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CONVOLUTION OF HANKEL TRANSFORM AND ITS APPLICATION TO AN INTEGRAL INVOLVING BESSEL FUNCTIONS OF FIRST KIND

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Abstract

In the paper a convolution of the Hankel transform is constructed. The convolution is used to the calculation of an integral containing Bessel functions of the first kind.

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1. Introduction

The convolution of a modified Hankel transform, introduced in [4], has been studied in [1], [4] in classical sense and in [7] in a space of generalized functions. For an another modified Hankel transform the other convolution in some space of functions is obtained (see [5]).

The present paper is devoted to propose a definition of a convolution and to prove the convolution property in the classical sense of the following standard Hankel transform (see [6], [8])

$$\mathcal{H}_\nu[f](x) = \int_0^\infty y J_\nu(yx) f(y)dy, \quad \mathrm{Re}(\nu) > -\frac{1}{2}. \tag{1}$$

As one of its applications, a formula of infinite interval of a product of Bessel functions of the first kind is established.

2. Convolution of Hankel Transform

Set

$$h(x) = \frac{2^{1-3\nu}x^{-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)} \iint_{u+v>x,|u-v|<x} \left[x^2-(u-v)^2\right]^{\nu-1/2} \left[(u+v)^2-x^2\right]^{\nu-1/2} \times (uv)^{1-\nu} f(u)g(v)du\,dv, \quad x\in(0,\infty). \tag{2}$$

The function $h(x)$ is called the Hankel convolution of the function $f(x)$ with the function $g(x)$. It is easy to see that the convolution is a commutative operator of f and g.

Let $L(R^+;\mu(x))$ be a class of integrable functions $f(x)$ with a weight $\mu(x)>0$ in $R^+=(0,\infty)$. The main aim of this section is to prove the following:

Theorem. *Let* $\mathrm{Re}(\nu) > \frac{1}{2}$ *and* $f(x), g(x) \in L(R_+; \sqrt{x})$. *Then the function* $h(x)$ *in* (2) *exists and there holds the convolution property*

$$\mathcal{H}_\nu[h](x) = x^{-\nu}\mathcal{H}_\nu[f](x)\mathcal{H}_\nu[g](x), \tag{3}$$

where $\mathcal{H}_\nu$ *is the Hankel transform* (1).

Proof. It is well known [6, (2.12.42.15)] that

$$\begin{aligned}&\int_0^\infty t^{1-\nu} J_\nu(xt)J_\nu(ut)J_\nu(vt)dt\\ &= \frac{2^{1-3\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}(xuv)^{-\nu}\left[x^2-(u-v)^2\right]_+^{\nu-1/2}\left[(u+v)^2-x^2\right]_+^{\nu-1/2},\end{aligned} \tag{4}$$

where $\mathrm{Re}(\nu) > -\frac{1}{2}$ and

$$\varphi_+(x) = \begin{cases} \varphi(x), & \varphi(x) \geqq 0, \\ 0, & \varphi(x) < 0. \end{cases}$$

Since

$$J_\nu(x) = \begin{cases} \sqrt{\dfrac{2}{\pi x}}\cos\left(x - \dfrac{\pi\nu}{2} - \dfrac{\pi}{4}\right) + O\left(x^{-3/2}\right), & (x \to +\infty), \\ O(x^\nu), & (x \to +0), \end{cases} \tag{5}$$

(see [3]) it is easy to conclude that there exists such a positive number C_1 independent of $x \in (0,\infty)$ that

$$|\sqrt{x}J_\nu(x)| < C_1, \quad x \in (0,\infty),$$

and $x^{-\nu}J_\nu(x) \in L(R_+)$ when $\mathrm{Re}(\nu) > \frac{1}{2}$. Therefore we have

$$\begin{aligned}\left|\int_0^N t^{1-\nu}J_\nu(xt)J_\nu(ut)J_\nu(vt)dt\right| &\leqq \frac{C_1^2}{\sqrt{uv}}\int_0^N \left|t^{-\nu}J_\nu(xt)\right|dt\\ &\leqq \frac{C_1^2 x^{\mathrm{Re}(\nu)-1}}{\sqrt{uv}}\int_0^\infty \left|t^{-\nu}J_\nu(t)\right|dt \leqq \frac{Cx^{\mathrm{Re}(\nu)-1}}{\sqrt{uv}},\end{aligned} \tag{6}$$

where C is independent of x, u, v and N. In particular, making use the formulas (2) and (4) with the help of the estimate (6) we have

$$|h(x)| \leqq Cx^{\mathrm{Re}(\nu)-1}\int_0^\infty\int_0^\infty \sqrt{uv}|f(u)g(v)|du\,dv < \infty,$$

since $f(x), g(x) \in L(R_+; \sqrt{x})$. Thus the function $h(x)$ in (2) exists. Furthermore, applying the Fubini theorem, we obtain

$$\begin{aligned} h(x) &= \int_0^\infty\int_0^\infty uvf(u)g(v)\int_0^\infty t^{1-\nu}J_\nu(xt)J_\nu(ut)J_\nu(vt)dt\,du\,dv\\ &= \int_0^\infty t^{1-\nu}J_\nu(xt)\int_0^\infty\int_0^\infty uvJ_\nu(ut)J_\nu(vt)f(u)g(v)du\,dv\,dt\\ &= \int_0^\infty tJ_\nu(xt)t^{-\nu}\mathcal{H}_\nu[f](t)\mathcal{H}_\nu[g](t)dt.\end{aligned} \tag{7}$$

Here we have used the existence of the Hankel transform $\mathcal{H}_\nu$ defined by (1) for functions from $L(R_+; \sqrt{x})$ (see [2], [8]). Moreover, we notice the fact

$$\mathcal{H}_\nu[f](x) = O(x^\nu), \quad (x \to +0) \quad \text{for} \quad f \in L\left(R_+; \sqrt{x}\right)$$

from [2, p. 74]. Therefore, if we set

$$k(t) = t^{-\nu}\mathcal{H}_\nu[f](t)\mathcal{H}_\nu[g](t), \tag{8}$$

we have
$$k(t) = O(t^{\nu}), \quad (t \to +0). \tag{9}$$
On the other hand we have
$$|\mathcal{H}_\nu[f](t)| \leqq \frac{1}{\sqrt{t}} \int_0^\infty \left|\sqrt{ut} J_\nu(ut) \sqrt{u} f(u)\right| du \leqq \frac{C}{\sqrt{t}}, \quad t \in (0, \infty), \tag{10}$$
and therefore,
$$k(t) = O\left(t^{-\nu-1}\right), \quad (t \to +\infty). \tag{11}$$
Since $\mathrm{Re}(\nu) > 1/2$, from (9), (11) we conclude that $k(t) \in L\left(R_+; \sqrt{t}\right)$. Therefore the formula (7) can be rewritten in the form
$$h(x) = \mathcal{H}_\nu[k](x).$$
Hence, by using the inversion formula of the Hankel transform in the class $L(R_+; \sqrt{x})$ (see [2], [8]):
$$\mathcal{H}_\nu \left[\mathcal{H}_\nu[k]\right](x) = k(x), \tag{12}$$
we obtain
$$k(x) = \mathcal{H}_\nu[h](x). \tag{13}$$
As $k(x)$ has the form (8), the formula (13) coincides with the formula (3). Thus the theorem is proved.

3. Application

As an application of Theorem we consider the integral
$$f_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(a_0, a_1, \cdots, a_n) = \int_0^\infty t^{\nu_0+1} J_{\nu_0}(a_0 t) \prod_{j=1}^n \left(t^2 + y_j^2\right)^{-\nu_j/2} J_{\nu_j}\left(a_j \sqrt{t^2 + y_j^2}\right) dt$$
with $a_j > 0$ $(j = 0, 1, \cdots, n)$ and $\mathrm{Re}(y_j) \geqq 0$ $(j = 1, 2, \cdots, n)$. We will prove that
$$f_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(a_0, a_1, \cdots, a_n) = 0$$
when
$$a_0 > a_1 + \cdots + a_n \quad \text{and} \quad \frac{1}{2} < \mathrm{Re}(\nu_0) < \sum_{j=1}^n \mathrm{Re}(\nu_j) + \frac{n-3}{2}.$$

We know that it is valid for $n = 1$ (see [6, (2.12.31.1)] for the case $\mathrm{Re}(y_1) = 0$, and [6, (2.12.35.12)] for the case $\mathrm{Re}(y_1) > 0$). Suppose that it is valid for every $k \leqq n$. We have to prove it for the case $k = n + 1$. Put
$$g_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(t, a_1, \cdots, a_n) = t^{\nu_0} \prod_{j=1}^n \left(t^2 + y_j^2\right)^{-\nu_j/2} J_{\nu_j}\left(a_j \sqrt{t^2 + y_j^2}\right).$$
By using (5) we have
$$\begin{aligned} g_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(t, a_1, \cdots, a_n) &= O(t^{\nu_0}), & (t \to +0) \\ &= O\left(t^{\nu_0 - \nu_1 - \cdots - \nu_n - n/2}\right), & (t \to +\infty). \end{aligned} \tag{14}$$
Suppose that
$$\frac{1}{2} < \mathrm{Re}(\nu_0) < \sum_{j=1}^n \mathrm{Re}(\nu_j) + \frac{n-3}{2}.$$
Then from (14) we conclude that $g_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(t, a_1, \cdots, a_n) \in L\left(R_+; \sqrt{t}\right)$. Therefore, by using the formula (10) we obtain
$$\mathcal{H}_{\nu_0}\left[g_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(t, a_1, \cdots, a_n)\right](x) = O\left(\frac{1}{\sqrt{x}}\right), \quad (x \to +0, x \to +\infty). \tag{15}$$
Since
$$\mathcal{H}_{\nu_0}\left[g_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(t, a_1, \cdots, a_n)\right](x) = f_{y_1, \cdots, y_n}^{\nu_0, \nu_1, \cdots, \nu_n}(x, a_1, \cdots, a_n),$$

the formula (15) can be read as

$$f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(x,a_1,\cdots,a_n)=O\left(\frac{1}{\sqrt{x}}\right),\quad (x\to+0,x\to+\infty).$$

But by the assumption we have

$$f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(x,a_1,\cdots,a_n)=0$$

when $x>a_1+\cdots+a_n$. Therefore

$$f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(x,a_1,\cdots,a_n)\in L\left(R_+;\sqrt{x}\right)$$

and by (12)

$$\mathcal{H}_{\nu_0}\left[f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(x,a_1,\cdots,a_n)\right](t)=g_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(t,a_1,\cdots,a_n).$$

Analogously, we have

$$f_{y_{n+1}}^{\nu_0,\nu_{n+1}}(x,a_{n+1})\in L\left(R_+;\sqrt{x}\right)$$

and

$$\mathcal{H}_{\nu_0}\left[f_{y_{n+1}}^{\nu_0,\nu_{n+1}}(x,a_{n+1})\right](t)=g_{y_{n+1}}^{\nu_0,\nu_{n+1}}(t,a_{n+1}).$$

under the conditions

$$\frac{1}{2}<\mathrm{Re}(\nu_0)<\mathrm{Re}(\nu_{n+1})-1.$$

Since

$$g_{y_1,\cdots,y_{n+1}}^{\nu_0,\nu_1,\cdots,\nu_{n+1}}(t,a_1,\cdots,a_{n+1})=t^{-\nu_0}g_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(t,a_1,\cdots,a_n)\,g_{y_{n+1}}^{\nu_0,\nu_{n+1}}(t,a_{n+1}),$$

then by using the theorem we obtain

$$\begin{aligned}f_{y_1,\cdots,y_{n+1}}^{\nu_0,\nu_1,\cdots,\nu_{n+1}}(x,a_1,\cdots,a_{n+1})&=\int_0^\infty tJ_{\nu_0}(xt)t^{-\nu_0}g_{y_{n+1}}^{\nu_0,\nu_{n+1}}(t,a_{n+1})\,g_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(t,a_1,\cdots,a_n)\,dt\\&=\mathcal{H}_{\nu_0}\left[t^{-\nu_0}\mathcal{H}_{\nu_0}\left[f_{y_{n+1}}^{\nu_0,\nu_{n+1}}(y,a_{n+1})\right](t)\cdot\mathcal{H}_{\nu_0}\left[f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(y,a_1,\cdots,a_n)\right](t)\right](x)\\&=\frac{2^{1-3\nu_0}x^{-\nu_0}}{\sqrt{\pi}\Gamma(\nu_0+1/2)}\iint_{u+v>x,|u-v|<x}\left(x^2-(u-v)^2\right)^{\nu_0-1/2}\left((u+v)^2-x^2\right)^{\nu_0-1/2}(uv)^{1-\nu_0}\\&\qquad\times f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(u,a_1,\cdots,a_n)\,f_{y_{n+1}}^{\nu_0,\nu_{n+1}}(v,a_{n+1})\,du\,dv.\end{aligned}\tag{16}$$

Since

$$f_{y_1,\cdots,y_n}^{\nu_0,\nu_1,\cdots,\nu_n}(u,a_1,\cdots,a_n)=0\quad\text{when}\quad u>a_1+\cdots+a_n$$

and

$$f_{y_{n+1}}^{\nu_0,\nu_{n+1}}(v,a_{n+1})=0\quad\text{when}\quad v>a_{n+1}$$

provided that

$$\frac{1}{2}<\mathrm{Re}(\nu_0)<\sum_{j=1}^{n}\mathrm{Re}(\nu_j)+\frac{n-3}{2},\quad \mathrm{Re}(\nu_0)-\mathrm{Re}(\nu_{n+1})<-1,\tag{17}$$

we conclude from (16) that

$$f_{y_1,\cdots,y_{n+1}}^{\nu_0,\nu_1,\cdots,\nu_{n+1}}(x,a_1,\cdots,a_{n+1})=0\quad\text{when}\quad x>a_1+\cdots+a_{n+1}\tag{18}$$

under (17).

The formula (18) can be analytically continued to the domain

$$-1<\mathrm{Re}(\nu_0)<\sum_{j=1}^{n+1}\mathrm{Re}(\nu_j)+\frac{n-3}{2}.$$

Thus we have proved

Corollary. *Let*

$$-1 < \mathrm{Re}(\nu_0) < \sum_{j=1}^{n} \mathrm{Re}(\nu_j) + \frac{n-3}{2}, \quad a_j > 0 \ (j = 1, \cdots, n)$$

with

$$a_0 > a_1 + \cdots + a_n$$

and

$$\mathrm{Re}(y_j) \geqq 0 \ (j = 1, \cdots, n).$$

Then

$$\int_0^\infty t^{\nu_0+1} J_{\nu_0}(a_0 t) \prod_{j=0}^{n} \left(t^2 + y_j^2\right)^{-\nu_j/2} J_{\nu_j}\left(a_j\sqrt{t^2+y_j^2}\right) dt = 0. \tag{19}$$

The formula (19) is a generalization of the formulae (2.12.44.7) (the case $y_1 = \cdots = y_n = 0$) and (2.12.44.8) (the case $\mathrm{Re}(y_1) > 0, \cdots, \mathrm{Re}(y_n) > 0$) in [6].

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