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A NOTE ON METRIC PRESERVING FUNCTIONS

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ABSTRACT The purpose of this note is to study modifications of the Euclidean metric on $\mathbb{R}$ with the following property There is a monotone sequence of closed balls with empty intersection

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1 INTRODUCTION

DEFINITION 1 We call a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ metric preserving iff $f(d) : M \times M \longrightarrow \mathbb{R}^+$ is a metric for every metric $d : M \times M \longrightarrow \mathbb{R}^+$, where (M, d) is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathcal{M}$ the set of all metric preserving functions (See Borsík [1], Borsík [2], Terpe [3])

The following result is well known (see Borsík [1], Terpe [3])

PROPOSITION 1 If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a concave function vanishing exactly at origin then it is metric preserving

It is well known that there is a complete metric space with the following property

There is a monotone sequence of closed balls with empty intersection. (1.1)

In Juza [4] such a metric space (which is not discrete) has been constructed by a modification of the Euclidean metric on $\mathbb{R}$, where $\mathbb{R}$ denotes the set of reals

For each $f \in \mathcal{M}$ denote by d_f the metric on $\mathbb{R}$ defined as follows

$$d_f(x, y) = f(|x - y|) \text{ for each } x, y \in \mathbb{R}$$

We call d_f a *modification* of the Euclidean metric on $\mathbb{R}$ (See Terpe [3])

EXAMPLE 1 Define $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows

$$f(x) = x, \text{ if } x \leq 2, f(x) = 1 + \frac{1}{x - 1} \text{ if } x > 2$$

In Juza [4] it is shown that $f \in \mathcal{M}$ and the metric space $(\mathbb{R}, d_f)$ has the property (1 1) The proof of (1 1) is based on the following property of the metric space $(\mathbb{R}, d_f)$:

For each compact set K there is a closed ball S and there is a compact set L such that $K \subseteq \mathbb{R} - S \subseteq L$ (1 2)

2 MAIN RESULTS

THEOREM 1 Let $f \in \mathcal{M}$. Suppose that there are $g, h: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that g, h are nonincreasing, and

they are not constant in each neighborhood of the point $+\infty$, (2.1)

$g(x) \leq f(x) \leq h(x)$ in some neighborhood of the point $+\infty$, (2.2)

$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x)$ (2.3)

Then the metric space $(\mathbb{R}, d_f)$ has the property (1.2)

PROOF Let $m \in \mathbb{N}$ be such that $g(x) \leq f(x) \leq h(x)$ for each $x \in [m, \infty)$. Put $d = \lim_{x \to +\infty} g(x)$. Evidently $d = \lim_{x \to +\infty} f(x) > 0$. Let K be a compact set. Put $s = \inf K - m$, $r = \sup K - s$, $\varepsilon = g(r)$. Since g is not constant on $(r, +\infty)$, there is $\xi > r$ such that $g(\xi) \neq \varepsilon$. Since g is nonincreasing, we have $\varepsilon \neq g(\xi) \leq g(r) = \varepsilon$. Therefore $g(\xi) < \varepsilon$. Since g is nonincreasing for each $x \geq \xi$ we get $g(x) \leq g(\xi)$. Thus $d = \lim_{x \to +\infty} g(x) \leq g(\xi) < \varepsilon$. Let $x \in [m, r]$. Then $f(x) \geq g(x) \geq g(r) = \varepsilon$. Therefore

$$\forall x \in [m, r] \quad f(x) \geq \varepsilon \tag{2.4}$$

Let $\delta \in (d, \varepsilon)$. Since $\lim_{x \to +\infty} h(x) = d < \varepsilon$, there is $t > r$ such that $h(t) < \delta$. Let $x \geq t$. Then $f(x) \leq h(x) \leq h(t) < \delta$. Thus

$$\forall x \in [t, \infty) \quad f(x) < \delta \tag{2.5}$$

Let S be a closed ball with the centre s and the radius δ. Put $L = [s - t, s + t]$. Now, we shall show that $K \subseteq \mathbb{R} - S$. Let $u \in K$. Then $|u - s| = u - s \in [m, r]$, and by (2.4) we get $d_f(u, s) = f(|u - s|) \geq \varepsilon > \delta$. Therefore $u \notin S$. Finally, we shall show that $\mathbb{R} - S \subseteq L$. Let $v \in \mathbb{R} - S$. Then $f(|v - s|) = d_f(v, s) > \delta$. By (2.5) we have $|v - s| < t$. Therefore $v \in L$.

EXAMPLE 2 Define $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows

$$f(x) = x, \text{ if } x \in [0, 1), \quad f(x) = \frac{1 + x + \sin^2(x - 1)}{2x}, \text{ if } x \in [1, \infty)$$

It is not difficult to verify that $f \in \mathcal{M}$ and the metric space $(\mathbb{R}, d_f)$ has the property (1.2) (which yields also the property (1.1)), however f is not monotone on every neighborhood of the point $+\infty$

EXAMPLE 3 Define $f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows, $f(x) = x$, if $x \in [0, 1]$, and $f(x) = \frac{1}{2}(x - 3n + 1 - |x - 3n + 1| + |x - 3n + \frac{1}{2} + \frac{1}{n+2}| + |x - 3n - \frac{1}{2} - \frac{1}{n+2}|)$, if $x \in (3n - 2, 3n + 1]$ $(n = 1, 2, 3, \ldots)$. It is not difficult to verify that $f \in \mathcal{M}$ and $(\mathbb{R}, d_f)$ is a metric space with the property (1.1), which has not the property (1.2). Indeed, the intersection of the sequence of closed balls $\{S_n\}_{n=1}^{\infty}$ (where S_n has the centre $x_n = 3 \cdot (2^{n-1} - 1)$ and the radius $\varepsilon_n = \frac{1}{2} + \frac{1}{2^n+1}$) is empty.

A characterization of metric preserving functions f such that the space $(\mathbb{R}, d_f)$ has the property (1.1) remains an open question.

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REFERENCES

1. BORSÍK, J. and DOBOŠ, J. Functions whose composition with every metric is a metric, Math. Slovaca 31(1981), 3-12 (in Russian).
2. BORSÍK, J. and DOBOŠ, J. On metric preserving functions, Real Analysis Exchange 13 (1987-88), 285-293
3. TERPE, F. Metric preservice fucntiosn, Proc. Conf. Topology and Measure IV, Greifswald, 1984, 189-197.
4. Juza, M. Poznámka o úplných metrických prostorech, Matematicko-fyzikálny časopis SAV, 6(1956), 143-148