^{\infty}$ and $C^1([0,\infty))$ functions $\{f_k\}_{k=1}^{\infty}$ converge to φ_{ij}, and f, respectively, uniformly on compact subsets of $[0,\infty)$ and the sequence $\{u_{0,k}\}$ converges to u_0 in $L_\infty(\Omega)$.

DEFINITION. A function $u \in L_\infty(Q_\infty)$ is a generalized solution of the initial-boundary value problem (1.1) if for each $T > 0$, the function u can be written as the weak $L_2(Q_T)$ limit of a sequence of classical solutions to an approximating sequence of problems for the problem (1.1).

The main purpose of this article is to prove the following three theorems.

THEOREM 1. (*Existence*) Suppose u_0 is a nonnegative function continuous on $\overline{\Omega}$. Then the initial-boundary value problem (1.1) has a nonnegative generalized solution u and $\|u(\cdot,t)\|_{\infty,\Omega} \leq \|u_0\|_{\infty,\Omega}$ for all $t \geq 0$.

We are also interested in establishing both necessary and sufficient conditions on the matrix Φ and the function f which ensure the existence of a finite extinction time for these weak solutions. The following two theorems are natural extensions of similar results for the isotropic case [6].

THEOREM 2. (*Necessity*) Let u_0 be a nontrivial nonnegative function continuous on $\overline{\Omega}$. If any nonnegative, bounded, generalized solution to the problem (1.1) has a finite extinction time, then (1.5) holds.

THEOREM 3. (*Sufficiency*) Let u_0 be a nonnegative function continuous on $\overline{\Omega}$. If either (1.3) holds or (1.4) holds for some $m \in \{1, 2, ..., n\}$, then any nonnegative, bounded, generalized solution of (1.1) has a finite extinction time.

It should be noted that there is a gap between necessity and sufficiency which the authors have been unable to fill. That is, we show that a sufficienct condition to ensure the existence of a finite extinction time is to have either strong absorption (i.e., f satisfies (1.3)) or fast diffusion in (at least) one direction (i.e., for some m, (1.4) holds). However, necessity requires only that some combination of the absorption and diffusion be "fast" (i.e., (1.5) holds). Thus, suppose that one has absorption and diffusion terms for which the integrals in (1.3) and (1.4) are infinite for all $m = 1, ..., n$ while inequality (1.5) holds. (Such functions do exist.) Does any solution to the resulting problem (1.1) have a finite extinction time? The authors have been unable to answer this question.

3 PROOF OF EXISTENCE.

PROOF OF THEOREM 1. Let $\{u_{0,k}\}$ be a sequence of $C_0^\infty(\Omega)$ functions converging uniformly on $\overline{\Omega}$ to u_0. Furthermore, we choose $u_{0,k}$ so that $\|u_{0,k}\|_{\infty,\Omega} \leq \|u_0\|_{\infty,\Omega} \equiv M$. Let $\{\varphi_{ij}^k\}_{k=1}^{\infty}$, $1 \leq i, j \leq n$, and $\{f_k\}$ be sequences of functions such that $\varphi_{ij}^k \in C^2([0,\infty))$, the matrix $\Phi_k'(s)$ (i.e., the matrix $(\frac{d}{ds}\varphi_{ij}^k(s))$) is positive definite on compact subsets of $[0,\infty)$, $f_k \in C^1([0,\infty))$ has a positive derivative everywhere and $\varphi_{ij}^k \to \varphi_{ij}$ and $f_k \to f$ uniformly on compact subsets of $[0,\infty)$ as $k \to \infty$.

The existence of the sequences $\{\Phi_k\}$ and $\{f_k\}$ are fairly easy to demonstrate. To produce the sequence $\{\Phi_k\}$, we set $\Psi_k(s) = \Phi'(1/k)$ for $s \leq 1/k$ and let $\Psi_k(s) = \Phi'(s)$ for $s \geq 1/k$. Let $\Phi_k(s) = \int_0^s \Psi_k(\sigma)d\sigma$. Clearly, the sequence $\{\Phi_k\}$ has the required properties (except for smoothness). In

particular, we note that

$$\lambda(\frac{1}{k})|\xi|^2 \le \xi^t \Phi_k'(s)\xi \le \Lambda'(\frac{1}{k})|\xi|^2 \tag{3.1}$$

for all $s \in [0, M]$ and $\xi \in \mathbf{R}^n$. An appropriate smooth approximation to this Φ_k will provide the required sequence. The sequence $\{f_k\}$ can be constructed similarly.

From Ladyzenskaja et al [10] (see p. 157) we know that the sequence of initial-boundary value problems given by (2.1) with Φ_k and f_k as just defined has a unique nonnegative classical solution on Q_T if $0 < T < \infty$.

We now establish that a classical solution to (2.1) exists on Q_∞. This is easy to do as follows: Let w_1 be a classical solution on Q_1 and define $u_1 = w_1$ on Q_1 and $u_1 = 0$ elsewhere. Similarly, let w_2 be a classical solution on Q_2 and define $u_2 = w_2$ on Q_2 and $u_2 = 0$ otherwise. Continue this process to produce a sequence $\{u_m\}$ for which $u_m = w_m$ on Q_m and $u_m = 0$ otherwise. Since the classical solution on Q_m is unique, we get $u_m = u_{m+1}$ on Q_m. Furthermore, we get $0 \le u_m \le u_{m+1} \le \|u_{0,k}\|_{\infty,\Omega} \le M$ on Q_∞ for all m.

Thus the sequence $\{u_m\}$ converges pointwise on Q_∞. Let $v = \lim_{m\to\infty} u_m$ and notice that $v = u_m$ on Q_m for all m. Thus, given any point in Q_∞, m may be chosen sufficiently large that $v = u_m$ at that point and thus v is a (not necessarily unique) nonnegative classical solution to (2.1) on Q_∞.

Since this classical solution obviously depends on k, we now let it be denoted by v_k and observe that $\|v_k(\cdot, t)\|_{\infty,\Omega} \le \|u_0\|_{\infty,\Omega}$ for all k and $t \in [0, \infty)$. To obtain a generalized solution to our problem, we let $T \in (0, \infty)$ and note that we need only prove that the sequence $\{v_k\}$ has a subsequence $\{v_{k_m}\}$ which converges weakly in $L_2(Q_T)$. Then, after reindexing the sequences $\{\varphi_{ij}^k\}$, $\{f_k\}$ and $\{u_{0,k}\}$ replacing k with k_m, we obtain the desired result. Thus, we let $T \in (0, \infty)$ and prove that the sequence $\{v_k\}$ has a subsequence $\{v_{k_m}\}$ which is weakly convergent in $L_2(Q_T)$. For ease of notation, we let $w = v_k$, multiply the differential equation in (2.1) by w and integrate over Q_t. Integration by parts produces

$$\frac{1}{2}\int_\Omega w^2(x,t)dx + \int_{Q_t}[\varphi_{ij}^k]'(w)w_{x_i}w_{x_j}dxd\tau + \int_{Q_t} wf_k(w)dxd\tau = \frac{1}{2}\int_\Omega u_{0,k}^2 dx.$$

valid for all $t \in [0, T]$. Using (3.1) and the positiveness of f_k, we get

$$\int_\Omega w^2(x,t)dx \le \int_\Omega u_{0,k}^2 dx$$

for all $t \in [0, T]$. Integrating over $[0, T]$, we have

$$\int_{Q_T} w^2 dxdt \le T\int_\Omega u_{0,k}^2 dx \le M^2 T|\Omega|$$

which, in turn, produces

$$\int_{Q_T} v_k^2 dxdt \le M^2 T|\Omega|.$$

Since the right side of this inequality is bounded independently of k, the reflexivity of $L_2(Q_T)$ yields the existence of a subsequence of $\{v_k\}$ which converges weakly in $L_2(Q_T)$. This completes the proof of Theorem 1.

4 PROOFS OF EXTINCTION RESULTS.

PROOF OF THEOREM 2. Let u be a nonnegative, bounded, generalized solution to (1.1). Let t_0 be the extinction time for u with $0 < t_0 < \infty$ and let $x_0 \in \Omega$ such that $u_0(x_0) > 0$. Choose

$r > 0$ and $\delta > 0$ so that $B \equiv \{x \in \Omega \mid |x - x_0| \leq r\} \subseteq \Omega$ and $u_0(x) \geq \delta$ for all $x \in B$. Let h be the function defined by

$$h(x) = \begin{cases} \exp\left(\dfrac{-|x - x_0|^2}{r^2 - |x - x_0|^2}\right) & \text{if } |x - x_0| < r \\ 0 & \text{if } |x - x_0| \geq r \ . \end{cases}$$

We note that h satisfies $0 \leq h(x) \leq 1$ on B, $\int_\Omega h(x)dx < \infty$, and $|\nabla h|^2 \leq Kh$ on Ω for some constant K depending only on the value of r. We define the sequence of nonnegative functions $\{a_k\}$ by $a_k(s) \equiv \Lambda_k(s) + f_k(s)$ where $\Lambda_k = tr(\Phi_k)$ and the sequences $\{\Phi_k\}$, $\{f_k\}$, and $\{u_{0,k}\}$ are defined as in the proof of Theorem 1. Now let $\{u_k\}$ be a corresponding sequence of classical solutions to (2.1) which converges weakly in $L_2(Q_{t_0+1})$ to u. We multiply (2.1) (with v_k replaced by u_k) by h, divide by $a_k(u_k) + \varepsilon$ ($\varepsilon > 0$) and integrate over Ω. After integrating by parts, we get

$$\frac{d}{dt} \int_\Omega h \int_0^{u_k} \frac{ds}{a_k(s) + \varepsilon} dx = -\int_\Omega \sum_{i,j=1}^n (\varphi_{ij}^k(u_k))_{x_i} \left(\frac{h}{a_k(u_k) + \varepsilon}\right)_{x_j} dx$$

$$-\int_\Omega \frac{f_k(u_k)h}{a_k(u_k) + \varepsilon} dx$$

$$= \int_\Omega \frac{a_k'(u_k)h}{(a_k(u_k) + \varepsilon)^2} \nabla u_k^T \Phi_k'(u_k) \nabla u_k \, dx$$

$$-\int_\Omega \frac{1}{a_k(u_k) + \varepsilon} \nabla u_k^T \Phi_k'(u_k) \nabla h \, dx$$

$$-\int_\Omega \frac{f_k(u_k)h}{a_k(u_k) + \varepsilon} dx. \tag{4.1}$$

We let H be defined by

$$H(x) = \begin{cases} \dfrac{|\nabla h(x)|}{h(x)^{1/2}} & \text{for } |x - x_0| < r \\ 0 & \text{for } |x - x_0| \geq r \ . \end{cases}$$

We note that H is continuous and bounded on $\overline{\Omega}$. (In fact, $0 \leq H(x) \leq K^{1/2}$ where the constant K comes from the properties of h above.) Using the fact that $h^{1/2}H = |\nabla h|$ on Ω and elementary estimates, the second integral on the right side of (4.1) may be estimated as

$$-\int_\Omega \frac{\nabla u_k^T \Phi_k'(u_k) \nabla h}{a_k(u_k) + \varepsilon} dx \geq -\int_\Omega \frac{1}{a_k(u_k) + \varepsilon} |\Phi_k'(u_k)\nabla u_k||\nabla h| \, dx$$

$$= -\int_\Omega \frac{h^{1/2}}{a_k(u_k) + \varepsilon} |\Phi_k'(u_k)\nabla u_k| H \, dx$$

$$= \int_\Omega \left(\frac{h^{1/2}|\Phi_k'(u_k)\nabla u_k|}{a_k(u_k) + \varepsilon} - \frac{H}{2}\right)^2 dx$$

$$-\int_\Omega \frac{|\Phi_k'(u_k)\nabla u_k|^2 h}{(a_k(u_k) + \varepsilon)^2} dx - \frac{1}{4}\int_\Omega H^2 \, dx$$

$$\geq -\int_\Omega \frac{|\Phi_k'(u_k)\nabla u_k|^2 h}{(a_k(u_k) + \varepsilon)^2} dx - \frac{1}{4}\int_\Omega H^2 \, dx$$

We now substitute this expression into (4.1) and use the fact that $f_k(u_k) \leq a_k(u_k) + \varepsilon$ to obtain

$$\frac{d}{dt} \int_\Omega h \int_0^{u_k} \frac{ds}{a_k(s) + \varepsilon} dx \geq \int_\Omega \frac{a_k'(u_k)h}{(a_k(u_k) + \varepsilon)^2} \nabla u_k^T \Phi_k'(u_k) \nabla u_k \, dx$$

$$-\int_\Omega \frac{|\Phi_k'(u_k)\nabla u_k|^2 h}{(a_k(u_k) + \varepsilon)^2} dx - \frac{1}{4}\int_\Omega H^2 \, dx - \int_\Omega h \, dx \ . \tag{4.2}$$

We use the positive definiteness of Φ_k', inequality (3.1), and the inequality $a_k'(s) \geq \Lambda_k'(s)$ in (4.2) to get (The argument u_k has been suppressed for ease of reading.)

$$\frac{d}{dt} \int_\Omega \int_0^{u_k} \frac{h\,ds}{a_k(s)+\varepsilon}\,dx \geq \int_\Omega \frac{h}{(a_k+\varepsilon)^2} \nabla u_k^T (\Phi_k')^{1/2}[a_k'I - \Phi_k'](\Phi_k')^{1/2}\nabla u_k\,dx - \int_\Omega H^2\,dx - \int_\Omega h\,dx$$

$$\geq -\frac{1}{4}\int_\Omega H^2\,dx - \int_\Omega h\,dx \equiv -K_0.$$

Integrating this inequality over the interval $[0, t]$ produces

$$\int_\Omega \int_0^{u_k(x,t)} \frac{h(x)ds}{\Lambda_k(s)+f_k(s)+\varepsilon}dx \geq \int_\Omega \int_0^{u_{0,k}(x)} \frac{h(x)ds}{\Lambda_k(s)+f_k(s)+\varepsilon}dx - K_0 t$$

for all $0 \leq t < \infty$. Thus, integrating this over the interval $[t_0, t_0+1]$, produces

$$\int_{t_0}^{t_0+1} \int_\Omega \int_0^{u_k(x,t)} \frac{h(x)ds}{\Lambda_k(s)+f_k(s)+\varepsilon}\,dx\,dt \geq \int_\Omega \int_0^{u_{0,k}(x)} \frac{h(x)ds}{\Lambda_k(s)+f_k(s)+\varepsilon}dx - K_0(t_0+1). \quad (4.3)$$

Since for fixed k and fixed $\varepsilon > 0$, the function F defined by

$$F(\sigma) = \int_0^\sigma \frac{ds}{\Lambda_k(s)+f_k(s)+\varepsilon}$$

.is concave, Jensen's inequality (see [11] p. 202) may be applied to the left side of (4.3) to get

$$|\Omega| \int_0^{U_k(t_0)} \frac{ds}{\Lambda_k(s)+f_k(s)+\varepsilon} \geq \int_\Omega h(x) \int_0^{u_{0,k}(x)} \frac{ds}{\Lambda_k(s)+f_k(s)+\varepsilon}dx - K_0(t_0+1).$$

where

$$U_k(t_0) = \frac{1}{|\Omega|}\int_{t_0}^{t_0+1}\int_\Omega h(x)u_k(x,t)\,dx\,dt.$$

We now let k approach ∞. The weak convergence of $\{u_k\}$ to u, the $L_\infty(\Omega)$ convergence of $\{u_{0,k}\}$ to u_0 as well as the uniform convergence of Λ_k and f_k to Λ and f, respectively, produces

$$|\Omega| \int_0^{U(t_0)} \frac{ds}{\Lambda(s)+f(s)+\varepsilon} \geq \int_\Omega h(x) \int_0^{u_0(x)} \frac{ds}{\Lambda(s)+f(s)+\varepsilon}dx - K_0(t_0+1).$$

where

$$U(t_0) = \frac{1}{|\Omega|}\int_{t_0}^{t_0+1}\int_\Omega h(x)u(x,t)\,dx\,dt.$$

Furthermore, since t_0 is the extinction time for u, we have $u(x,t) = 0$ for almost all $(x,t) \in \Omega \times [t_0, \infty)$. Therefore, the last inequality yields

$$\int_\Omega h(x) \int_0^{u_0(x)} \frac{ds}{\Lambda(s)+f(s)+\varepsilon}dx \leq K_0(t_0+1).$$

By our choice of the set B and the function h, this inequality produces

$$\int_B h(x) \int_0^\delta \frac{ds}{\Lambda(s)+f(s)+\varepsilon}dx \leq \int_B h(x) \int_0^{u_0(x)} \frac{ds}{\Lambda(s)+f(s)+\varepsilon}dx \leq K_0(t_0+1)$$

and hence

$$\int_0^\delta \frac{ds}{\Lambda(s)+f(s)+\varepsilon} \leq \frac{K_0(t_0+1)}{\int_B h(x)dx} < \infty.$$

Now let $\varepsilon \downarrow 0$ and the proof of Theorem 2 is complete.

PROOF OF THEOREM 3. Suppose $0 \leq u_0 \leq M$. If (1.3) holds, then the proof of the existence of a finite extinction time is quite similar to that of the isotropic case (see Theorem 1 of [5]) and is therefore omitted.

Thus, suppose (1.1) holds for some m. For ease of notation, we let $\psi = \varphi_{mm}$. Choose $r > 0$ so that $\overline{\Omega} \subseteq \{x \in \mathbf{R}^n \mid |x_i| < r, \ 1 \leq i \leq n\}$ and let z_0 be a $C^\infty(\mathbf{R})$ function such that $z_0(x_m) = M$ for $|x_m| < r$ and $z_0(-r-1) = z_0(r+1) = 0$. Let z be a mild solution (as in [9] and [12]) to the initial-boundary value problem

$$\frac{\partial}{\partial t} z = \frac{\partial^2}{\partial x_m^2} \psi(z) \qquad \text{for } |x_m| < r+1, \quad t > 0$$
$$z(-r-1, t) = 0 \qquad \text{for} \qquad\qquad t \geq 0$$
$$z(r+1, t) = 0 \qquad \text{for} \qquad\qquad t \geq 0$$
$$z(x_m, 0) = z_0(x_m) \qquad \text{for } |x_m| \leq r+1 \qquad .$$

From [1], we know that z has a finite extinction time, T_0. We shall show that $u = 0$ a.e. for $t \geq T_0$. Thus let $T \in (T_0, \infty)$. Since u is a generalized solution of (1.1), there exist sequences $\{\varphi_{ij}^k\}_{k=1}^\infty$ and $\{f_k\}$ of well-behaved functions converging uniformly on compact subsets of $[0, \infty)$ to φ_{ij}, $1 \leq i, j \leq n$, and to f, respectively, such that the sequence $\{v_k\}$ of classical solutions to (2.1) converges weakly in $L_2(Q_T)$ to the solution u. Let z_k be the unique classical solution of $(\psi_k \equiv \varphi_{mm}^k)$

$$\frac{\partial}{\partial t} z_k = \frac{\partial^2}{\partial x_m^2} \psi_k(z_k) \qquad \text{for } |x_m| < r+1, \quad t > 0$$
$$z_k(-r-1, t) = 0 \qquad \text{for} \qquad\qquad t \geq 0$$
$$z_k(r+1, t) = 0 \qquad \text{for} \qquad\qquad t \geq 0$$
$$z_k(x_m, 0) = z_0(x_m) \qquad \text{for } |x_m| \leq r+1,$$

From Sacks [12] (see Proposition 2.1 of [12]), we know that $z_k \to z$ in $C([0, T]; L^1(-r-1, r+1))$. Also, since z_k satisfies

$$L_k z_k = 0 \ \leq \ f_k(v_k) = L_k v_k \qquad \text{in } Q_T$$
$$v_k \ \leq \ z_k \qquad \text{on } \partial\Omega \times [0, T]$$
$$v_k(x, 0) \ \leq \ z_0(x) \qquad \text{on } \overline{\Omega}$$

where L_k is the operator defined by

$$L_k w \equiv \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \varphi_{ij}^k(w) - \frac{\partial w}{\partial t},$$

the results from Protter and Weinberger [13] (see pp. 187-188) yield $v_k \leq z_k$ on Q_T for all k. Hence

$$\int_{T_0}^T \int_\Omega v_k(x, t) dx \leq \int_{T_0}^T \int_\Omega z_k(x_m, t) dx$$

We now let $k \to \infty$ to get

$$\int_{T_0}^T \int_\Omega u(x, t) dx \leq \int_{T_0}^T \int_\Omega z(x_m, t) dx \quad .$$

Thus, since z vanishes for $t \geq T_0$, we must have $u = 0$ a.e. on $\Omega \times [T_0, T]$. Since T was chosen arbitrarily from (T_0, ∞), we must have $u = 0$ a.e. on $\Omega \times [T_0, \infty)$. This completes the proof of Theorem 3.

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