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AROUND THE FURUTA INEQUALITY - THE OPERATOR INEQUALITIES $(AB^2A)^{3/4} \leq ABA \leq A^3$

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Abstract: For positive operators A and B with A invertible it is shown that $(AB^2A)^{1/2} \leq A^2$ implies $(AB^2A)^{3/4} \leq ABA$. The inequalities in the title for $0 \leq B \leq A$ are then derived as a consequence.

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1. INTRODUCTION.

In this paper operator inequalities centered around the celebrated Furuta inequality are considered. As motivation, we begin with a brief account of the origin of the inequalities in the title.

We consider (bounded, linear) operators acting on a Hilbert space. For an operator A, we write $A \geq 0$ (or $0 \leq A$) if A is a positive operator. For positive operators A and B, we write $A \geq B$ (or $B \leq A$) if $A - B \geq 0$.

It is well-known that $0 \leq B \leq A$ implies that $B^r \leq A^r$ for every real number r with $0 \leq r \leq 1$. Thus, $0 \leq B \leq A$ implies $B^{1/2} \leq A^{1/2}$. But, in general, $0 \leq B \leq A$ does not necessarily imply that $B^2 \leq A^2$. In [1], the following conjecture was raised:

$$\text{If } 0 \leq B \leq A, \text{ then } (AB^2A)^{1/2} \leq A^2. \tag{1}$$

This conjecture was answered affirmatively by Furuta [2]. Indeed, Furuta proved a more general inequality that contains inequality (1) as a special case:

THE FURUTA INEQUALITY. For p, $r \geq 0$ and $q \geq 1$ with $p + 2r \leq (1 + 2r)q$,

$$0 \leq B \leq A \text{ implies } (A^rB^pA^r)^{1/q} \leq A^{(p+2r)/q}.$$

Setting $p = q = 2$ and $r = 1$, The Furuta inequality becomes (1). Furuta also observed that setting $p = 2$, $r = 1$ and $q = 4/3$, a stronger inequality resulted:

$$\text{If } 0 \leq B \leq A, \text{ then } (AB^2A)^{3/4} \leq A^3. \tag{2}$$

That (2) implies (1) can readily be seen by taking the 2/3-power of both sides of (2).

After the appearance of Furuta's original paper [2], Kamei [3] gave a direct proof of inequalities (1) and (2) using the special notion of *operator means*. More recently, Furuta [4] constructed the operator function $G_r(p) = (A^r B^p A^r)^{(1+2r)/(p+2r)}$ for $A \geq B \geq 0$, $r \geq 0$ and $p \in [1, \infty)$, and showed that $G_r(p)$ is a decreasing function on $[1, \infty)$. In particular, $G_1(2) \leq G_1(1)$. This yielded the following improvement of (2):

$$\text{If } 0 \leq B \leq A, \text{ then } (AB^2A)^{3/4} \leq ABA \leq A^3. \tag{3}$$

The main result of this paper is to show that for positive operators A and B with A invertible, the inequality $(AB^2A)^{3/4} \leq ABA$ is a consequence of the inequality $(AB^2A)^{1/2} \leq A^2$. We also establish inequality (3) as a corollary of our result by giving a new proof of (1) which appears to be simpler than those of [3], [5] and [6]. Our proof is completely elementary. It was inspired by the work of Pedersen and Takesaki [7].

2. The Main Result.

Theorem. *Suppose A and B are positive operators with A invertible. Then*

$$(AB^2A)^{1/2} \leq A^2 \quad \textit{implies} \quad (AB^2A)^{3/4} \leq ABA.$$

Proof. Let $T = A^{-1}(AB^2A)^{1/2}A^{-1}$. The assumptions imply that $0 \leq T \leq I$, the identity operator. Simple calculation shows that $B^2 = TA^2T$. Now

$$\begin{aligned}
[A^{-1}(AB^2A)^{3/4}A^{-1}]^2 &= [A^{-1}(ATA)^{3/2}A^{-1}]^2 \\
&= A^{-1}(ATA)^{1/2}(ATA)A^{-2}(ATA)(ATA)^{1/2}A^{-1} \\
&= A^{-1}(ATA)^{1/2}(AT^2A)(ATA)^{1/2}A^{-1} \\
&\leq A^{-1}(ATA)^{1/2}(ATA)(ATA)^{1/2}A^{-1} \\
&= A^{-1}(ATA)^2A^{-1} = TA^2T = B^2
\end{aligned}$$

Taking square roots, we have $A^{-1}(AB^2A)^{3/4}A^{-1} \leq B$ and hence $(AB^2A)^{3/4} \leq ABA$, and the proof is completed.

Corollary 1. *If $0 \leq B \leq A$, then $(AB^2A)^{3/4} \leq ABA \leq A^3$.*

Proof. Without loss of generality, assume that A is invertible. In view of the theorem, it suffices to establish the inequality $(AB^2A)^{1/2} \leq A^2$. Again we employ the idea of Pedersen and Takesaki. Let $S = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$. Since $0 \leq B \leq A$, $0 \leq S \leq I$ and $B = SAS$. Thus

$$\begin{aligned}
(AB^2A)^{1/2} &= (A(SAS)^2A)^{1/2} = (ASAS^2ASA)^{1/2} \\
&\leq (ASA^2SA)^{1/2} = ASA \leq A^2.
\end{aligned}$$

This completes the proof.

The following improvement of (1) is a consequence of Corollary 1.

Corollary 2. *If $0 \leq B \leq A$, then $(AB^2A)^{1/2} \leq (ABA)^{2/3} \leq A^2$.*

Corollary 3. *Suppose A and B are positive operators with A invertible.*

(*a*) *Then, $(AB^2A)^{3/4} = ABA$ if the operator $T = A^{-1}(AB^2A)^{1/2}A^{-1}$ is a projection.*

(*b*) *If, in addition B is invertible, then $(AB^2A)^{3/4} = ABA$ if and only if $A = B$.*

PROOF. (*a*) If T is a projection then $T^2 = T$. In this case, the only " $\leq$ " in the main body of the proof of the theorem becomes " = ".

(*b*) If $(AB^2A)^{3/4} = ABA$, then the occurrence of " $\leq$ " mentioned in (a) again becomes " = ". If B is also invertible, then T is invertible. Consequently, $T^2 = T = I$ and hence $A = B$.

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