Internat. J. Math. & Math. Sci.
VOL. 19 NO. 1 (1996) 1-8

APPROXIMATION BY WEIGHTED MEANS OF WALSH-FOURIER SERIES

F. MÓRICZ*
Bolyai Institute, University of Szeged
Aradi vértanúk tere 1, 6720 Szeged, HUNGARY

B. E. RHOADES**
Department of Mathematics, Indiana University
Bloomington, Indiana 47405, U S A

(Received April 26, 1993 and in revised form February 20,1995)

ABSTRACT. We study the rate of approximation to functions in L^p and, in particular, in $\mathrm{Lip}(\alpha, p)$ by weighted means of their Walsh-Fourier series, where $\alpha > 0$ and $1 \le p \le \infty$. For the case $p = \infty, L^p$ is interpreted to be C_W, the collection of uniformly W continuous functions over the unit interval $[0,1)$. We also note that the weighted mean kernel is quasi-positive, under fairly general conditions.

1991. Mathematics Subject Classification. Primary 41A30.

KEY WORDS AND PHRASES: Walsh system, Walsh-Fourier series, weighted mean, rate of convergence, Lipschitz class, Walsh-Dirichlet kernel, Walsh-Fejér kernel, quasi-positive kernel.

1. INTRODUCTION.

We consider the Walsh orthonormal system $\{w_k(x) : k \ge 0\}$ defined on the unit interval $I := [0,1)$ using the Paley enumeration (see [4]).

Let $\mathcal{P}_n$ denote the collection of Walsh polynomials of order less than n; that is, functions of the form

$$P(x) := \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \ge 1$ and $\{a_k\}$ is any sequence of real numbers.

the approximation by Walsh polynomials in the norms of $L^p := L^p(I), 1 \le p < \infty$, and $C_W := C_W(I)$. The class C_W is the collection of all functions $f : I \to R$ that are uniformly continuous from the dyadic topology of I into the usual topology of R; briefly, uniformly $W-$continuous. The dyadic topology is generated by the collection of dyadic intervals of the form

$$I_m := [k2^{-m}, (k+1)2^{-m}), \qquad k = 0, 1, \dots, 2^m - 1; \qquad m = 0, 1 \dots$$

For C_W we shall write L^∞. Set

$$\|f\|_p := \left\{ \int_0^1 |f(x)|^p dx \right\}^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty := \sup\{|f(x)| : x \in I\}.$$

The best approximation of a function $f \in L^p, 1 \le p \le \infty$, by polynomials in $\mathcal{P}_n$ is defined by

$$E_n(f, L^p) := \inf_{P \in \mathcal{P}_n} \|f - P\|_p.$$

For $f \in L^p$, the modulus of continuity is defined by

$$\omega_p(f,\delta) := \sup\{\|f(\cdot\dot{+}t) - f(\cdot)\|_p : |t| < \delta\},$$

where $\delta > 0$ and $\dot{+}$ denotes dyadic addition. For $\alpha > 0$, the Lipschitz classes in L^p are defined by

$$\mathrm{Lip}(\alpha, p) := \{f \in L^p : \omega_p(f,\delta) = \mathcal{O}(\delta^\alpha) \quad \text{as} \quad \delta \to 0\}.$$

Concerning further properties and explanations, we refer the reader to [3], whose notations are adopted here as well.

2. MAIN RESULTS.

For $f \in L^1$, its Walsh–Fourier series is defined by

$$\sum_{k=0}^{\infty} a_k w_k(x), \quad \text{where} \quad a_k := \int_0^1 f(t) w_k(t) dt. \tag{2.1}$$

The nth partial sum of the series in (2.1) is

$$s_n(f,x) := \sum_{k=0}^{n-1} a_k w_k(x), \qquad n \geq 1,$$

which can also be written in the form

$$s_n(f,x) = \int_0^1 f(x\dot{+}t) D_n(t) dt,$$

where

$$D_n(t) := \sum_{k=0}^{n-1} w_k(t), \qquad n \geq 1,$$

is the Walsh-Dirichlet kernel of order n.

Throughout, $\{p_k : k \geq 1\}$ will denote a sequence of nonnegative numbers, with $p_1 > 0$. The weighted means for series (2.1) are defined by

$$t_n(f,x) := \frac{1}{P_n} \sum_{k=1}^{n} p_k s_k(f,x),$$

where

$$P_n := \sum_{k=1}^{n} p_k, \qquad n \geq 1.$$

We shall always assume that

$$\lim_{n\to\infty} P_n = \infty,$$

which is the condition for regularity.

The representation

$$t_n(f,x) = \int_0^1 f(x\dot{+}t) L_n(t) dt \tag{2.2}$$

plays a central role in the sequel, where

$$L_n(t) := \frac{1}{P_n} \sum_{k=1}^{n} p_k D_k(t), \qquad n \geq 1, \tag{2.3}$$

is the weighted mean kernel.

THEOREM 1. Let $f \in L^p, 1 \le p \le \infty, n := 2^m + k, 1 \le k \le 2^m, m \ge 1$.
(i) If $\{p_k\}$ is nondecreasing and satisfies the condition

$$\frac{np_n}{P_n} = \mathcal{O}(1), \tag{2.4}$$

then

$$\|t_n(f) - f\|_p \le \frac{3}{P_n} \sum_{j=0}^{m-1} 2^j p_{2^{j+1}-1} \omega_p(f, 2^{-j}) + \mathcal{O}(\omega_p(f, 2^{-m})). \tag{2.5}$$

(ii) If $\{p_k\}$ is nonincreasing, then

$$\|t_n(f) - f\|_p \le \frac{3}{P_n} \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}) + \mathcal{O}(\omega_p(f, 2^{-m})). \tag{2.6}$$

THEOREM 2. Let $f \in \mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \le p \le \infty$. Then for $\{p_k\}$ nondecreasing,

$$\|t_n(f) - f\|_p = \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ \mathcal{O}(n^{-1} \log n) & \text{if } \alpha = 1, \\ \mathcal{O}(n^{-1}) & \text{if } \alpha > 1; \end{cases} \tag{2.7}$$

for $\{p_k\}$ nonincreasing,

$$\|t_n(f) - f\|_p = \mathcal{O}\left(\frac{1}{P_n} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} p_{2^j} + 2^{-\alpha m} \right). \tag{2.8}$$

Given two sequences $\{p_k\}$ and $\{q_k\}$ of nonnegative numbers, we write $p_k \asymp q_k$ if there exist two positive constants C_1 and C_2 such that

$$C_1 q_k \le p_k \le C_2 q_k \quad \text{for all} \quad k \quad \text{large enough.}$$

We present two particular cases for nonincreasing $\{p_k\}$.
Case (i): $p_k \asymp (\log k)^{-\beta}$ for some $\beta > 0$. Then $P_n \asymp n(\log n)^{-\beta}$. It follows from (2.8)

$$\|t_n(f) - f\|_p = \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } 0 < \alpha < 1 \text{ and } \beta > 0, \\ \mathcal{O}(n^{-1} \log n) & \text{if } \alpha = 1 \text{ and } 0 < \beta < 1, \\ \mathcal{O}(n^{-1} \log n \log\log n) & \text{if } \alpha = \beta = 1, \\ \mathcal{O}(n^{-1} (\log n)^{\beta}) & \text{if } \alpha = 1 \text{ and } \beta = 1, \\ & \text{or if } \alpha > 1 \text{ and } \beta > 0. \end{cases}$$

Case (ii): $p_k \asymp k^{-\beta}$ for some $0 < \beta \le 1$. Then $P_n \asymp n^{1-\beta}$ if $0 < \beta < 1$ and $P_n \asymp \log n$ if $\beta = 1$. The case $\beta > 1$ is unimportant since $P_n = \mathcal{O}(1)$. By (2.8),

$$\|t_n(f) - f\|_p = \begin{cases} \mathcal{O}(n^{-\alpha}) & \text{if } \alpha + \beta < 1, \\ \mathcal{O}(n^{\beta-1} \log n + n^{-\alpha}) & \text{if } \alpha + \beta = 1, \\ \mathcal{O}(n^{\beta-1}) & \text{if } \alpha + \beta > 1 \text{ and } \beta > 1, \\ \mathcal{O}((\log n)^{-1}) & \text{if } \beta = 1, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

REMARK 1. The slower P_n tends to infinity, the worse is the rate of approximation.

REMARK 2. Watari [6] has shown that a function $f \in L^p$ belongs to $\mathrm{Lip}(\alpha, p)$ for some $\alpha > 0$ and $1 \le p \le \infty$ if and only if

$$E_n(f, L^p) = \mathcal{O}(n^{-\alpha}).$$

Thus, for $0 < \alpha < 1$, the rate of approximation to functions f in the class $\mathrm{Lip}(\alpha, p)$ by $t_n(f)$ is as good as the best approximation.

REMARK 3. For $\alpha > 1$, the rate of approximation by $t_n(f)$ in the class $\mathrm{Lip}(\alpha, p)$ cannot be improved too much.

THEOREM 3. If for some $f \in L^p, 1 \le p \le \infty$,

$$\|t_{2^m}(f) - f\|_p = o(P_{2^m}^{-1}) \quad \text{as} \quad m \to \infty, \tag{2.9}$$

then f is a constant.

If $p_k = 1$ for all k, then the $t_n(f, x)$ are the $(C, 1)$ - means (i.e., the first arithmetic means) of the series in (2.1). In this case, Theorem 2 was proved by Yano [8] for $0 < \alpha < 1$ and by Jastrebova [1] for $\alpha = 1$; Theorem 3 also reduces to a known result (see e.g. [5, p. 191]).

3. AUXILIARY RESULTS

Let

$$K_n(t) := \frac{1}{n} \sum_{k=1}^{n} D_k(t), \qquad n \ge 1 \tag{3.1}$$

by the Walsh-Fejer kernel.

LEMMA 1. (see [7]). Let $m \ge 0$ and $n \ge 1$. Then $K_{2^m}(t) \ge 0$ for each $t \in I$,

$$\int_0^1 |K_n(t)| dt \le 2, \qquad \text{and} \qquad \int_0^1 K_{2^m}(t) dt = 1.$$

LEMMA 2. Let $n := 2^m + k, 1 \le k \le 2^m$, and $m \ge 1$. Then for $L_n(t)$ defined in (2.3),

$$\begin{aligned} P_n L_n(t) = &- \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(p_{2^{j+1}-i} - p_{2^{j+1}-i-1}) K_i(t) \\ &- \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) 2^j p_{2^j} K_{2^j}(t) \\ &+ \sum_{j=0}^{m-1} (P_{2^{j+1}-1} - P_{2^j-1}) D_{2^{j+1}}(t) \\ &+ (P_n - P_{n-k-1}) D_{2^m}(t) + r_m(t) \sum_{i=1}^{k} p_{2^m+i} D_i(t), \end{aligned} \tag{3.2}$$

where the $r_j(t)$ are the Rademacher functions.

Proof. From (2.3)

$$\begin{aligned} P_n L_n(t) &= \sum_{i=1}^{2^m-1} p_i D_i(t) + \sum_{i=2^m}^{2^m+k} p_i D_i(t) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} p_{2^j+i} D_{2^j+i}(t) + \sum_{i=0}^{k} p_{2^m+i} D_{2^m+i}(t) \\ &= \sum_{j=0}^{m-1} \sum_{i=0}^{2^j-1} p_{2^j+i} (D_{2^j+i}(t) - D_{2^j+1}(t)) \\ &+ \sum_{j=0}^{m-1} D_{2^j+1}(t) \sum_{i=0}^{2^j-1} p_{2^j+i} + \sum_{i=0}^{k} p_{2^m+i} D_{2^m+i}(t). \end{aligned} \tag{3.3}$$

We will make use of formula (3.4) of [3]:

$$D_{2^j+1}(t) - D_{2^j+i}(t) = r_j(t)w_{2^j-1}(t)D_{2^j-i}(t), \qquad 0 \le i < 2^j,$$

and the formula in line 4 from below of [4, p. 46]:

$$D_{2^m+i}(t) = D_{2^m}(t) + r_m D_i(t), \qquad 1 \le i \le 2^m.$$

Substituting these into (3.3) yields

$$\begin{aligned} P_n L_n(t) = &- \sum_{j=0}^{m-1} r_j(t) w_{2^j-1}(t) \sum_{i=0}^{2^j-1} p_{2^j+i} D_{2^j-i}(t) \\ &+ \sum_{i=0}^{m-1} (P_{2^{j+1}} - P_{2^j-1}) D_{2^j-i}(t) \\ &+ (P_n - P_{n-k-1}) D_{2^m}(t) + r_m(t) \sum_{i=1}^{k} p_{2^m+i} D_i(t). \end{aligned}$$

Hence (3.2) follows, by noting that

$$D_i(t) = iK_i(t) - (i-1)K_{i-1}(t), \qquad i \ge 1, \;\; K_0(t) := 0,$$

(see (3.1)) and accordingly

$$\begin{aligned} \sum_{i=0}^{2^j-1} p_{2^j+i} D_{2^j-i}(t) &= \sum_{i=1}^{2^j} p_{2^{j+1}-i} D_i(t) \\ &= \sum_{i=1}^{2^j-1} i(p_{2^{j+1}-i} - p_{2^{j+1}-i-1}) K_i(t) + 2^j p_{2^j} K_{2^j}(t). \end{aligned}$$

Motivated by (3.2), we define a linear operator R_n by

$$R_n(t) := \frac{1}{P_n} \sum_{i=1}^{k} p_{2^m+i} D_i(t), \tag{3.4}$$

where $n := 2^m + k, 1 \le k \le 2^m$, and $m \ge 1$. A Sidon type inequality of [2] implies that R_n as well as the weighted mean kernel L_n defined in (2.3) are quasi-positive.

LEMMA 3. Let $\{p_k\}$ be a sequence of nonnegative numbers either nondecreasing and satisfying condition (2.4) or merely nonincreasing, and let R_n be defined by (3.4). Then there exists a constant C such that

$$I_n := \int_0^1 |R_n(t)| dt \le C, \qquad n \ge 1. \tag{3.5}$$

PROOF. By [2, Lemma 1 for $p = 2$],

$$I_n \le \frac{4k^{1/2}}{P_n} \left(\sum_{i=1}^{k} p_{2^m+i}^2 \right)^{1/2}$$

Due to monotonicity,

$$I_n \le \begin{cases} \frac{4kp_n}{P_n} \le \frac{2np_n}{P_n} & \text{if } \{p_k\} \text{ is nondecreasing,} \\ \frac{4kp_{2^m+1}}{P_n} \le 4 & \text{if } \{p_k\} \text{ is nonincreasing.} \end{cases}$$

By (2.4), hence (3.5) follows.

LEMMA 4 (see [3]). If $g \in \mathcal{P}_{2^m}, f \in L^p$, where $m \geq 0$ and $1 \leq p \leq \infty$, then

$$\left\| \int_0^1 r_m(t) g(t)[f(\cdot \dot{+} t) - f(\cdot)]dt \right\|_p \leq 2^{-1}\omega_p(f, 2^{-m})\|g\|_1.$$

4. PROOFS OF THEOREMS 1-3.

PROOF OF THEOREM 1. We shall present the details only for $1 \leq p < \infty$. By (2.2), (3.2), and the usual Minkowski inequality,

$$\begin{aligned} P_n\|t_n(f) - f\|_p &= \left\{ \int_0^1 \left| \int_0^1 P_n L_n(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &\leq \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t) g_j(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &+ \sum_{j=0}^{m-1} \left\{ \int_0^1 \left| \int_0^1 r_j(t) h_j(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &+ \sum_{j=0}^{m-1} (P_{2^{j+1}-1} - P_{2^j-1}) \left\{ \int_0^1 \left| \int_0^1 D_{2^{j+1}}(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &+ (P_n - P_{n-k-1}) \left\{ \int_0^1 \left| \int_0^1 D_{2^m}(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &+ P_n \left\{ \int_0^1 \left| \int_0^1 r_m(t) R_n(t)[f(x \dot{+} t) - f(x)]dt \right|^p dx \right\}^{1/p} \\ &=: I_{1n} + I_{2n} + I_{3n} + I_{4n} + I_{5n}, \qquad \text{say,} \end{aligned} \tag{4.1}$$

where

$$g_j(t) := w_{2^j-1}(t) \sum_{i=1}^{2^j-1} i(p_{2^{j+1}-i} - p_{2^{j+1}-i-1})K_i(t),$$

$$h_j(t) := 2^j p_{2^j} w_{2^j-1}(t) K_{2^j}(t).$$

From Lemma 1,

$$\begin{aligned} \int_0^1 |g_j(t)|dt &\leq \sum_{i=1}^{2^j-1} i|p_{2^{j+1}-i} - p_{2^{j+1}-i-1}| \int_0^1 |K_i(t)|dt \\ &\leq 2 \sum_{r=2^j+1}^{2^{j+1}-1} (2^{j+1} - r)|p_r - p_{r-1}| =: A_j, \qquad \text{say,} \end{aligned}$$

If $\{p_k\}$ is nondecreasing, we have

$$\begin{aligned} A_j &= 2^{j+2} \sum_{r=2^j+1}^{2^{j+1}-1} (p_r - p_{r-1}) - 2 \sum_{r=2^j+1}^{2^{j+1}-1} (rp_r - (r-1)p_{r-1}) + 2 \sum_{r=2^j+1}^{2^{j+1}-1} p_{r-1} \\ &= 2^{j+2}(p_{2^{j+1}-1} - p_{2^j}) - 2[(2^{j+1} - 1)p_{2^{j+1}-1} - 2^j p_{2^j}] + 2(P_{2^{j+1}-2} - P_{2^j-1}) \\ &< 2(P_{2^{j+1}-1} - P_{2^j-1}) \leq 2^{j+1} p_{2^{j+1}-1}. \end{aligned}$$

If $\{p_k\}$ is nonincreasing, we have

$$A_j = 2^{j+2} \sum_{r=2^j+1}^{2^{j+1}-1} (p_{r-1} - p_r) + 2 \sum_{r=2^j+1}^{2^{j+1}-1} (rp_r - (r-1)p_{r-1}) - 2 \sum_{r=2^j+1}^{2^{j+1}-1} p_{r-1} < 2^{j+1} p_{2^j}.$$

Thus, by Lemma 4, for $\{p_k\}$ nondecreasing,

$$I_{1n} \le \sum_{j=0}^{m-1} 2^j p_{2^{j+1}-1} \omega_p(f, 2^{-j}), \tag{4.2}$$

and for $\{p_k\}$ nonincreasing,

$$I_{1n} \le \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}). \tag{4.3}$$

Again, by Lemmas 1 and 4,

$$\int_0^1 |h_j(t)|dt \le 2^j p_{2^j} \int_0^1 K_{2^j}(t)dt = 2^j p_{2^j},$$

whence

$$I_{2n} \le 2^{-1} \sum_{j=0}^{m-1} 2^j p_{2^j} \omega_p(f, 2^{-j}). \tag{4.4}$$

Since

$$D_{2^m}(t) = \begin{cases} 2^m & \text{if } t \in [0, 2^{-m}), \\ 0 & \text{if } t \in [2^{-m}, 1) \end{cases}$$

(see, e.g., [5, p.7]), by the generalized Minkowski inequality,

$$\begin{aligned} I_{3n} &\le \sum_{j=0}^{m-1} (P_{2^{j+1}-1} - P_{2^j-1}) \int_0^1 D_{2^{j+1}}(t) \left\{ \int_0^1 |f(x \dot{+} t) - f(x)|^p dx \right\}^{1/p} dt \\ &\le \sum_{j=0}^{m-1} (P_{2^{j+1}-1} - P_{2^j-1}) \omega_p(f, 2^{-j-1}) \end{aligned} \tag{4.5}$$

and

$$I_{4n} \le (P_n - P_{n-k-1}) \omega_p(f, 2^{-m}). \tag{4.6}$$

Note that

$$P_{2^{j+1}-1} - P_{2^j-1} \le \begin{cases} 2^j p_{2^{j+1}-1} & \text{if } \{p_k\} \text{ is nondecreasing,} \\ 2^j p_{2^j} & \text{if } \{p_k\} \text{ is nonincreasing.} \end{cases} \tag{4.7}$$

By Lemmas 3 and 4,

$$\begin{aligned} I_{5n} &\le 2^{-1} P_n \omega_p(f, 2^{-m}) \int_0^1 |R_n(t)|dt \\ &\le 2^{-1} C P_n \omega_p(f, 2^{-m}). \end{aligned} \tag{4.8}$$

Combining (4.1) – (4.8) yields (2.5) and (2.6).

PROOF OF THEOREM 2. For $\{p_k\}$ nondecreasing we have, from (2.4) and (2.5),

$$\begin{aligned} \|t_n(f) - f\|_p &= \mathcal{O}\left(\frac{p_{2^m}}{P_n} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-\alpha m} \right) \\ &= \mathcal{O}\left(2^{-m} \sum_{j=0}^{m-1} 2^{(1-\alpha)j} + 2^{-\alpha m} \right). \end{aligned}$$

Hence (2.7) follows easily.

For $\{p_k\}$ nonincreasing, (2.8) is immediate.

PROOF OF THEOREM 3. By a theorem of Watari [6]

$$\|s_{2^m}(f) - f\|_p \le 2E_{2^m}(f, L^p).$$

Thus, from (2.9),

$$\|s_{2^m}(f) - f\|_p = o(P_{2^m}^{-1}). \tag{4.9}$$

Clearly,

$$\begin{aligned} P_{2^m}\{s_{2^m}(f,x) - t_{2^m}(f,x)\} &= \sum_{k=1}^{2^m} p_k\{s_{2^m}(f,x) - s_k(f,x)\} \\ &= \sum_{k=1}^{2^m-1} p_k \sum_{i=k}^{2^m-1} a_i w_i(x) \\ &= \sum_{i=1}^{2^m-1} P_i a_i w_i(x). \end{aligned}$$

Now (2.9) and (4.9) imply

$$\lim_{m\to\infty} \Big\| \sum_{i=1}^{2^m-1} P_i a_i w_i(x) \Big\|_p = 0.$$

Since the L^p-norm dominates the L^1-norm for $p > 1$, it follows that for $j \ge 1$,

$$\begin{aligned} |P_j a_j| &= \lim_{m\to\infty} \Big| \int_0^1 w_j(x) \sum_{i=1}^{2^m-1} P_i a_i w_i(x) dx \Big| \\ &\le \lim_{m\to\infty} \Big\| \sum_{i=1}^{2^m-1} P_i a_i w_i(x) \Big\|_1 = 0. \end{aligned}$$

Hence we conclude that $a_j = 0$ for all $j \ge 1$. Therefore, $f = a_0$, a constant.

---

* This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant#234.

** This research was completed while the author was a Fulbright scholar at the Bolyai Institute, University of Szeged, Hungary, during the fall semester in the academic year 1992/93.

REFERENCES

1. JASTREBOVA, M.A., On approximation of functions satisfying the Lipschitz condition by arithmetic means of their Walsh–Fourier series, *Mat. Sb.* **71** (1966), 214–226.
2. MÓRICZ, F. and SCHIPP, F., On the integrability and L^1–convergence of Walsh series with coefficients of bounded variation, *J. Math. Anal. Appl.* **146** (1990), 99–109.
3. MÓRICZ, F. and SIDDIQI, A.H., Approximation by Nörlund means of Walsh-Fourier series, *J. Approx. Theory* **70** (1992), 375–389.
4. PALEY, R.E.A.C., A remarkable system of orthogonal functions, *Proc. London, Math. Soc.* **34** (1932), 241-279.
5. SCHIPP, F., WADE, W.R., and SIMON, P., "Walsh Series. An introduction to Dyadic Harmonic Analysis", Adadémiai Kiadó, Budapest, 1990.
6. WATARI, C., Best approximation by Walsh polynomials, *Tôhoku Math. J.* **15** (1963), 1–5.
7. YANO, SH., On Walsh series, *Tôhoku Math. J.* **3** (1951), 223–242.
8. YANO, SH., On approximation by Walsh functions, *Proc. Amer. Math. Soc.* **2** (1951), 962-967.