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EXTREME POINTS AND CONVOLUTION PROPERTIES OF SOME CLASSES OF MULTIVALENT FUNCTIONS

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ABSTRACT. This paper deals with the extreme points of closed convex hulls of the classes of multivalent functions related to Ruscheweyh derivatives and then these are used to determine the coefficient bounds Finally, we investigate convolution conditions and other properties of the functions in these classes

KEY WORDS AND PHRASES: Multivalent, Hadamard product, starlike, Ruscheweyh derivative
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1. INTRODUCTION

An analytic s-function is said to be *p-valent* if it assumes each value not more than p-times and some value exactly p-times Let $M(p)$, $p \geq 1$ integer, denote the family of all functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1 1}$$

which are analytic and p-valent in the unit disk $\triangle = \{z : |z| < 1\}$ Let $S_p^*(\alpha)$ denote the class of functions of the form (1 1) which satisfy the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > p\alpha, \quad \text{and} \quad \int_0^{2\pi} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) d\theta = 2\pi p \tag{1 2ab}$$

for $0 \leq \alpha < 1$ and $z \in \triangle$ A function in $S_p^*(\alpha)$ is called a *p-valent starlike* of order α in $\triangle$ The class $S_p^*(\alpha')$, where $\alpha' = \alpha p$, $0 \leq \alpha' < p$, was introduced by Goluzina [1] A function f of the form (1 1) is said to be in $C_p(\alpha)$ if $zf'(z)/p$ is in $S_p^*(\alpha)$ A function in $C_p(\alpha)$ is called a *p-valent convex* of order α in $\triangle$ We observe that $S_p^*(\alpha) \subset S_p^*(0) = S_p^*$, $C_p(\alpha) \subset C_p(0) = C_p$ Goodman [2] introduced the classes S_p^* and C_p In the same paper, he observed that these are subclasses of $M(p)$ Besides, note that $S_1^*(\alpha) = S^*(\alpha)$ and $C_1(\alpha) = C(\alpha)$, where $S^*(\alpha)$ and $C(\alpha)$ consist of the functions which are, respectively, starlike of order α and convex of order α in $\triangle$, see, for example [3] Also, we notice that there are different ways of extending starlike and convex concepts to p-valent functions and Hummel [4] has made an extensive study of the various possibilities

If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ are two power series in $\triangle$, then its Hadamard product is defined as $(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k$ in $\triangle$ For $f \in M(p)$, we write

$$D^{n+p-1}f(z)=\frac{z^p}{(1-z)^{n+p}}*f(z),\quad n\geq -p+1 \tag{1 3}$$

The operator $D^{n+p-1}f$ is called the $(n+p-1)$th order *Ruscheweyh derivative* of f Let $R_n(p,\alpha)$ denote the subclasses of functions f in $M(p)$ which satisfy the conditions

$$\mathrm{Re}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right)>p\alpha \quad\text{and}\quad \int_0^{2\pi}\mathrm{Re}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right)d\theta=2\pi p \tag{1 4ab}$$

for $0\leq\alpha<1$ and $z\in\triangle$ The condition (1 4b) implies that $D^{n+p-1}f(z)$ has p roots in $\triangle$ We observe that $R_{-p+1}(p,\alpha)=S_p^*(\alpha)$, $R_0(1,\alpha)=S^*(\alpha)$, and $R_1(1,\alpha)=C(\alpha)$ The class $R_n(1,\alpha)=R_n(\alpha)$ was introduced and studied by the author [5] Moreover, $R_n(p,\alpha)\subset R_n(p,0)\subset K_n(p)$ for $n\geq p$, where

$$K_n(p)=\left\{f\in M(p):\mathrm{Re}\,\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}>1/2,\ z\in\triangle\right\}$$

are subclasses of $M(p)$ studied by Goel and Sohi [6] Note that $K_n(1)$ was introduced by Ruscheweyh [7]

The fundamental viewpoint of considering convex hulls and extreme points of $S^*(\alpha)$ and $C(\alpha)$ were first given in [8] and [9] The author and Silverman [10] have studied the extreme points of the closed convex hull of $R_n(\alpha)$ Earlier, some of the concepts of convex hulls and extreme points were extended to multivalent functions in [11-13], and others

In the present paper, we determine the extreme points of the closed convex hulls of $R_n(p,\alpha)$ and $K_n(p)$ These are then used to determine the coefficient bounds Finally, we investigate convolution conditions and other properties of the functions in $R_n(p,\alpha)$

In the sequel, we denote the closed convex hull of a family F by $co\,F$ Also, let $E(co\,F)$ denote the set of all extreme points of F

2. EXTREME POINTS

LEMMA 2.1. [12] $E\big(co\,S_p^*(\alpha)\big)$ consists of the functions given by

$$\frac{z^p}{(1-xz)^{2(1-\alpha)p}}=z^p+\sum_{k=p+1}^{\infty}\frac{(2p-2\alpha p)_{k-p}x^{k-p}}{(k-p)!}z^k \tag{2 1}$$

$|x|=1$, $z\in\triangle$, where $(a)_k=a(a+1)(a+2)\ldots(a+k-1)$

THEOREM 2.1. The extreme points of $co\,R_n(p,\alpha)$, $0\leq\alpha<1$, are given by the functions

$$f_x(z)=z^p+\sum_{k=p+1}^{\infty}\frac{(2p-2\alpha p)_{k-p}(n+p-1)!}{(k+n-1)!}x^{k-p}z^k,\quad |x|=1,\quad z\in\triangle. \tag{2 2}$$

PROOF. We first notice that the operator $D^{n+p-1}:f\rightarrow D^{n+p-1}f$ is an isomorphism from $R_n(p,\alpha)$ to $S_p^*(\alpha)$ and consequently it preserves extreme points Also, we observe that

$$D^{n+p-1}f(z)=\frac{z^p}{(1-z)^{n+p}}*f(z)=z^p+\sum_{k=p+1}^{\infty}\binom{n+k-1}{n+p-1}a_kz^k. \tag{2 3}$$

Therefore, from Lemma 2 1 we find that the extreme points of $R_n(p,\alpha)$ are given by

$$z^p+\sum_{k=p+1}^{\infty}\binom{n+k-1}{n+p-1}^{-1}\frac{(2p-2\alpha p)_{k-p}}{(k-p)!}x^{k-p}z^k.$$

This simplifies to (2 2) and the proof is complete

REMARK 2.1. The special case of $p=1$ in Theorem 2 1 yields the extreme points of $co\,R_n(\alpha)$ found in [10]

REMARK 2.2. Letting $n = -p+1$ and $\alpha = 0$ in Theorem 2 1, we obtain the extreme points of $co\, S_p^*$ found by Hallenbeck and Livingston [11]

COROLLARY 2.1. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ is in $R_n(p,\alpha)$, then

$$|a_k| \le \frac{(2p-2\alpha p)_{k-p}(n+p-1)!}{(k+n-1)!}, \quad k \ge p+1 \tag{2.4}$$

with equality for

$$f_x(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(2p-2\alpha p)_{k-p}(n+p-1)!}{(k+n-1)!} x^{k-p} z^k, \quad |x| = 1, \quad z \in \triangle.$$

COROLLARY 2.2. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ is in S_p^*, then

$$|a_k| \le \frac{(2p)(2p+1)...(p+k-1)}{(k-p)!}, \quad k \ge p+1.$$

The above result of Goodman [2] may be found by letting $\alpha = 0$, $n = -p+1$ in Corollary 2 1

COROLLARY 2.3. If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ is in $R_n(p,\alpha)$, then

$$|f(z)| \le r^p + \sum_{k=p+1}^{\infty} \frac{(2p-2\alpha p)_{k-p}(n+p-1)!}{(k+n-1)!} r^k,$$

with equality for $f_x(z)$ at $z = \overline{x}r$.

In [10], the author and Silverman found the extreme points of the closed convex hull of $K_n(p)$, when $p = 1$ In the next theorem, we find the corresponding result when $p \ge 1$

LEMMA 2.2. [6] $K_n(p) \subset K_{n-1}(p)$ for every $n \ge -p+1$

THEOREM 2.2. The extreme points of $coK_n(p)$ are

$$\left\{ \frac{z^p}{1-xz} = z^p + \sum_{k=p+1}^{\infty} x^{k-p} z^k : |x| = 1, z \in \triangle \right\} \quad \text{for all} \quad n \ge -p+1.$$

PROOF. For $g(z) = z^p/(1-xz)$, we observe that

$$D^{n+p-1} g(z) = \frac{z^p}{(1-xz)^{n+p}} * \frac{z^p}{1-xz} = \frac{z^p}{(1-xz)^{n+p}}.$$

Therefore,

$$\operatorname{Re} \frac{D^{n+p} g(z)}{D^{n+p-1} g(z)} = \operatorname{Re} \frac{1}{1-xz} > 1/2.$$

It implies that $g \in K_n(p)$ for every $n \ge -p+1$ We thus have

$$\left\{ \frac{z^p}{1-xz} : |x| = 1, z \in \triangle \right\} \subset K_n(p).$$

But by Lemma 2 2, $K_n(p) \subset K_{-p+1}(p)$ Since

$$\begin{aligned} K_{-p+1}(p) &= \left\{ f \in M(p) : \operatorname{Re} \frac{D^1 f}{D^2 f} > \frac{1}{2} \right\} \\ &= \left\{ f \in M(p) : \operatorname{Re} \frac{zf' - (p-1)f}{f} > 1/2 \right\} = S_p^*\left(\frac{2p-1}{2p}\right), \end{aligned}$$

we have

$$\left\{\frac{z^p}{1-xz} : |x|=1, z \in \triangle\right\} \subset S_p^*\left(\frac{2p-1}{2p}\right)$$

On the other hand, the extreme points of $co\, S_p^*\left(\frac{2p-1}{2p}\right)$ are $\left\{\frac{z^p}{1-xz} : |x|=1, z \in \triangle\right\}$, from Lemma 2 1, and the result follows

COROLLARY 2.4 If $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in K_n(p)$, then $|a_k| \leq 1$ for $k \geq p+1$

3. INCLUSION RELATIONS

LEMMA 3.1. [14] Let ω be a non-constant and analytic function in $|z| < r < 1$, $\omega(0) = 0$ If $|\omega|$ attains its maximum value on the circle $|z| = r$ at z_0, then $z_0\omega'(z_0) = k\omega(z_0)$, where $k(\geq 1)$ is any real number

THEOREM 3.1. $R_{n+1}(p,\alpha) \subset R_n(p,\alpha)$ for all $\alpha(0 \leq \alpha < 1)$, and $n \geq -p+1$

PROOF. Let $f \in R_{n+p}(p,\alpha)$ Define an analytic function $\omega(z)$ in $\triangle$ such that $\omega(0) = 0$, $\omega(z) \neq -1$ for all $z \in \triangle$ by

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = p\left(\frac{1+(2\alpha-1)\omega(z)}{1+\omega(z)}\right). \tag{3 1}$$

Using the identity

$$z\left(D^{n+p-1}f(z)\right)' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z), \tag{3 2}$$

we can rewrite (3 1) as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{n+p+((2\alpha-1)p+n)\omega(z)}{(n+p)(1+\omega(z))}. \tag{3 3}$$

Taking logarithmic differentiation of (3 3), we get

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = p\left(\frac{1+(2\alpha-1)\omega(z)}{1+\omega(z)}\right) - \frac{2p(1-\alpha)z\omega'(z)}{(1+\omega(z))(n+p+(n+(2\alpha-1)p)\omega(z))}. \tag{3 4}$$

We claim that $|w(z)| < 1$ for all $z \in \triangle$ For otherwise, by Lemma 3 1, there exists a point $z_0 \in \triangle$ such that $z_0\omega'(z) = k\,\omega(z_0)$ with $|\omega(z_0)| = 1$ and $k \geq 1$ Applying this result to (3 4), we obtain

$$\mathrm{Re}\left(\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)}\right) \leq p\alpha - \frac{kp(1-\alpha)}{2(n+\alpha p)} \leq p\alpha \quad \text{for each} \quad n \geq -p+1.$$

This proves that $f \notin R_{n+1}(p,\alpha)$, which contradicts the hypothesis We thus conclude that $|\omega(z)| < 1$ for all $z \in \triangle$ and hence $f \in R_n(p,\alpha)$ This completes the proof

In view of Theorem 3 1, it immediately follows that $R_n(p,\alpha) \subset S_p^*(\alpha)$ for all $n \geq -p+1$ For α fixed and $n = n(\alpha)$ sufficiently large, we shall now show that $R_n(p,0) \subset C_p(\alpha)$ We need the following

LEMMA 3.2. Let $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in M(p)$, and let $0 \leq \alpha < 1$ The function f is in $C_p(\alpha)$ if

$$\sum_{k=p+1}^{\infty} k(k-\alpha p)|a_k| \leq p^2(1-\alpha). \tag{3 5}$$

PROOF. It suffices to show that

$$\left|1+\frac{zf''(z)}{f'(z)} - p\right| \leq p(1-\alpha). \tag{3 6}$$

We have

$$\begin{aligned}
&|zf''(z) - (p-1)f'(z)| - p(1-\alpha)|f'(z)| \\
&\le \left|\sum_{k=p+1}^{\infty} k(k-p)a_k\right| - p(1-\alpha)\left(p - \sum_{k=p+1}^{\infty} k|a_k|\right) \\
&\le \sum_{k=p+1}^{\infty} k(k-p)|a_k| - p^2(1-\alpha) + p(1-\alpha)\sum_{k=p+1}^{\infty} k|a_k| \\
&= \sum_{k=p+1}^{\infty} k(k-\alpha p)|a_k| - p^2(1-\alpha) \le 0,
\end{aligned}$$

which proves (3 6) This completes the proof

THEOREM 3.2. For fixed α, $0 \le \alpha < 1$, $p \ge 1$ integer

$$R_n(p,0) \subset C_p(\alpha) \quad \text{for any} \quad n \ge n_0 = \frac{2(p+1)^4}{p(1-\alpha)}.$$

PROOF. Let $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ be in $R_n(p,0)$ We observe that

$$f \in R_n(p,0) \iff D^{n+p-1} f \in S_p^*. \tag{3 7}$$

In view of (2 3), and as a consequence of Corollary 2 2 of Theorem 2 1, we obtain

$$\left|\binom{n+k-1}{n+p-1} a_k\right| \le \frac{(2p)_{k-p}}{(k-p)!},$$

which yields

$$|a_k| \le \frac{(2p)_{k-p}}{(k-p)!} \binom{n+k-1}{n+p-1}^{-1} \quad \text{for all} \quad k \ge p+1. \tag{3 8}$$

Therefore, because of Lemma 3 2, we only need to prove that

$$\sum_{k=p+1}^{\infty} k^2|a_k| \le \sum_{k=p+1}^{\infty} k^2 \frac{(2p)_{k-p}}{(k-p)!} \binom{n+k-1}{n+p-1}^{-1} \le p^2(1-\alpha) \quad \text{for all} \quad n \ge n_0.$$

Since $\sum_{k=p+1}^{\infty} (1/k^2) < 1$, it suffices to establish that

$$\sum_{k=p+1}^{\infty} \frac{k^2(2p)_{k-p}}{(k-p)!} \binom{n+k-1}{n+p-1}^{-1} \le p^2(1-\alpha) \sum_{k=p+1}^{\infty} (1/k^2) \tag{3 9}$$

for all $n \ge n_0$ We notice that (3 9) holds if

$$d_k = \frac{k^4(2p)_{k-p}}{(k-p)!} \binom{n+k-1}{n+p-1}^{-1} \le p^2(1-\alpha) \tag{3 10}$$

for all $n \ge n_0$, $k \ge p+1$ But

$$\binom{n+k-1}{n+p-1}^{-1} = \frac{(n+p-1)!\,(k-p)!}{(n+k-1)!}$$

is a decreasing function of $n(\ge -p+1)$ Therefore, we need only prove (3 10) for $n = n_0$ Since for $n = n_0$,

$$d_{p+1} = \frac{2p(p+1)^4}{n_0 + p} \le p^2(1-\alpha)$$

is true for all $p \geq 1$, it follows that (3 10) holds for $k = p+1$ Thus the proof of the theorem will be completed by proving that d_k is a decreasing function of $k(\geq p+1)$ for $n = n_0$, that is, if

$$\frac{d_{k+1}}{d_k} = \frac{(k+1)^4(k+p)}{k^4(k+n_0)} \leq 1,$$

which is equivalent to

$$g(k) = k^4(n_0 - 4 - p) - k^3(6+4p) - k^2(4+6p) - k(1+4p) - p \geq 0$$

Since

$$n_0 - 4 - p \geq \frac{2(p+1)^4 - p(4+p)}{p},$$

we have

$$g(k) \geq \left(\frac{2(p+1)^4}{p} - 15 - 16p\right)k^4 = \left[\frac{2(p+1)}{p}\left((p+1)^3 - 8p\right) + 1\right]k^4 > 0$$

for all $p \geq 1$ The proof is complete

4. CONVOLUTION CONDITIONS

LEMMA 4.1. Let $f \in M(p)$ Then $f \in S_p^*(\alpha)$, $0 \leq \alpha < 1$, $p \geq 1$, if and only if $f(z) * \left[(z^p + Bz^{p+1})/(1-z)^2\right] \neq 0 \; (0 \leq |z| < 1, |x| = 1)$, where

$$B = \frac{x+1 - 2p(1-\alpha)}{2p(1-\alpha)}. \tag{4 1}$$

PROOF. Since $zf'(z)/f(z)$ at $z = 0$ is p, therefore

$$\mathrm{Re}\left(\frac{\frac{zf'(z)}{f(z)} - \alpha p}{p - \alpha p}\right) > 0,$$

which is equivalent to

$$\frac{\frac{zf'(z)}{f(z)} - \alpha p}{p - \alpha p} \neq \frac{x-1}{x+1}, \quad |x| = 1, \quad x \neq -1.$$

This simplifies to

$$(zf'(z) - \alpha p f(z))(x+1) - (p - \alpha p)(x-1)f(z) \neq 0 \tag{4 2}$$

in the annulus $\rho < |z| < 1$ for some $\rho (0 < \rho < 1)$ For $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, it is easy to show that

$$f(z) * \frac{z^p}{(1-z)^2} = z^p + \sum_{k=p+1}^{\infty} (k - p + 1)a_k z^k = zf'(z) - (p-1)f(z), \tag{4 3}$$

and

$$f(z) * \frac{z^p}{1-z} = f(z). \tag{4 4}$$

Therefore, (4 2) is equivalent to

$$f(z) * \left[\left\{\frac{z^p}{(1-z)^2} + (p-1)\frac{z^p}{1-z} - \alpha p \frac{z^p}{1-z}\right\}(x+1) - (p - \alpha p)(x-1)\frac{z^p}{1-z}\right] \neq 0$$

that is

$$f(z) * \left(\frac{2p(1-\alpha)z^p + (x + 1 - 2p(1-\alpha))z^{p+1}}{(1-z)^2} \right) \neq 0$$

This proves (4 1)

THEOREM 4.1. The function f is in $R_n(p, \alpha)$ if and only if

$$f(z) * \left(\frac{z^p + \frac{(n+p)x + n + p(2\alpha - 1)}{2p(1-\alpha)} z^{p+1}}{(1-z)^{n+p+1}} \right) \neq 0 \tag{4 5}$$

for $0 < |z| < 1$, $|x| = 1$.

PROOF. Since $f \in R_n(p, \alpha)$ if and only if $D^{n+p-1} f \in S_p^*(\alpha)$, an application of (1 3) to Lemma 4 1 yields

$$f(z) * \left(h(z) * \left(\frac{z^p}{(1-z)^2} + \frac{Bz^{p+1}}{(1-z)^2} \right) \right) \neq 0, \tag{4 6}$$

where $h(z) = z^p/(1-z)^{n+p}$ But in view of (4 3) and (4 4), we may write

$$\begin{aligned}
h(z) * \left(\frac{z^p}{(1-z)^2} + \frac{Bz^{p+1}}{(1-z)^2} \right) &= h(z) * \frac{z^p}{(1-z)^2} + Bh(z) * \frac{z^{p+1}}{(1-z)^2} \\
&= zh'(z) - (p-1)h(z) + B(zh'(z) - ph(z)) \\
&= (B+1)zh'(z) - (p - 1 + Bp)h(z) \\
&= (B+1)\left(\frac{pz^p}{(1-z)^{n+p}} + \frac{(n+p)z^{p+1}}{(1-z)^{n+p+1}} \right) - (p - 1 + Bp)\frac{z^p}{(1-z)^{n+p}} \\
&= \frac{z^p + (-1 + (B+1)(n+p))z^{p+1}}{(1-z)^{n+p+1}}.
\end{aligned}$$

Substituting the value of B from (4 1), simplifying, the result then follows from (4 6)

5. CONCLUDING COMMENTS

It would be possible to obtain additional information and solutions to extremal problems for the family $R_n(p, \alpha)$ if one gets $f_x(z)$ in (2 2) into closed form For example, using the closed form, viz $z^p/(1 - xz)^{2p}$, for $n = -p + 1$ and $\alpha = 0$, Hallenbeck and Livingston [11] found coefficient estimates for functions majorized by or subordinate to functions in S_p^* Note that the closed form of (2 2) even for the special case of $p = 1$ is an open problem (see, [10])

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