FRACTIONAL INTEGRATION OPERATOR OF VARIABLE ORDER IN THE HOLDER SPACES $H^{\lambda(x)}$

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ABSTRACT. The fractional integrals $I_{a+}^{\alpha(x)}\varphi$ of variable order $\alpha(x)$ are considered. A theorem on mapping properties of $I_{a+}^{\alpha(x)}$ in Holder-type spaces $H^{\lambda(x)}$ is proved, this being a generalization of the well known Hardy-Littlewood theorem.

KEYWORDS: Fractional integration, variable fractional order, mapping properties, Holder continuous functions, Hardy-Littlewood theorem.

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1. INTRODUCTION

In the paper [1] the authors introduced and investigated the fractional integrals

$$I_{a+}^{\alpha(x)}\varphi = \frac{1}{\Gamma[\alpha(x)]} \int_{a}^{x} \varphi(t)(x-t)^{\alpha(x)-1} \, dt \tag{1}$$

of variable order $\alpha(x) > 0$, and considered the corresponding versions of fractional differentiation as well.

In this paper we prove the theorem on the behaviour of the operator $I_{a+}^{\alpha(x)}$ in the generalized Holder spaces $H^{\lambda(x)}$ the order of which also

depends on the point x . This is a generalization of the Hardy — Littlewood

theorem, well — known in the case of constant orders $\alpha(x)=\alpha=$const   and

$\lambda(x)=\lambda=$const (Hardy and Littlewood [2]; see also Samko et al [3], p. 53-54).

   Our interest in integration and differentiation of a variable order is

motivated not only by the desire to generalize the classical notion, but by

some far reaching goals as well. There is the well known theory of fractional

Sobolev type spaces , see its elements e.g. in [3], sections 26-27. These

spaces consist of functions whose smoothness property can be expressed either

globally or locally in terms of the existence of fractional derivatives. The

smoothness property of a function may, however, vary from point to point. The

construction of the corresponding Sobolev type spaces is an open question. The

notion introduced in (1) is the appropriate tool for this purpose. In this

paper we deal only with the question of improving the smoothness property,

expressed in terms of the Holder type condition, by the operator (1), and the

theorem proved may be considered as a starting point for further

investigations of functions with varying order of smoothness. In Section 1 we

give all required definitions and some auxilliary lemmas, while Section 2

contains the statement and the proof of the main result.

   In what follows the letter c may denote different positive constants.

2. PRELIMINARIES

   Let $\Omega = [a,b]$, $-\infty < a < b < \infty$. The following is a generalization of

the Holder space H^{λ} , $0 < \lambda \le 1$.

   DEFINITION 1. We say that $f(x) \in H^{\lambda(x)}(\Omega)$ where $\lambda(x)$ is a positive

(not necessarily continuous) function, $0 < \lambda(x) \le 1$, if

$$|f(x+h)-f(x)| \le c|h|^{\lambda(x)} \tag{2}$$

for all x , $x+h \in [a,b]$.

   It is easily seen that (2) implies that

$$|f(x+h)-f(x)| \le c|h|^{\lambda(x+h)}$$

So, it is not difficult to show that the definition of the class $H^{\lambda(x)}(\Omega)$ by

(2) is equivalent to the definition by means of the following symmetrical

inequality

$$|f(x_1)-f(x_2)| \leq c|x_1-x_2|^{\max\{\lambda(x_1),\lambda(x_2)\}} \quad . \tag{3}$$

It is easily seen that $H^{\lambda(x)}(\Omega)$ is a ring with respect to the usual multiplication. It is a Banach space with respect to the norm

$$\|f\|_{H^{\lambda(x)}} = \sup_{x\in\Omega} \sup_{\substack{|h|\leq 1 \\ h+x\in\Omega}} \frac{|f(x+h)-f(x)|}{|h|^{\lambda(x)}} \simeq$$

$$\sup_{x_1,x_2\in\Omega} \frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^{\max\{\lambda(x_1),\lambda(x_2)\}}} \tag{4}$$

where $\simeq$ denotes the equivalence: $f \simeq g \Leftrightarrow c_1 f \leq g \leq c_2 f$, $c_1 > 0$, $c_2 > 0$.

Generalizing Definition 1 we give the following

DEFINITION 2. We say that $f(x) \in H^{\lambda(x),\mu(x)}(\Omega)$ where $\lambda(x)$ and $\mu(x)$ are given functions, $0 < \lambda(x) \leq 1$, $-\infty < \mu(x) < \infty$, if

$$|f(x+h)-f(x)| \leq c|h|^{\lambda(x)} \left(\ln \frac{1}{|h|}\right)^{\mu(x)} , \quad |h| < \frac{1}{2} ,$$

under assumption that $x, x+h \in \Omega$.

We shall need the following auxilliary assertions.

LEMMA 1. Let the function $\lambda(x) \in C(\Omega)$ satisfy the condition

$$|\lambda(x+h)-\lambda(x)| \leq \frac{A}{1+\ln\frac{b-a}{h}} , \quad A = \text{const} > 0 \tag{5}$$

for all x, $x+h \in \Omega$. Then the function

$$g_{s,t}(x) = (x-a)^{\lambda[t+s(x-a)]-\lambda(t)} , \tag{6}$$

where $x,t,t+s(x-a) \in \Omega$, $s \in [0,1]$, is bounded from zero and infinity:

$$0 < d^{-A} \leq g_{s,t}(x) \leq d^A < \infty \tag{7}$$

with a constant $d = \max\{e,b-a,1/(b-a)\}$ not depending on s,t and x.

PROOF. Since

$$\ln g_{s,t}(x) = \{\lambda[t+s(x-a)] - \lambda(t)\} \ln(x-a), \quad \text{by (5) we have}$$

$$|\ln g_{s,t}(x)| \le \frac{A|\ln(x-a)|}{1+\ln\dfrac{b-a}{s(x-a)}} \le \frac{A|\ln(x-a)|}{1+\ln\dfrac{b-a}{x-a}}$$

Simple calculations show that the maximum of the right-hand side is A $\max\{1,|\ln(b-a)|\}$. Really, let $f(y) = A\,|y|(1+c-y)^{-1}$, where $-\alpha \quad y \le c = \ln(b-a)$. Suppose $b-a \ge 1$ first. For $y>0$ we have $f(y) = Ay(1+c-y)^{-1} \le Ay \le A\ln(b-a)$. If $y<0$, then $f(y) = A|y|(1+c+|y|)^{-1} \le A$. Therefore,

$$f(y) \le A \max \{ 1, \ln(b-a) \}$$

in the case $c = \ln(b-a) > 0$. Let now $c < 0$. Then $y < 0$ and $f(y) = A - A(1+c)(1+c-y)^{-1}$ if $1+c \ge 0$. If $1+c < 0$, we have $f'(y) = -A(1+c)(1+c-y)^{-2} > 0$, so that $f(y) \le f(c)$ for $y \le c$ which gives $f(y) \le A|c| = A\,|\ln(b-a)|$. So,

$$f(y) \le A \max \{ 1, |\ln(b-a)| \}$$

in all cases. Therefore, $|\ln_{s,t} g(x)| \le A \max\{1,|\ln(b-a)|\}$, whence (6) follows.

LEMMA 2. For any function $\alpha(x)$ such that $0 \le \alpha(x) \le 1$, the inequality

$$\left|(x+h)^{\alpha(x)} - x^{\alpha(x)}\right| \le h^{\alpha(x)}, \ h>0, \ x>0 \tag{8}$$

holds.

PROOF. By dividing all members in (8) by $h^{\alpha(x)}$, we see that the inequality (8) is equivalent to the inequality $f(\zeta) \le 1$ with $f(\zeta) = (1+\zeta)^{\alpha(x)} - \zeta^{\alpha(x)} > 0$ for all $\zeta > 0$, which is evident because $f'(\zeta) \le 0$ and $f(0) = 1$.

LEMMA 3. Let $0 < \alpha \le 1$ and $-\alpha < \lambda < 1 - \alpha$. Then

$$\int_0^\alpha t^\lambda [t^{\alpha-1} - (t+1)^{\alpha-1}]\, dt = \frac{\sin \alpha\pi}{\sin (\alpha+\lambda)\pi}\, B(\alpha, 1+\lambda) \tag{9}$$

PROOF. Let $A(\lambda,\alpha)$ denote the left-hand side in (9) . After the substitution $t+1 = S^{-1}$ we have

$$A(\lambda,\alpha) = \int_0^1 (1-s)^\lambda \ \frac{(1-s)^{\alpha-1}-1}{s^{1+\lambda+\alpha}} \ ds \qquad (10)$$

Hence

$$A(\lambda,\alpha) = \int_0^1 (1-s)^\lambda \ \frac{(1-s)^{\alpha-1}+(1-s)^\mu}{s^{1+\lambda+\alpha}} \ ds + \int_0^1 (1-s)^\lambda \ \frac{(1-s)^\mu-1}{s^{1+\lambda+\alpha}} \ ds$$

with μ not determined as yet. Hence, by (10)

$$A(\lambda,\alpha) = A(\lambda+\mu,\alpha-\mu) + \int_0^1 (1-s)^\lambda \ \frac{(1-s)^\mu-1}{s^{1+\lambda+\alpha}} \ ds$$

The second integral here is evaluated by means of analytical continuation with respect to μ :

$$\int_0^1 (1-s)^\lambda \ \frac{(1-s)^\mu-1}{s^{1+\lambda+\alpha}} \ ds \ = \ B(\lambda+\mu+1, \ -\lambda-\alpha) - B(\lambda+1, \ -\lambda-\alpha)$$

under the appropriate conditions on the parameters μ and λ . So,

$$A(\lambda,\alpha) = A(\lambda+\mu,\alpha-\mu) + \Gamma(-\lambda-\alpha) \left[\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\mu+1-\alpha)} - \frac{\Gamma(\lambda+1)}{\Gamma(1-\alpha)}\right]$$

Simple calculations show that $A(0,\alpha) = 1/\alpha$. So, choosing $\mu=-\lambda$ we have

$$A(\lambda,\alpha) = \frac{1}{\lambda+\alpha} + \Gamma(-\lambda-\alpha) \left[\frac{1}{\Gamma(1-\alpha-\lambda)} - \frac{\Gamma(\lambda+1)}{\Gamma(1-\alpha)}\right]$$

which gives (9) after easy calculations.

3. THE MAIN THEOREM

Considering the fractional integral $I_{a+}^{\alpha(x)}\varphi$, defined in (1), of the function $\varphi(x) \in H^{\lambda(x)}$, we shall assume the following conditions on $\alpha(x)$ and $\lambda(x)$ to be satisfied:

i) $0 < \alpha(x) \leq 1$ with $m = \inf\limits_{x \in \Omega} \alpha(x) > 0$

and $\alpha(x) \in H^{\mu(x)}(\Omega)$, $0 < \delta \leq \mu(x) \leq 1$;

ii) $0 < \lambda(x) \leq 1$ and $\lambda(x)$ satisfies the condition (5);

iii) $\alpha(x) + \lambda(x) \leq 1$.

THEOREM. Let the conditions i) – iii) be satisfied and let $\varphi(x) \in H^{\lambda(x)}(\Omega)$. Then $I_{a+}^{\alpha(x)}\varphi$ has the form

$$I_{a+}^{\alpha(x)}\varphi = \frac{\varphi(a)}{\Gamma[1+\alpha(x)]} (x-a)^{\alpha(x)} + f(x) \qquad (11)$$

where

$$f(x) \in H^{\gamma(x)}(\Omega) , \ \gamma(x) = \min\{\lambda(x)+\alpha(x), \ \mu(x)\} \qquad (12)$$

if $\max\limits_{x \in \Omega}[\lambda(x)+\alpha(x)] < 1$ and

$$f(x) \in H^{\gamma(x),1}(\Omega) \qquad (13)$$

if $\max\limits_{x \in \Omega}[\lambda(x)+\alpha(x)] = 1$. In both cases

$$|f(x)| \leq c(x-a)^{\lambda(x)+\alpha(x)} \leq c(x-a)^{\lambda(a)+\alpha(a)} \qquad (14)$$

REMARK. The assertion (13) can be exactified:

$$|f(x+h)-f(x)| \leq \frac{c}{1-\alpha(x)-\lambda(x)} \ |h|^{\gamma(x)} \qquad (15)$$

for all $x \in \Omega$ such that $\alpha(x) + \lambda(x) < 1$ and

$$|f(x+h)-f(x)| \leq c|h|^{\gamma(x)} \ln \frac{1}{|h|} , \ |h| < 1 \qquad (16)$$

for those x which give the equality $\alpha(x) + \lambda(x) = 1$, c being a positive constant not depending on x and h (see the proof of the theorem).

PROOF OF THE THEOREM. From the conditions i) and ii) it follows that $\gamma(x) =: \frac{1}{\Gamma[\alpha(x)]} \in H^{\mu(x)}(\Omega)$. Really, $|\gamma(x+h)-\gamma(x)| \leq |\alpha(x+h) - \alpha(x)|$

$\max\limits_{a \leq x \leq b} (|\Gamma'(x)/\Gamma(x)|) \leq c \ |h|^{\mu(x)}$. So, it is sufficient to consider the

integral

$$g(x) = \int_a^x (x-t)^{\alpha(x)-1} \varphi(t) dt$$

we have to prove that

$$g(x) = \frac{\varphi(a)}{\alpha(x)} (x-a)^{\alpha(x)} + f_0(x) \qquad (17)$$

where

$$f_0(x) \in H^{\lambda(x)+\alpha(x)} \quad \text{or} \quad f_0(x) \in H^{\lambda(x)+\alpha(x),1} \qquad (18)$$

if $\max[\lambda(x)+\alpha(x)]<1$ or $\max[\lambda(x)+\alpha(x)]=1$, respectively (with the

exactification (15)-(16), if we will). The derivation itself of the equality

(17) with the function

$$f_0(x) = \int_a^x \frac{\varphi(t)-\varphi(a)}{(x-t)^{1-\alpha(x)}} \ dt$$

is obvious. For the function $f_0(x)$ we prove first the estimate (14). We have

$$|f_0(x)| \leq \|\varphi\|_{H^{\lambda(x)}} \int_a^x (x-t)^{\alpha(x)-1} (t-a)^{\lambda(t)} dt$$

$$= c(x-a)^{\alpha(x)+\lambda(a)} \int_0^1 (1-s)^{\alpha(x)-1} s^{\lambda[a+s(x-a)]} \ g_{s,a}(x) \ ds$$

where $g_{s,a}(x)$ - is the function (6). By lemma 1 we have

$$|f_0(x)| \leq c_1 (x-a)^{\alpha(x)+\lambda(a)} \int_0^1 (1-s)^{m-1} \ ds = c_2 (x-a)^{\alpha(x)+\lambda(a)}$$

Hence, to obtain the estimate (14), it remains to observe that

$$(x-a)^{\alpha(x)+\lambda(a)} \leq c(x-a)^{\alpha(x)+\lambda(x)} \leq c_1 (x-a)^{\alpha(a)+\lambda(a)} \leq c_2 (x-a)^{\alpha(x)+\lambda(x)}$$

which follows from the lemma 1 (we remark that the condition (5) for

$\lambda(x)+\alpha(x)$ is fulfilled because it is satisfied for $\lambda(x)$ by the assumption in ii) and for $\alpha(x)$ by the assumption in i)).Thus, (14) is proved.

To prove the statements (18) we consider the difference $f_0(x+h) - f_0(x)$ taking h positive. (If $h < 0$, by denoting $x+h = x_1$, $x = x_1+(-h)$ we reduce the consideration to the case of positive increment). We represent this difference as

$$f_0(x+h) - f_0(x) = \int_{-h}^{x-a} \frac{\varphi_0(x-t)dt}{(t+h)^{1-\alpha(x+h)}} - \int_0^{x-a} \frac{\varphi_0(x-t)dt}{t^{1-\alpha(x)}}$$

$$= \int_{-h}^{x-a} \varphi_0(x-t) \left[(t+h)^{\alpha(x+h)-1} - (t+h)^{\alpha(x)-1}\right] dt$$

$$+ \left[\int_{-h}^{x-a} \frac{\varphi_0(x-t)dt}{(t+h)^{1-\alpha(x)}} - \int_0^{x-a} \frac{\varphi_0(x-t)dt}{t^{1-\alpha(x)}}\right] = J_0+J \quad, \qquad (19)$$

with $\varphi_0(t) = \varphi(t) - \varphi(a)$. We estimate J_0 first. We have

$$|J_0| \le \max |\varphi_0(t)| \int_{-h}^{x-a} \left|(t+h)^{\alpha(x+h)-1} - (t+h)^{\alpha(x)-1}\right| dt$$

$$= c \int_0^{x+h-a} \left|t^{\alpha(x+h)-1} - t^{\alpha(x)-1}\right| dt$$

Since $t^u-t^v = t^\xi \ln (u-v)$ with ξ between u and v , we obtain

$$|J_0| \le c|\alpha(x+h)-\alpha(x)| \int_0^{x+h-a} t^{\xi-1}|\ln t| dt$$

with ξ between $\alpha(x+h)$ and $\alpha(x)$ so that $\xi \ge m > 0$. Since $t^{\xi-1} \le At^{m-1}$ with $A = \max\{1,(b-a)^{1-m}\}$, we have

$$|J_0| \le cAh^{\mu(x)} \int_0^{2(b-a)} t^{m-1}|\ln t|\,dt = c_1 h^{\mu(x)} \qquad (20)$$

As regards the term J in (19) it should be decomposed similarly to the proof of the Hardy – Littlewood theorem for the case $\alpha(x) = \alpha = $ const , $\lambda(x) = \lambda = $ const , see Samko et al [3]. We have

$$J = J_1 + J_2 + J_3$$

with

$$J_1 = \varphi_0(x) \int_{x-a}^{x-a+h} (t+h)^{\alpha(x)-1}\,dt \;, \qquad J_2 = \int_{-h}^{0} \frac{\varphi_0(x-t)-\varphi_0(x)}{(t+h)^{1-\alpha(x)}}\,dt \;,$$

$$J_3 = \int_0^{x-a} [(t+h)^{\alpha(x)-1} - t^{\alpha(x)-1}]\,[\varphi_0(x-t) - \varphi_0(x)]\,dt \;.$$

The estimate of J_1. In the case $h \le x-a$, for the term

$$J_1 = \frac{\varphi_0(x)}{\alpha(x)} \left[(x-a+h)^{\alpha(x)} - (x-a)^{\alpha(x)}\right]$$

we use the inequality

$$|\varphi_0(x)| \le \|\varphi\|_{H^{\lambda(x)}} (x-a)^{\lambda(x)} \qquad (21)$$

and the inequality $(1+t)^{\beta}-1 \le \beta t$, with $0 \le \beta \le 1$ and $t > 0$. We have

$$|J_1| \le c\frac{(x-a)^{\lambda(x)+\alpha(x)}}{\alpha(x)} \left|\left(1 + \frac{h}{x-a}\right)^{\alpha(x)} - 1\right|$$

$$\qquad (22)$$

$$\le c(x-a)^{\lambda(x)+\alpha(x)} \frac{h}{x-a} = c(x-a)^{\lambda(x)+\alpha(x)-1} h \le c\,h^{\lambda(x)+\alpha(x)} \;.$$

If $x-a \leq h$ we use (21) again and have

$$|J_1| \leq c \ \frac{(x-a)^{\lambda(x)}}{\alpha(x)} \left| (x+h-a)^{\alpha(x)} - (x-a)^{\alpha(x)} \right| \ .$$

Hence, by (8)

$$|J_1| \leq \frac{c}{m} (x-a)^{\lambda(x)} h^{\alpha(x)} \leq \frac{c}{m} h^{\lambda(x)+\alpha(x)} \ .$$

The estimate of J_2. The estimate of J_2 is completely like in the case of constant order:

$$|J_2| \leq c \int_{-h}^{0} \frac{|t|^{\lambda(x)} \ dt}{(t+h)^{1-\alpha(x)}} = c_1 h^{\lambda(x)+\alpha(x)} \tag{23}$$

The estimate of J_3. We have

$$|J_3| \leq c \int_{0}^{x-a} t^{\lambda(x)} \left| (t+h)^{\alpha(x)-1} - t^{\alpha(x)-1} \right| \ dt$$

$$= c \ h^{\lambda(x)+\alpha(x)} \int_{0}^{\frac{x-a}{h}} t^{\lambda(x)} \left[t^{\alpha(x)-1} - (t+1)^{\alpha(x)-1} \right] \ dt \tag{24}$$

If $x-a \leq h$, we have

$$|J_3| \leq c \ h^{\lambda(x)+\alpha(x)} \int_{0}^{1} t^{\lambda(x)} \left[t^{\alpha(x)-1} + (t+1)^{\alpha(x)-1} \right] \ dt$$

$$\leq c \ h^{\lambda(x)+\alpha(x)} \int_{0}^{1} (t^{m-1}+1) dt = c \ h^{\lambda(x)+\alpha(x)}$$

Let $x-a \geq h$. Then

$$|J_3| \le c \ h^{\lambda(x)+\alpha(x)} \int_0^\alpha t^{\lambda(x)} \left[t^{\alpha(x)-1} - (t+1)^{\alpha(x)-1} \right] dt$$

under the assumption that $\lambda(x) + \alpha(x) < 1$. Then by the lemma 3 we have

$$|J_3| \le c \ h^{\lambda(x)+\alpha(x)} \ \frac{\sin[\alpha(x)\pi]}{\sin[\alpha(x)\pi+\lambda(x)\pi]} \ .$$

because $B(\alpha(x),1+\lambda(x)) \le$ const. Since $1/(\sin\pi t) \le c(1-t)^{-1}$, we have

$$|J_3| \le \frac{c_1 h^{\lambda(x)+\alpha(x)}}{1-\alpha(x)-\lambda(x)} \tag{25}$$

If $\lambda(x)+\alpha(x) = 1$, we split the integration in (24) from 0 to 1 and from 1 to (x-a)/h and havehave from (24)

$$|J_3| \le c_2 h \left[1 + \int_1^{\frac{x-a}{h}} t^{-1} \ dt \right]$$

since $\left| t^{\alpha(x)-1} - (t+1)^{\alpha(x)-1} \right| \le c \ t^{\alpha(x)-2}$ for $t > 1$. So,

$$|J_3| \le c \ h \left[1 + \ln \frac{x-a}{h} \right] \le c_1 h \ \ln \frac{1}{h} \tag{26}$$

because $x-a \ge h$.

Gathering the estimates (20), (22), (23), (24), (25) and (26) we obtain the inequalities (15) − (16) which proves the theorem.

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REFERENCES

1 . ROSS,B. AND SAMKO,S. Integration and differentiation to a variable
        fractional order, Integral Transforms and Special Functions,
        1(1993), N 3.

2. HARDY,G.H. AND LITTLEWOOD,J.E. Some properties of fractional integrals, I,
        Math. Zeit., 27(1928), N 4, 565-606.

3. SAMKO,S.,KILBAS A. AND MARICHEV,O.   Integrals and   Derivatives of
        Fractional Order. Theory and Applications. Gordon & Breach Sci.
        Publ., 1993 (to appear).   (Russian edition by "Nauka i Tekhnika",
        Minsk, 1987).