Internat. J. Math. & Math. Sci.
VOL. 17 NO. 4 (1994) 759-770

EXISTENCE OF WEAK SOLUTIONS FOR ABSTRACT HYPERBOLIC-PARABOLIC EQUATIONS

MARCONDES RODRIGUES CLARK
Universidade Federal da Paraíba - Campus II - DME - CCT
58.109-970 - Campina Grande - Paraíba - Brasil
e-mail:mclark at brufpb2.bitnet

(Received July 30, 1992 and in revised form April 19, 1993)

ABSTRACT

In this paper we study the Existence and Uniqueness of solutions for the following Cauchy problem:

$$\begin{aligned} &A_2u''(t) + A_1u'(t) + A(t)u(t) + M(u(t)) = f(t), \quad t \in (0,T) \\ &u(0) = u_0; A_2u'(0) = A_2^{\frac{1}{2}}u_1; \end{aligned} \tag{1}$$

where A_1 and A_2 are bounded linear operators in a Hilbert space $H, \{A(t)\}_{0\leq t\leq T}$ is a family of self-adjoint operators, M is a non-linear map on H and f is a function from $(0,T)$ with values in H.

As an application of problem (1) we consider the following Cauchy problem:

$$\begin{aligned} &k_2(x)u'' + k_1(x)u' + A(t)u + u^3 = f(t) \text{ in } Q, \\ &u(0) = u_0; k_2(x)u'(0) = k_2(x)^{\frac{1}{2}}u_1 \end{aligned} \tag{2}$$

where Q is a cylindrical domain in $\mathbb{R}^4$; k_1 and k_2 are bounded functions defined in an open bounded set $\Omega \subset \mathbb{R}^3$,

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x,t)\frac{\partial}{\partial x_i});$$

where a_{ij} and $a'_{ij} = \frac{\partial}{\partial t}u_{ij}$ are bounded functions on Ω and f is a function from $(0,T)$ with values in $L^2(\Omega)$.

KEY WORDS AND PHRASES: Existence of weak solutions, Nonlinear equation, Cauchy problem, Existence and Uniqueness.

AMS Subject Classifications:35L15

INTRODUCTION

Let $T > 0$ be a positive real number and Ω be a bounded open set of $\mathbb{R}^n$, with smooth boundary Γ. In the cylinder $Q = \Omega \times (0,T)$, Bensoussan et al. [01], studied the homogeneization for the following Cauchy problem:

$$\begin{aligned} &k_2(x)u'' + k_1(x)u' - \Delta u = f \text{ in } Q. \\ &u(x,0) = u_0(x) \text{ e } k_2(x)u'(x,0) = k^{\frac{1}{2}}2(x)u_1(x), \qquad x \in \Omega \end{aligned} \tag{3}$$

Many authors have been investigating the existence of solution for non-linear equations associated with problem (3),

see: Larkin [04], Lima [05], Medeiros [07-09],Melo [10], Maciel [11], Neves [12] and Vagrov [15].

Other interesting results relative to existence of a solution for a non-linear equation associated with the equation of the problem (3) can be found in the work of J'orgens [03]

In this is work he proved the existence of classical solution by iterative methods for the mixed problem associated to the equation

$$u_{tt} - \Delta u + F'(|u|^2)u = 0,$$

in open domain of $\mathbb{R}^3$, with the hypotesis $F(0) = 0$ and $|F'(s)| \leq a[b + F(s)]^\alpha$ where a, b and α are positive constants with $\alpha < \frac{2}{3}$.

In Section 1, we establish some notation for the function spaces and conditions for A_1, A_2, $\{A(t)\}_{0\leq t\leq T}$, M and f in equation (1). In Section 2, we state our main results and we prove the assertions made. In the final Section we make an application of problem (1).

1. PRELIMINARIES

We will assume that standard function spaces are known: $C^k(\Omega)$, $L^p(Q)$; $H^k(\Omega)$, $H_0^k(\Omega)$, $C^k(0,T;X)$, $L^p(0,T;X)$ where X is a Banach space.

Let H be a real Hilbert space, with inner product and the norm denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively.

We consider here the following assumptions:

i) $A_2 : H \to H$, a positive symmetric operator

ii) $A_1 : H \to H$, a symmetric operator such that:

$$(A_1 u, u) \geq \beta |u|^2, \quad 0 < \beta \epsilon \mathbb{R}, \quad \text{for all } u \in H.$$

iii) Let $\{A(t);\ t \geq 0\}$ be a family of self-adjoint linear operators of H, such that there exists a constant $\alpha > 0$, satisfying $(A(t)u, u) \geq \alpha|u|^2$ for all $u \in D(A(t))$, where we assume that the domain $D(A(t))$ of $A(t)$ is constant, i.e, $D(A(t)) = D(A(s))\ \forall\, t,\ s \geq 0$. It is known from the spectral theory for self-adjoint opeators that there exists only one positive self-adjoint operator $A^{\frac{1}{2}}(t)$ such that:

$$D(A(t)) \subseteq D(A^{\frac{1}{2}}(t)).$$

From assumption iii) we have, see Medeiros [09], that $D(A^{\frac{1}{2}}(t))$ is constant.

Let $V_t = D(A^{\frac{1}{2}}(t))$ with inner product $((\cdot))$ and associated norm $\|\cdot\|_t$. Therefore $\|u\|_t^2 = |A^{\frac{1}{2}}(t)u|^2 \geq \alpha|u|^2$.

So that, V_t is a Hilbert space, dense and embedded in $H (V_t \hookrightarrow H)$, and V_t is isomorphic with V_0, $\forall\, t$.

iv) $A(t)$ is continuously strongly differentiable.

v) For $u \in D(A(0))$, we assume that there exists a real $\gamma > 0$, independent from t, such that:

$$(A'(t)u, u) \leq \gamma\|u\|_0^2, \quad \forall\, t \in [0,T]$$

vi) We assume that the embedding $V_0 \hookrightarrow H$ is compact. Therefore, the spectrum of the operator $A(t)$ is discret.

Identifying H with his dual H', we have the immersions:

$V_0 \hookrightarrow H \hookrightarrow V_0'$; where each space is dense on the following one.

In this work, we use the symbol $< \cdot, \cdot >$, to denote the duality between V_0' and V_0. Sometimes it means an application of a Vector distribution to a real test function.

vii) Let M be an operator of V_0 in H satisfying the following conditions:

a) M is monotone, hemi-continuous and bounded (in the sense of taking bounded sets of V_0 into bounded sets of H).

b) There exists a constant $\sigma > 0$ so that

$$\int_0^T (M(u(s)), u'(s))ds \geq -\sigma \; \forall \, t \in \{0,T] \text{ and } \forall \, u \in E_{\bar{C}}$$

where $E_{\bar{C}}$ denotes the set $\{u \in L^\infty(0,T;V_0); u' \in L^2(0,T;H) \text{ and } \|u(0)\|_0 \leq \bar{C}\}$

2.1 The Main Results

Theorem - 1: (Existence) Under the above assumptions (i-vii) and considering

$$f \in L^2(0,T;H) \tag{2.1}$$

$$u_0 \in V_0 \tag{2.2}$$

$$u_1 \in H, \tag{2.3}$$

then there exists a function u defined in $(0,T)$ with values in V_0 such that:

$$u \in L^\infty(0,T;V_0) \tag{2.4}$$

$$u' \in L^2(0,T;H), \tag{2.5}$$

besides this, u is a solution of problem (1) in the following way:

$$\begin{aligned} &-\int_0^T (A_2u'(t), \Phi'(t)v)dt + \int_0^T (A_1u'(t), \Phi(t)v)dt + \\ &+ \int_0^T (A^{\frac{1}{2}}(t)u(t), A^{\frac{1}{2}}(t)\Phi(t)v)dt + \\ &+ \int_0^T (M(u(t)), \Phi(t)v)dt = \int_0^T (f(t), \Phi(t)v)dt, \quad \forall \, v \in V_0 \end{aligned} \tag{2.6}$$

and $\forall \, \Phi \in C_0^1(0,T)$.

$$u(0) = u_0 \tag{2.7}$$

$$A_2u'(0) = A_2^{\frac{1}{2}}u_1. \tag{2.8}$$

For the uniqueness we need the following condition on M:

viii) Given $C > 0$, there exists $K > 0$, which depends on C, such that:

$$|M(u) - M(v)| \leq K|u - v|$$

for all u, $v \in V$ whenever $\|u\|_0 \leq C$ and $\|v\|_0 \leq C$.

Theorem - 2. (Uniqueness) Suppose that the operators $A_1, A_2, A(t)$ satisfy the conditions of Theorem-1 and (viii), respectively, and M maps functions of $L^\infty(0,T;V_0)$ into functions of $L^2(0,T;H)$. Then, there exists at most one function u in the class

$$u \in L^\infty(0,T;V_0), u' \in L^2(0,T;H),$$

and u is a solution of problem (1) in the sense (2.6) - (2.8) of Theorem-1.

Remark 2.1

From (2.4), (2.5) and (2.6) we obtain that $A_2u'' \in L^2(0,T;V_0')$ and this together with (2.4) (2.5) imply that the initial conditions (2.7) (2.8) make sense.

2.2 Proof of the Theorems

In this part we use the followin result:

Lema 1. Let $u \in L^2(0,T;H)$, $u' \in L^2(0,T;V_0')$ with v, and $v' \in L^2(0,T;V_0)$. Then

$$\frac{d}{dt} < u, v >=< u', v > +(u, v').$$

For the proof of this lemma see Tanabe, [13].

We apply the standard Galerking approximate procedure. Let (w_ν) be a base of $D(A(0))$ that it is a base of H, by density. From the assumption (i), we have $((A_2 + \lambda I)^{\frac{1}{2}} w_\nu)$ is also a base of H; where $\lambda > 0$ is a constant. Let $V_m(0)$ be a subspace of $D(A(0))$ generated by the first-m vectors $w_1, \dots, w_m$, and $V_m^\lambda(0)$ the subspace generated by first-m vectors $(A_2 + \lambda I)^{\frac{1}{2}} w_1, \dots, (A_2 + \lambda I)^{\frac{1}{2}} w_m$.

We put $u_{\lambda m}(t) = \sum_{i=1}^{m} g_{i\lambda} m(t) w_i$ as a solution of the approximate perturbed problem:

$$\begin{aligned} &((A_2 + \lambda I)u''_{\lambda m}(t) + A_1 u'_{\lambda m}(t) + A(t)u_{\lambda m}(t) + (M(u_{\lambda m}(t)), v) = \\ &= (f(t), v), \ \ \forall\, v \in V_m(0). \end{aligned} \tag{2.9}$$

$$u_{\lambda m}(0) = u_{0m}; \ \text{ where } u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \to u_0 \tag{2.10}$$

strongly in V_0

$$u'_{\lambda m}(0) = u_{1\lambda m}; \ \ \text{ where } u_{1\lambda m} = \sum_{i=1}^{m} \beta_{i\lambda m} w_i \tag{2.11}$$

where the coefficient $\beta_{i\lambda m}$ denotes the coordinates of the vector $P_{\lambda m} u_1$, the orthogonal projection of the vector u_1 upon the subspace $V_m^\lambda(0)$ in relation to the base $((A_2 + \lambda I)^{\frac{1}{2}} w_\nu)$, such that:

$$P_{\lambda m} u_1 = \sum_{i=1}^{m} \beta_{i\lambda m} (A_2 + \lambda I)^{\frac{1}{2}} w_i.$$

We have that $P_{\lambda m} u_1 \to u_1$ strongly in H and satisfies

$$|P_{\lambda m} u_1| \le |u_1| \ \ \forall\, m \ \ \text{e} \ \ \forall\, \lambda > 0.$$

System (2.9) - (2.11) is equivalent to a system of non-linear ordinary differential equations, which has a solution $u_{\lambda m}(t)$ by using Caratheodory's theorem ,see Coddington - Levinson [02]; defined in an interval $[0, t_m)$, with $t_m < T$, for each $m \in \mathbb{N}$.

2.3 - "A priori" Estimates

In (2.9) taking $v = 2u'_{\lambda m}(t)$ we have:

$$\begin{aligned} & \frac{d}{dt}|(A_2+\lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)|^2 + 2(A_1 u'_{\lambda m}(t), u'_{\lambda m}(t)) + \\ + \; & 2(A^{\frac{1}{2}}(t)u_{\lambda m}(t), A^{\frac{1}{2}}(t)u'_{\lambda m}(t)) + 2(M(u_{\lambda m}(t)), u'_{\lambda m}(t)) = \\ = \; & 2(f(t), u'_{\lambda m}(t)). \end{aligned}$$

Using the above assumptions, we have,

$$\begin{aligned} & |(A_2+\lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)|^2 + \beta\int_0^t |u'_{\lambda m}(s)|^2 ds + \|u_{\lambda m}(t)\|_t^2 \leq \\ \leq \; & 2\sigma + |P_{\lambda m}u_1|^2 + \|u_{0m}\|_0^2 + \int_0^t (A'(s)u_{\lambda m}(s), u_{\lambda m}(s))ds + \\ + \; & \frac{1}{\beta}\int_0^t |f(s)|^2 ds. \end{aligned}$$

From (2.1), (2.10) and (2.11), there exists a constant $C^{(*)}$ such that

$$\begin{aligned} & |(A_2+\lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)|^2 + \beta\int_0^t |u'_{\lambda m}(s)|^2 ds + \|u_{\lambda m}(t)\|_t^2 \leq C + \\ + \; & \int_0^t (A'(s)u_{\lambda m}(s), u_{\lambda m}(s))ds. \end{aligned}$$

(*) Let us denote by C various constants.

It is not difficult to prove that the function $g(t) = \|u_{\lambda m}(t)\|_t^2$ is continuous. So that from Gronwall's inequality, from $V_t \cong V_0$, and from the assumption (v), we conclude that:

$$\|u_{\lambda m}(t)\|_0 \leq C \tag{2.12}$$

independently from $\lambda > 0$ $m \in \mathbb{N}$ and of $t \in [0, t_m)$. So that, we have

$$|(A_2+\lambda I)^{\frac{1}{2}}u'_{\lambda m}(t)|^2 + \beta\int_0^t |u'_{\lambda m}(s)|^2 ds + \|u_{\lambda m}(t)\|_t^2 \leq C \tag{2.13}$$

independently from $\lambda > 0$, $m \in \mathbb{N}$ and of $t \in [0, t_m)$.

Therefore, from (2.12), (2.13) and by Carathéodory Theorem there exists a solution in all interval $[0, T]$.

So we obtain the following estimates:

$$\|u_{\lambda m}\|_{L^\infty(0,T;V_0)} \leq C, \ \forall\, \lambda > 0, \ m \in \mathbb{N}. \tag{2.14}$$

$$\|u'_{\lambda m}\|_{L^2(0,T;H)} \leq C, \ \forall\, \lambda > 0, \ m \in \mathbb{N}. \tag{2.15}$$

Where C is a constant independent of $m \in \mathbb{N}$ and $\lambda > 0$. From the estimate (2.14) and noting that M is bounded it follows that

$$\|M(u_{\lambda m})\|_{L^\infty(0,T;H)} \leq C, \ \forall\, \lambda > 0, \ m \in \mathbb{N}. \tag{2.16}$$

The estimates (2.14) - (2.16), imply that there exists a subsequence of $(u_{\lambda m})$, still denoted by $(u_{\lambda m})$, and a function u_λ such that

$$u_{\lambda m} \rightarrow u_\lambda \ \text{ weak-star in } L^\infty(0,T;V_0). \tag{2.17}$$

$$u'_{\lambda m} \rightarrow u'_\lambda \ \text{ weak in } L^2(0,T;H) \tag{2.18}$$

$$A^{\frac{1}{2}}(t)u_{\lambda m} \to A^{\frac{1}{2}}(t)u_{\lambda} \quad \text{weak-star in } L^{\infty}(0,T;H) \tag{2.19}$$

$$(A_2+\lambda I)u'_{\lambda m} \to (A_2+\lambda I)u'_{\lambda} \quad \text{weak in } L^2(0,T;H) \tag{2.20}$$

$$A_1 u'_{\lambda m} \to A_1 u'_{\lambda} \quad \text{weak in } L^2(0,T;H) \tag{2.21}$$

$$M(u'_{\lambda m}) \to \chi \quad \text{weak-star in } L^{\infty}(0,T;H) \tag{2.22}$$

The fact that $A^{\frac{1}{2}}(t)$; A_1 and A_2 are weakly closed operators of $L^2(0,T;H)$ was used in (2.19), (2.20) and (2.21).

2.4 - The Nonlinear Term

Since $H \hookrightarrow V_0'$ continuously, it follows from (2.15) that:

$$\|u'_{\lambda m}\|_{L^2(0,T;V_0')} \le C, \text{ independently of } \lambda > 0 \text{ and} m \in I\!N. \tag{2.23}$$

From (2.4), (2.23) and by the compact embedding from V_0 in H, it follows from the Lemma of Aubin-Lions, see Lions [06], that:

$$u_{\lambda m} \to u_{\lambda} \quad \text{strong in } L^2(0,T;H). \tag{2.24}$$

For $v \in L^2(0,T;V)$ and $\Theta > 0$ a real number, by the monotonicity of M we have:

$$\int_0^T (M(u_\lambda + \Theta v) - M(u_{\lambda m}),\ u_\lambda + \Theta v - u_{\lambda m})dt \ge 0.$$

From this inequality, taking the limit $m \to \infty$ and using the convergences (2.22) and (2.24) we get:

$$\int_0^T (M(u_\lambda + \Theta v) - \chi, v)dt \ge 0, \quad \forall\, v \in L^2(0,T;V).$$

It follows, by the hemicontinuity of M, that,

$$M(u_\lambda) = \chi. \tag{2.25}$$

By multiplying both sides of (2.9) by $\Phi \in C_0^\infty(0,T)$, integrating from $t=0$ to $t=T$, passing to the limit and using the convergences (2.19) - (2.22) we obtain,

$$\begin{aligned} &-\int_0^T ((A_2+\lambda I)u'_\lambda, \Phi' v)dt + \int_0^T (A_1 u'_\lambda, \Phi v)dt + \\ &+\int_0^T (A^{\frac{1}{2}}(t)u_\lambda, A^{\frac{1}{2}}(t)\Phi v)dt + \int_0^T (M(u_\lambda), \Phi v)dt = \\ &= \int_0^T (f, \Phi v)dt, \quad \forall\ \Phi \in C_0^\infty(0,T),\ \forall\, v \in V. \end{aligned} \tag{2.26}$$

Since the linear combinations of $w_1, \ldots, w_m$ are dense in $D(A(0))$, it follows that the above equality, remains valid for all $v \in D(A(0))$ and for all $\Phi \in C_0^\infty(0,T)$ also. So that, u_λ is a solution of the perturbed problem in the sense given in (2.6).

From this we have that

$$((A_2+\lambda I)u'_\lambda)' = -A_1 u'_\lambda - A(t)u_\lambda - M(u_\lambda) + f \in L^2(0,T;V_0'). \tag{2.27}$$

Noticing that the estimates (2.14) - (2.16) are independent of $\lambda > 0$, we obtain the same convergences (2.17) - (2.22) and also the equality (2.25) replacing $u_{\lambda m}$ by u_λ by and u_λ by u.

By the above arguments, taking the limit in (2.26) we have that u satisfies (2.4)-(2.6).

From (2.6) we have,

$$(A_2u')' + A_1u' + A(t)u + M(u) = f \quad \text{in} \quad L^2(0,T;V_0'). \tag{2.28}$$

$$(A_2u')' \in L^2(0,T;V_0'). \tag{2.29}$$

2.5 - The Inicial Conditions

The proof of the initial conditions (2.7) and (2.8) are obtained by the convergences (2.17), (2.18). Let $\Phi \in C^1([0,T])$ with $\Phi(0) = 1$, $\Phi(T) = 0$, and $v \in V_0$. Then by (2.17) and using Lemma 1, with $\Phi v \in V_0$, we obtain

$$- < (A_2 + \lambda I)u_\lambda'(0), v > - \int_0^T ((A_2 + \lambda I)u_\lambda', \Phi' v)dt +$$
$$+ \int_0^T (A_1u_\lambda', \Phi v)dt + \int_0^T < A(t)u_\lambda, \Phi v > dt +$$
$$+ \int_0^T (M(u_\lambda), \Phi v)dt = \int_0^T (f, \Phi v)dt.$$

Taking the limit in the above equality, we obtain

$$\begin{aligned} &- < A_2^{\frac{1}{2}}u_1, v > - \int_0^T (A_2u', \Phi' v)dt + \int_0^T (A_1u', \Phi v)dt + \\ &+ \int_0^T < A(t)u, \Phi v > dt + \int_0^T (M(u), \Phi v)dt = \int_0^T (f, \Phi v)dt. \end{aligned} \tag{2.30}$$

Integrating by parts $- \int_0^T (A_2u_\lambda', \Phi' v)dt$, observing (2.29) and using Lemma-1, we get from (2.28) and (2.30) that:

$$< A_2u'(0), v > = < A_2^{\frac{1}{2}}u_1, v >, \quad \forall\, v \in V.$$

From this it follows the proof of Theorem 1.

Remark 1. We obtain the same Theorem 1 by considering:

$$M : L^2(0,T;V_0) \to L^2(0,T;H)$$

pseudo-monotone and satisfying condition (vii) (see Lions, [06]).

3.- PROOF OF THEOREM 2

If u and v satisfy Theorem-1, then $w = u - v$ satisfies:

$$(A_2w')' + A_1w' + A(t)w + M(u) - M(v) = 0 \;\text{ in }\; L^2(0,T;V_0'). \tag{3.1}$$

$$w(0) = 0, \;\; A_2w'(0) = 0. \tag{3.2}$$

We'll prove that $w = 0$ in $[0,T]$.

We observe that the solution $u'(t) \in H$ and $(A_2u')'(t) \in V'$. Therefore it doesn't make sense the duality between these vectors. In this case, we'll use the method introduced by Visik-Ladyzenskaja [14].

For each s with $0 < s < T$, we'll consider the function $z(t)$ given by:

$$z(t) = \begin{cases} -\displaystyle\int_t^s w(\xi)d\xi & \text{if } 0 \le t \le s \\ 0 & \text{if } s < t \le T \end{cases} \tag{3.3}$$

We have that $z(s) = 0$, $z'(t) = w(t)$ for $0 \le t \le s$ and $z(t) \in V_0$ for each $t \in [0,T]$.

Defining $w_1(t)$ by, $w_1(t) = \int_0^t w(\gamma)d\gamma$, we have $z(t) = w_1(t) - w_1(s)$, $0 \le t \le s$. Taking the duality of (3.1) with (3.3) and integrating from $t = 0$ to $t = T$, we obtain

$$\int_0^T < (A_2 w')', z > dt + \int_0^T (A_1 w', z)dt + \int_0^T < A(t)w, z > dt + \\ + \int_0^T (M(u) - M(v), z)dt = 0. \tag{3.4}$$

We have that:

$$\int_0^T < (A_2 w')', z > dt = -\frac{1}{2}(A_2 w(s), w(s))$$

$$\int_0^T (A_1 w', z)dt = -\int_0^s (A_1 w, w)dt.$$

$$\int_0^T A^{\frac{1}{2}}(t)z)dt =$$

$$= \frac{1}{2}\int_0^s \frac{d}{dt}\|z(t)\|_t^2 dt - \frac{1}{2}\int_0^s (A'(t)z(t), z(t))dt =$$

$$= -\frac{1}{2}|A^{\frac{1}{2}}(0)w_1(s)|^2 - \frac{1}{2}\int_0^s (A'(t)z(t), z(t))dt.$$

Substituting the above equalities in (3.4) we have:

$$\frac{1}{2}|A_2^{\frac{1}{2}} w(s)|^2 + \int_0^s (A_1 w, w)dt + \frac{1}{2}|A^{\frac{1}{2}}(0)w_1(s)|^2 =$$

$$= \int_0^s (M(u) - M(v), z)dt - \frac{1}{2}\int_0^s (A'(t)z(t), z(t))dt.$$

By using hypotheses ii), iii), v), viii) in the above equality, we obtain:

$$\frac{1}{2}|A_2^{\frac{1}{2}} w(s)|^2 + \beta\int_0^s |w(t)|^2 dt + \frac{\alpha}{2}|w_1(s)|^2 \le$$

$$\le \int_0^s \mu|w(t)||z(t)|dt + \frac{\gamma}{2}\int_0^s |z(t)|^2 dt \le \int_0^s \mu|w(t)||w_1(t)|dt$$

$$+ \int_0^s \mu|w(t)||w_1(s)|dt + \frac{\gamma}{2}\int_0^s |z(t)|^2 dt.$$

By applying the inequality $ab \le \frac{\lambda a^2}{2} + \frac{b^2}{2\lambda}$, $\forall\, \lambda > 0$, in the above inequality one has:

$$(\beta - \mu^2\lambda)\int_0^s |w(t)|^2 dt + \left[\frac{\alpha}{2} - (\frac{1}{2\lambda} + \gamma)s\right]|w_1(s)|^2 \le$$

$$(\frac{1}{2\lambda} + \gamma)\int_0^s |w_1(t)|^2 dt, \ \forall\, \lambda > 0 \text{ such that } \beta - \mu^2\lambda > 0$$

and $\frac{\alpha}{2} - (\frac{1}{2\lambda + \gamma})s > 0$. If we choose $\lambda > 0$ such that $\beta - \mu^2\lambda = \frac{\beta}{2}$, that is, $\lambda = \frac{\beta}{2\mu^2}$ and s_0 such that $\frac{\alpha}{2} - (\frac{1}{2\lambda} + \gamma)s_0 = \frac{\alpha}{4}$, that is, $s_0 = \frac{\alpha\lambda}{2(1 + 2\lambda\gamma)}$, we obtain from the above equality:

$$\frac{\beta}{2}\int_0^s |w(t)|^2 dt + \frac{\alpha}{4}|w_1(s)|^2 \le (\frac{\mu^2}{\beta} + \gamma)\int_0^s |w_1(t)|^2 dt \tag{3.5}$$

$\forall\, s \in [0, s_0]$. Gronwall's inequality implies that $w_1(s) = 0$ for all $s \in [0, s_0]$. Which implies $w_1(s) = 0 \ \forall\, s \in [0, s_0]$, consequently $w(t) = 0$ for all $t \in [0, s_0]$.

Using the same argument in $[0, s_0]$ for the Cauchy problem:

$$\left[\begin{array}{l} (A_2w')' + A_1w' + A(t)w + M(u) - M(v) = 0 \\ w(s_0) = 0, \ A_2w'(s_0) = 0 \end{array}\right.$$

we obtain that $w(t) = 0$, for all $t \in [s_0, 2s_0]$.

After a finite number of steps we conclude $w(t) = 0$ in $[0,T]$ and the proof of the Theorem 2 is completed.

3. EXAMPLES

1) Let Ω be a regular bounded open subset of $\mathbb{R}^n$ and $H = L^2(\Omega)$, $V = H_0^1(\Omega)$.

Let us define the functions k_1, $k_2 \in L^\infty(\Omega)$ such that $k_1(x) \geq \beta > 0$ a.e. and $k_2(x) \geq 0$ a.e. in Ω where β is a constant.

We define the operators A_1 and A_2 in $L^2(\Omega)$ by

$$(A_1u)(x) = k_1(x)u(x), \ \ (A_2u)(x) = k_2(x)u(x)$$

and consider

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x,t)\frac{\partial}{\partial x_i})$$

being the domain of $A(t)$ the space $H^2(\Omega) \cap H_0^1(\Omega)$ which is dense in $L^2(\Omega)$; where $a_{ij} = a_{ji}$ and

$$a'_{ij} = \frac{\partial}{\partial t}a_{ij} \in L^\infty(\Omega \times (0,T)), \ \ \forall\, 1 \leq i, \ j \leq n.$$

Then $A(t)$ is a family of self-adjoint operators.

We also assume that:

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \gamma(|\xi_1|^2 + \ldots + |\xi_n|^2);$$

$(x,t) \in Q$, $0 < \gamma \in \mathbb{R}$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$; then, by Poincaré-Friedrichs inequality implies that $(A(t)u, u) \geq \alpha|u|^2$, for all $u \in D = D(A(t))$ and for some constant $\alpha > 0$.

Noting that

$$|((A(t) - A(t_0))u, u)| \leq \sum_{i,j=1}^{n} \int_\Omega |a_{ij}(x,t) - a_{ij}(x,t_0)|.|\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j}|dx,$$

being $a_{ij} \in L^\infty(Q)$, we have that there exists the $\lim_{t \to t_0}(t - t_0)^{-1}(A(t)u - A(t_0)u)$ in norm of $L^2(\Omega)$.

Therefore $A(t)$ is continuously strongly differentiable.

Being $A'(t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a'_{ij}(x,t)\frac{\partial}{\partial x_i})$ with $a'_{ij} = a'_{ji} \in L^\infty(Q)$ $\forall\, 1 \leq i, \ j \leq n$, we have $|(A'(t)u, u)| \leq supess_Q|a'_{ij}(x,t)|\|u\|^2_{H_0^1}$; where we used Cauchy-Schwarz and Poincaré-Friedrichs inequalities. Then we obtain $(A'(t)u, u) \leq \gamma\|u\|^2$, where $\|\cdot\|$ denote the norm in $H_0^1(\Omega) \cap H^2(\Omega)$.

It is well known that $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$ compactly.

Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(s) = s^3$, and $M : H_0^1(\Omega) \to L^2(\Omega)$ a operator defined by $(Mu)(x) = F(u(x))$.

Due to the properties of F it follows that M is monotone, hemicontinuous bounded and

$$\int_0^t (M(u(s)), u'(s))ds \geq -\sigma, \ \ \forall\, t \in [0,T]$$

for all $u \in E_c$ where E_c is the set $\{u \in L^\infty(0,T;H_0^1(\Omega)),\ \ u' \in L^2(0,T;L^2(\Omega))$ and $\|u(0)\| \leq C\}$. The constant σ depends an C.

Let us prove the two last properties. Being

$$|Mu|^2 = \int_\Omega |(Mu)(x)|^2 dx = \int_\Omega |u^3(x)|^2 dx =$$
$$\int_\Omega |u(x)|^6 dx = \|u\|^6_{L^6(\Omega)},$$

it follows from Sobolev inequalities, $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$ $(n \geq 3)$. Therefore $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ $(n = 3)$ and, $|Mu|^2 \leq c\|u\|^6$. So that, M is bounded.

Let $g(\tau) = \int_0^\tau F(r)dr$. Then $g(\tau) \geq 0,\ \forall\, \tau \in \mathbb{R}$, and for $u \in E_c$,

$$\int_0^t (M(u(s)), u'(s)ds = \int_0^t \int_\Omega u^3(x,s)\frac{\partial u}{\partial s}(x,s)dxds =$$
$$\int_0^t \int_\Omega F(u(x,s))\frac{\partial u}{\partial s}(x,s)dxds = \int_0^t \int_\Omega \frac{\partial g}{\partial s}(u(x,s))dxds =$$
$$= \int_\Omega g(u(x,t))dx - \int_\Omega g(u(x,0))dx \geq -\int_\Omega g(u(x,0))dx =$$
$$= -\frac{1}{4}\int_\Omega [u(x,0)]^4 dx = -\frac{1}{4}\int_\Omega |u(x,0)||u(x,0)|^3 dx \geq$$
$$\geq -\left[\int_\Omega |u(x,0)|^2 dx\right]^{\frac{1}{2}} \cdot \left[\int_\Omega |u(x,0)|^6 dx\right]^{\frac{1}{2}} \geq -\sigma.$$

Therefore one has studied the existence and uniqueness of solutions of the mixed problem for the equation

$$k_2(x)u'' + k_1(x)u' + A(t)u + u^3 = f.$$

2)In the same scheme we have analogous results for the equations

$$k_2(x)u'' + k_1(x)u' + A(t)u + M(u) = f$$

where $(Mu)(x) = F(u(x))$ here $F(s)$ is defined by

$$F(s) = \begin{cases} sign(s)\dfrac{s^2}{1+s^2} & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases}$$

Acknowledgments. I would like to express my thanks to Professors L. A. Medeiros and M. Milla Miranda for their encouragement and valuable suggestions.

REFERENCES

[01] BENSOUSSAN, A.; LIONS, J .L.; PAPANICOLAU, G., *Perturbations et aumentation des conditions initials - simgular Perturbation and Boundary Layer Theory* - Springer-Verlag, Lyon,(1976) pp. 10-29.

[02] CODDINGTON, E. & LEVINSON, N.,*Theory of ordinary differential equations.* McGraw-Hill, New York (1955)

[03] J'ORGENS, K.,*Das Anfangswertproblem im Grobem f'ur eine Klasse nichtlinearer Wellengleichungen* , Math Zeitshr. 77, (1961) pp. 295-308.

[04] LARKIN, N.A., *Boundary problems in the large for a class of hyperbolic equations* Siberian Math. Journal, vol. 18, n$^{\underline{o}}$ 6,(1977) pp. 1003-1006.

[05] LIMA, O., *Existence and Uniquiness of an Abstract Nonlinear Hyperbolic-Parabolic Equation.* Applicable Analysis, vol.24, (1987) pp 101-116.

[06] LIONS, J.L., *Quelques Méthodes de Resolution des Problemes aux Limites non Linéaires* - Dunod, Gauthier - Villars, Paris.(1969)

[07] MEDEIROS, L.A., *Non-linear Hyperbolic-Parabolic Partial Differential Equations-Funkcialaj Ekvacioj*, n$^{\underline{o}}$ 23, (1978)pp. 151-158.

[08] MEDEIROS, L.A., *Remarks on hyperbolic-parabolic Partial Differential Equations* - 11$^{\underline{o}}$ Seminário Brasileiro de Análise - (1978)pp. 1-12.

[09] MEDEIROS, L.A., *On the initial and boundary value problem for non-linear wave equations in Hilbert space* - Transl. of the Amer. Math. Soc. - Vol. 136,(1969) pp. 305-327.

[10] MELLO, E.A., *Soluções Fracas de um problema Hiperbólico-Parabólico não-linear*, Tese de Doutorado - IN-URFJ.(1983)

[11] MACIEL, A.B., *Soluções de uma Classe de equações Hiperbólica-Parabólica com não linearidade de tipo contínuo*, 23$^{\underline{o}}$ SBA, (1986) pp. 25-31.

[12] NEVES, B.P., *Perturbações e aumento das condições iniciais de equações hiperbólica-parabólica não Linear* 9$^{\underline{o}}$ SBA, (1979) pp. 109-123.

[13] TANABE, H., *Equations of evolution.* Osaka University.(1977)

[14] VISIK, M.; LADYZENSKAJA, O., *On boundary value problem for PDE and certain classe of operators equations.* Amer. Math. Soc. Transl. Série 2, vol. 10.(1958)

[15] VAGROV, V.N., *On the Cauchy for some parabólic-hyperbolic equations* - Soviet. Math. Dokl, vol. 14-5, (1973) pp. 1396-1400.