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SOME RESULTS ON BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Existence results for a second order boundary value problem for functional differential equations, are given. The results are based on the nonlinear Alternative of Leray-Schauder and rely on a priori bounds on solutions. These results are generalizations of recent results from ordinary differential equations and complete our earlier results on the same problem.

KEY WORDS AND PHRASES. Boundary value problems, functional differential equations,a priori bounds, Leray-Schauder alternative.
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1.INTRODUCTION.

The purpose of this paper is to provide existence results for second order boundary value problems (BVP for short) for functional differential equations. Before we refer our BVP, for the convenience of the reader, we employ the standar setting for functional differential equations, [3].

Let $r \geq 0$ be given and let $C = C([-r,0], R^n)$ denote the space of continuous functions that map the interval $[-r,0]$ into R^n. For $\phi \in C$, the norm of φ is defined by

$$\|\phi\| = sup\{|\phi(\theta)| : \quad -r \leq \theta \leq 0\},$$

where $|.|$ denotes any convenient norm in R^n. If $x : [-r,T] \to R^n$, $T > 0$ is continuous, then for each $t \in [0,T]$, $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta), -r \leq \theta \leq 0$.

We consider the following BVP

$$x''(t) + f(t, x_t, x'(t)) = 0, \quad t \in [0,T] \tag{E}$$

$$\begin{aligned} \alpha_0 x_0 - \alpha_1 x'(0) &= \varphi \\ \beta_0 x(T) + \beta_1 x'(T) &= \eta \end{aligned} \tag{BC}$$

where $f: [0,T] \times C \times R^n \to R^n$ is a continuous function, $\varphi \in C$, $\eta \in R^n$ and $\alpha_0, \alpha_1, \beta_0, \beta_1$ are positive real constants.

At first glance it might appear to the reader that the first boundary condition in (BC) is an unusual condition since it connects the history x_0 with a single value $x'(0)$. This condition does, however, arise in a natural way and is suggested by the well posedness of the BVP (E)-(BC), since the function f depends on the term x_t and simultaneously, on the values of derivatives at the present time.

It should be noted that the BVP (E)-(BC) has been studied earlier in [4] under various condition on f.

The basic existence theorem for the BVP (E)-(BC), which relies on the a priori bounds, has been proved recently in [4] and it is not given here. We emphasize here that in the proof of the existence theorem in [4] we have not used the standard assumption that f maps bounded sets of $[0,T] \times C \times R^n$ into bounded sets in R^n.

Our main purpose in this paper is to give conditions on f wich imply the needed a priori bounds. These conditions are different from those given in [4]. More precisely, the a priori bounds for the solutions x and its derivatives x' are obtained via L^2-estimates(See [5]) and a Nagumo type condition analogous to that used in [2] for ordinary differential equations.

The results of this paper are not comparable with those of [4] and seems to be new even when (E) is an ordinary differential equation, i.e. $r = 0$.

In what follows $\|.\|$ stands for the L^2-norm defined by

$$\|x\| = \left(\int_0^T |x(t)|^2 dt \right)^{1/2}$$

and $< .,. >$ stads for the Euclidean inner product in R^n.

For subsequent use we shall state here the following inequalities.

LEMMA 1.1. (a) For any function $x \in C'[0,T]$

$$\|x\| \leq \frac{2T}{\pi}\|x'\| + \frac{1}{2}\sqrt{T}|x(0)| + \frac{1}{2}\sqrt{T}|x(T)|.$$

(b) For any function $x \in C'[0,T]$

$$\|x\|^2 \leq \frac{T^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T}\|x'\|^2 + \frac{\sqrt{T}}{\sqrt{2}}[|x(0)|^2 + |x(T)|^2], \quad T = 2.$$

The above inequalities follows by essentially the same reasoning as in lemmas 2.3 and 2.4 of [5]. Obviously in the case $T = 2$ we can use the inequality (a) instead of (b).

2. MAIN RESULTS

Now we present our main result on the existence of solutions of the BVP (E)-(BC).

THEOREM 2.1. Let $f: [0,T] \times C \times R^n \to R^n$ be a continuous function. Assume that:

(H_1) *There exist nonnegative constants* A *and* B *with*

$$B < min\left\{ \left(1 - \frac{AT^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T}\right)\frac{2\pi}{4T+\pi\sqrt{T}}, \quad \frac{4}{\sqrt{T}}\left(\frac{\alpha_0}{\alpha_1} - \frac{A\sqrt{T}}{\sqrt{2}}\right), \quad \frac{4}{\sqrt{T}}\left(\frac{\beta_0}{\beta_1} - \frac{A\sqrt{T}}{\sqrt{2}}\right)\right\}$$

such that

$$< u(0), f(t,u,v) > \leq A|u(0)|^2 + B|u(0)||v|$$

for all $t \in [0,T], u \in C$ and $v \in R^n$.

(H_2) *There exists a continuous function* $h: R^+ \to R^+$ *and a constant*

N such that

$$< v, f(t,u,v) > \leq h(|v|^2)|v|^2$$

for all $t \in [0,T]$, $v \in R^n$ and $u \in C$ with $\|u\| \leq M$ and

$$\int_{\frac{M^2}{T}}^{N} \frac{ds}{h(s)} \geq 2M^2$$

where

$$M = max\{\sqrt{kc^{-1}}, \sqrt{kc_0^{-1}} + \frac{|\phi(0)|}{2\alpha_1 c_0}, \sqrt{kc_1^{-1}} + \frac{|\eta|}{2\beta_1 c_1}\}$$

$$c = 1 - \frac{AT^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T} - \frac{2BT}{\pi} - \frac{1}{2}B\sqrt{T}$$

$$c_0 = \frac{\alpha_0}{\alpha_1} - \frac{A\sqrt{T}}{\sqrt{2}} - \frac{1}{4}B\sqrt{T}$$

$$c_1 = \frac{\beta_0}{\beta_1} - \frac{A\sqrt{T}}{\sqrt{2}} - \frac{1}{4}B\sqrt{T}$$

$$k = \frac{|\phi(0)|^2}{4\alpha_0\alpha_1} + \frac{|\eta|^2}{4\beta_0\beta_1}$$

Then the BVP (E)-(BC) has at least one solution.

PROOF. To prove the existence of solutions of the BVP (E)-(BC) we apply the Nonlinear Alternative of Leray-Schauder in the manner applied in [4]. To do this we need to verify that the set of all posible solutions of the family of BVP $(E_\lambda) - (BC)$, where

$$x''(t) + \lambda f(t, x_t, x'(t)) = 0, \quad t \in [0,T] \qquad (E_\lambda)$$

is a priori bounded by a constant independent of λ.

Let x be a solution of $(E_\lambda) - (BC)$. By taking the inner product of (E_λ) with $x(t)$, integrating by parts over $[0,T]$ and using the fact that

$$\begin{aligned}\int_0^T < x(t), x''(t) > dt &= < x(T), x'(T) > - < x(0), x'(0) > - \|x'\|^2 \\ &= < x(T), \frac{\eta - \beta_0 x(T)}{\beta_1} > - < x(0), \frac{\alpha_0 x(0) - \phi(0)}{\alpha_1} > - \|x'\|^2 \\ &= < x(T), \frac{\eta}{\beta_1} > - \frac{\beta_0}{\beta_1}|x(T)|^2 - \frac{\alpha_0}{\alpha_1}|x(0)|^2 + < x(0), \frac{\phi(0)}{\alpha_1} > - \|x'\|^2.\end{aligned}$$

we obtain by (H_1)

$$\begin{aligned}\|x'\|^2 + \frac{\alpha_0}{\alpha_1}|x(0)|^2 + \frac{\beta_0}{\beta_1}|x(T)|^2 &\leq \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)| + \int_0^T < x(t), f(t, x_t, x'(t)) > dt \\ &\leq \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)| + \int_0^T < x_t(0), f(t, x_t, x'(t)) > dt \\ &\leq \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)| + A\|x\|^2 + B\|x\|\|x'\|.\end{aligned}$$

Lemma 1.1 implies

$$\begin{aligned}
&\|x'\|^2 + \frac{\alpha_0}{\alpha_1}|x(0)|^2 + \frac{\beta_0}{\beta_1}|x(T)|^2 \\
&\quad \le \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)| + \frac{AT^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T}\|x'\|^2 \\
&\quad\quad + \frac{A\sqrt{T}}{\sqrt{2}}[|x(0|^2 + |x(T)|^2] \\
&\quad\quad + B\|x'\|\{\frac{2T}{\pi}\|x'\| + \frac{1}{2}\sqrt{T}|x(0)| + \frac{1}{2}\sqrt{T}|x(T)|\} \\
&\quad \le \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)| + \frac{AT^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T}\|x'\|^2 \\
&\quad\quad + \frac{A\sqrt{T}}{\sqrt{2}}[|x(0)|^2 + |x(T)|^2] \\
&\quad\quad + \frac{2BT}{\pi}\|x'\|^2 + \frac{1}{2}B\sqrt{T}\{\frac{1}{2}|x(0)|^2 + \frac{1}{2}\|x'\|^2\} \\
&\quad\quad + \frac{1}{2}B\sqrt{T}\{\frac{1}{2}|x(T)|^2 + \frac{1}{2}\|x'\|^2\}
\end{aligned}$$

Consequently

$$\begin{aligned}
&(1-\frac{AT^2}{\pi^2}\frac{2+\sqrt{2T}}{2-T} - \frac{2BT}{\pi} - \frac{1}{2}B\sqrt{T})\|x'\|^2 \\
&\quad + (\frac{\alpha_0}{\alpha_1} - \frac{A\sqrt{T}}{\sqrt{2}} - \frac{1}{4}B\sqrt{T})|x(0)|^2 \\
&\quad + (\frac{\beta_0}{\beta_1} - \frac{A\sqrt{T}}{\sqrt{2}} - \frac{1}{4}B\sqrt{T})|x(T)|^2 \\
&\le \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)|
\end{aligned}$$

or

$$\begin{aligned}
&c\|x'\|^2 + \left(\sqrt{c_0}|x(0)| - \frac{|\phi(0)|}{2\alpha_1\sqrt{c_0}}\right)^2 + \left(\sqrt{c_1}|x(T)| - \frac{|\eta|}{2\beta_1\sqrt{c_1}}\right)^2 \\
&\le \frac{|\phi(0)|^2}{4\alpha_0\alpha_1} + \frac{|\eta|^2}{4\beta_0\beta_1} = k.
\end{aligned}$$

The last inequality implies that

$$\|x'\| \le M, \quad |x(0)| \le M, \quad \textit{and} \quad |x(T)| \le M \tag{$\star$}$$

Therefore for every $t \in [0,T]$

$$\begin{aligned}
|x(t)| &\le |x(0)| + |\int_0^t x'(s)ds| \\
&\le |x(0)| + \sqrt{T}\|x'\| \le M + \sqrt{T}M = (1+\sqrt{T})M,
\end{aligned}$$

which is the required a priori bound on x on the interval $[0,T]$.

Next we shall prove that x is bounded on $[-r,0]$. From the first boundary condition we have:

$$\alpha_1|x'(0)| \le |\phi(0)| + \alpha_0 M \le \|\phi\| + \alpha_0 M$$

and consequently

$$\|x\| \le \frac{1}{\alpha_0}[\|\phi\| + \alpha_1|x'(0)|] \le \frac{2}{\alpha_0}\|\phi\| + M.$$

Therefore

$$max\{|x(t)| : -r \leq t \leq T\} \leq M_0 = max\{\frac{2}{a_0}\|\phi\| + M_*(1+\sqrt{T})M\}.$$

Also, $(\star)$ implies, by the mean value theorem, that there exists $t_0 \in [0,T]$ such that

$$T|x'(t_0)|^2 \leq M^2$$

or

$$|x'(t_0)|^2 \leq \frac{M^2}{T}.$$

Now, taking the inner product of (E_λ) with $x'(t)$ we have by (H_2)

$$|\frac{d}{dt}|x'(t)|^2| \leq 2h(|x'(t)|^2)|x'(t)|^2$$

or

$$|\frac{d}{dt}\int_0^{|x'(t)|^2}\frac{ds}{h(s)}| \leq 2|x'(t)|^2$$

Integrating the above inequality we get

$$\begin{aligned}\int_0^{|x'(t)|^2}\frac{ds}{h(s)} &\leq \int_0^{|x'(t_0)|^2}\frac{ds}{h(s)} + 2\int_0^T |x'(t)|^2 dt \\ &\leq \int_0^{|x'(t_0)|^2}\frac{ds}{h(s)} + 2M^2 \\ &\leq \int_0^{\frac{M^2}{T}}\frac{ds}{h(s)} + \int_{\frac{M^2}{T}}^N \frac{ds}{h(s)} = \int_0^N \frac{ds}{h(s)}.\end{aligned}$$

Hence

$$|x'(t)| \leq \sqrt{N}, \quad t \in [0,T].$$

Consequently the required a priori bounds are established and the results follows.

THEOREM 2.2. Let $f: [0,T] \times C \times R^n \to R^n$ be a continuous function. Assume that (H_1) holds and moreover

(H'_2) *There exist a continuous function* $h: R^+ \to R^+$ a constant $N > 0$ and nonnegative constants q_i, $i = 1,\ldots,6$ such that

$$< v, f(t,u,v) > \leq h(|v|^2)(q_1|u(0)|^2 + q_2|v|^2 + q_3|u(0)||v| + q_4|u(0)| + q_5|v| + q_6)$$

for all $t \in [0,T], v \in R^n$ and $u \in C$ with $\|u\| \leq M$ and

$$\int_{\frac{M^2}{T}}^N \frac{ds}{h(s)} \geq 2Q$$

where

$$Q = q_1(1+\sqrt{T})^2M^2 + q_2M^2 + q_3(1+\sqrt{T})M^2 + q_4(1+\sqrt{T})\sqrt{T}M + q_5\sqrt{T}M + q_6.$$

Then the BVP (E)-(BC) has at least one solution.

PROOF. Since the first part of the proof is similar to that of Theorem 2.1 we give only the part of the proof which concerns the a priori bound of $max\{|x'(t)| : 0 \le t \le T\}$. Taking the inner product of (E_λ) with $x'(t)$ we have by (H'_2)

$$|\frac{d}{dt}|x'(t)|^2| \le 2h(|x'(t)|^2(q_1|x(t)|^2 + q_2|x'(t)|^2 + q_3|x(t)||x'(t)| + q_4|x(t)| + q_5|x'(t)| + q_6)$$

or

$$|\frac{d}{dt}\int_0^{|x'(t)|^2}\frac{ds}{h(s)}| \le 2(q_1|x(t)|^2 + q_2|x'(t)|^2 + q_3|x(t)||x'(t)| + q_4|x(t)| + q_5|x'(t)| + q_6)$$

Integrating the above inequality, and using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\int_0^{|x'(t)|^2}\frac{ds}{h(s)} &\le \int_0^{|x'(t_0)|^2}\frac{ds}{h(s)} + \{q_1\int_0^T|x(t)|^2dt + q_2\int_0^T|x'(t)|^2dt \\ &\quad + q_3\int_0^T|x(t)||x'(t)|dt + q_4\int_0^T|x(t)|dt + q_5\int_0^T|x'(t)|^2dt + q_6\} \\ &\le \int_0^{|x'(t_0)|^2}\frac{ds}{h(s)} + 2\{q_1\|x\|^2 + q_2\|x'\|^2 + q_3\|x\|\|x'\| \\ &\quad + q_4\sqrt{T}\|x\| + q_5\sqrt{T}\|x'\| + q_6\} \\ &\le \int_0^{|x'(t_0)|^2}\frac{ds}{h(s)} + 2Q \\ &\le \int_0^{\frac{M^2}{T}}\frac{ds}{h(s)} + \int_{\frac{M^2}{T}}^{N}\frac{ds}{h(s)} = \int_0^N\frac{ds}{h(s)}.\end{aligned}$$

Hence

$$|x'(t)| \le \sqrt{N}, \quad t \in [0,T]$$

which completes the proof.

Now we present some examples to illustrate how the above results may be used to yield existence of solutions of specific boundary value problems.

EXAMPLE 2.3. We consider the following BVP

$$x''(t) + g(t)x(t)F(t, x_t, x'(t)) = 0, \quad t \in [0,1] \tag{ϵ}$$

$$\begin{aligned}x_0 - x'(0) &= \phi \\ x(1) + x'(1) &= 2\end{aligned} \tag{bc}$$

where $g : [0,1] \to R$ is a continuous and positive function and F: $[0,1] \times C \times R^n \to R^n$ a bounded function with bound K.

Here $f(t,u,v) = g(t)u(0)F(t,u,v), T = 1, \alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 1$ and $\eta = 2$.

Then we have

$$< u(0), f(t,u,v) >= g(t)u^2(0)F(t,u,v) \le g_0K|u(0)|^2$$

i.e (H_1) holds with $A = g_0K, g_0 = max\{g(t) : t \in [0,1]\}$ and $B = 0$. Without loss of generality we can choose the functions g and F in such a way that $A = 1$.

By an easy calculation we find for $\phi(0) = 2$, $k = 2$, $c_1 = 1 - \frac{1}{\sqrt{2}} = c_0$, $c = 1 - \frac{2+\sqrt{2}}{\pi^2} \simeq 3.7$.

We remark also that

$$< v, f(t,u,v) >= g(t)u(0)F(t,u,v)v \leq g_0 MK|v| = |v|^2 (\frac{M}{\sqrt{|v|^2}})$$

This means that (H_2) holds with $h(s) = \frac{M}{\sqrt{s}}$. (The condition $\int_{M^2}^{N} \frac{ds}{h(s)} \geq 2M^2$ is obvious).
Therefore the BVP (e)-(bc) has at least one solution by Theorem 2.1.

EXAMPLE 2.4. It is easy to see, as in the previous example, that the BVP

$$x''(t) + g(t)x(t)F(t, x_t, x'(t)) + q(t)|x'(t)| = 0, \quad t \in [0, \frac{1}{9}] \tag{e_1}$$

$$\begin{aligned} x_0 - x'(0) &= \varphi \\ x(1) + x'(1) &= \sqrt{2} \end{aligned} \tag{bc}_1$$

where $q : [0, \frac{1}{9}] \to R$ is a continuous and positive function, has at least one solution, for $\varphi(0) = \sqrt{2}$ and $B = 1$.

3. CONCLUDING REMARKS

In [4] the BVP (E)-(BC) has been studied under the following conditions, which are briefly reproduced here.

(A_1) There exist a constant $M > 0$ such that $|u(0)| > M$ and $< u(0), v >= 0$ implies $< u(0), f(t,u,v) >< 0$ for all $t \in [0,T]$ and $v \in R^n$.

(A_2) $< u(0), f(t,u,v) >\leq k_1|v|^2 + k_2$
$| < v, f(t,u,v) > | \leq (k'_1|v|^2 + k'_2)|v|$ for all $t \in [0,T]$, $u \in C$ and $v \in R^n$.

(A_3) $|f(t,u,v)| \leq q(t)\Omega(|v|)$ for all $t \in [0,T]$, $u \in C$ and $v \in R^n$.

Let us add in the above list the assumptions (H_1) and (H_2) of Theorem 2.1.

(H_1) $< u(0), f(t,u,v) >\leq A|u(0)|^2 + B|u(0)||v|$ for all $t \in [0,T]$, $u \in C$ and $v \in R^n$.

(H_2) $< v, f(t,u,v) >\leq h(|v|^2)|v|^2$ for all $t \in [0,T]$, $u \in C$ and $v \in R^n$.

We also remind that:

The BVP (E)-(BC) has at least one solution if

(A_1) and (A_2) hold, [4. Th. 4.1]

(A_1) and (A_3) hold, [4. Th. 4.2]

(H_1) and (H_2) (or (H'_2)) hold, Th. 2.1 (or Th. 2.2)

The following questions are immediately arisen.

Has the BVP (E)-(BC) a solution if

1) (A_1) and (H_2) (or (H'_2)) hold ?

2) (H_1) and (A_2) hold ?

3) (H_1) and (A_3) hold ?

The answer in all of the above questions is positive. Indeed the cases 2) and 3) are obvious, since every one of conditions (H_1), (A_2) and A_3) gives independently a priori bound on x or x'.

Some comments are needed for the case 1). By taking the inner product of (E_λ) with $x(t)$, integrating by parts over $[0,T]$ and using (A_1) we get

$$\|x'\|^2 + \frac{\alpha_0}{\alpha_1}|x(0)|^2 + \frac{\beta_0}{\beta_1}|x(T)|^2 \leq \frac{|\phi(0)|}{\alpha_1}|x(0)| + \frac{|\eta|}{\beta_1}|x(T)|$$

or

$$\|x'\|^2 + \Big(\sqrt{\frac{\alpha_0}{\alpha_1}}|x(0)| - \frac{|\phi(0)|}{2\sqrt{\alpha_0\alpha_1}}\Big)^2 + \Big(\sqrt{\frac{\beta_0}{\beta_1}}|x(T)| - \frac{|\eta|}{2\sqrt{\beta_0\beta_1}}\Big)^2$$
$$\leq \frac{|\phi(0)|^2}{4\alpha_0\alpha_1} + \frac{|\eta|^2}{4\beta_0\beta_1}.$$

This implies the existence of the bound M and the rest of the proof is essentially the same as in Theorem 2.1.

We summarize the above discussion in the following

THEOREM 3.1. The BVP (E)-(BC) has at least one solution if one of the following pairs of conditions holds:

1) (A_1) and (A_2) 2) (A_1) and (A_3)

3) (A_1) and (H_2) (or (H'_2) 4) (H_1) and (A_2)

5) (H_1) and (A_3) 6) (H_1) and (H_2) (or (H'_2)

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