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CHARACTERISTIC POLYNOMIALS OF SOME WEIGHTED GRAPH BUNDLES AND ITS APPLICATION TO LINKS

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ABSTRACT. In this paper, we introduce weighted graph bundles and study their characteristic polynomial. In particular, we show that the characteristic polynomial of a weighted K_2 ($\overline{K}_2$)-bundles over a weighted graph Γ_ω can be expressed as a product of characteristic polynomials two weighted graphs whose underlying graphs are Γ As an application, we compute the signature of a link whose corresponding weighted graph is a double covering of that of a given link.

KEY WORDS AND PHRASES. Graphs, weighted graphs, graph bundles, characteristic polynomials, links, signature.
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1. INTRODUCTION.

Let Γ be a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $\boldsymbol{R}$ be the field of real numbers. A *weighted graph* is a pair $\Gamma_\omega = (\Gamma, \omega)$, where Γ is a graph and $\omega: V(\Gamma) \cup E(\Gamma) \to \boldsymbol{R}$ is a function. We call Γ the *underlying graph* of Γ_ω and ω the *weight function* of Γ_ω. In particular, if $\omega(E(\Gamma)) \subset \{1, -1\}$ and $\omega(V(\Gamma)) = \{0\}$, then we call Γ_ω a *signed graph.*

Let $V(\Gamma) = \{u_1, \cdots, u_n\}$. The *adjacency matrix* of Γ_ω is an $n \times n$ matrix $A(\Gamma_\omega) = (a_{ij})$ defined as follows:

$$a_{ij} = \begin{cases} \omega(e) & \text{if } e = u_i u_j \in E(\Gamma) \text{ and } i \neq j, \\ \omega(u_i) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq i, j \leq n$.

The characteristic polynomial $P(\Gamma_\omega; \lambda) = |\lambda I - A(\Gamma_\omega)|$ of the adjacency matrix $A(\Gamma_\omega)$ is called the *characteristic polynomial* of the weighted graph Γ_ω. A root of $P(\Gamma_\omega; \lambda)$ is called an *eigenvalue* of Γ_ω.

Note that if the weight function $\mathcal{L}$ of Γ is defined by $\mathcal{L}(e) = -1$ for $e \in E(\Gamma)$ and $\mathcal{L}(u) = deg(u)$ for $u \in V(\Gamma)$, where $deg(u)$ denotes the degree of u, that is, the number of edges incident to u, then the weighted adjacency matrix $A(\Gamma_{\mathcal{L}})$ is called the *Laplacian matrix* of Γ. We call $\mathcal{L}$ the *Laplacian function* of Γ. The number of spanning trees of a connected graph Γ is the value of any cofactor of $A(\Gamma_{\ell})$ [*Matrix tree theorem*] and is equal to the value $\frac{1}{n}\prod_{\lambda\neq 0}\lambda$, where λ runs through all non-zero eigenvalues of $A(\Gamma_{\ell})$. Moreover, the eigenvalues of $A(\Gamma_{\ell})$ may be used to calculate the radius of gyration of a Gaussian molecule. For more applications of the eigenvalues of $A(\Gamma_{\ell})$, the reader is suggested to refer [5].

2. WEIGHTED GRAPH BUNDLES.

First, we introduce a weighted graph bundle. Every edge of a graph Γ gives rise to a pair of oppositely directed edges. We denote the set of directed edges of Γ by $D(\Gamma)$. By e^{-1} we mean the reverse edge to an edge $e \in D(\Gamma)$. For any finite group G, a *G-voltage assignment* of Γ is a function $\phi: D(\Gamma)\to G$ such that $\phi(e^{-1}) = \phi(e)^{-1}$ for all $e \in D(\Gamma)$. We denote the set of all G-voltage assignments of Γ by $C^1(\Gamma;G)$. Let Λ be another graph and let $\phi \in C^1(\Gamma;Aut(\Lambda))$, where $Aut(\Lambda)$ is the group of all graph automorphisms of Λ. Now, we construct a graph $\Gamma\times^{\phi}\Lambda$ as follows: $V(\Gamma\times^{\phi}\Lambda) = V(\Gamma)\times V(\Lambda)$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $\Gamma\times^{\phi}\Lambda$ if either $u_1u_2 \in D(\Gamma)$ and $v_2 = \phi(u_1u_2)v_1$ or $u_1 = u_2$ and $v_1v_2 \in E(\Lambda)$. We call $\Gamma\times^{\phi}\Lambda$ the *Λ-bundle over Γ associated with ϕ* and the natural map $p^{\phi}:\Gamma\times^{\phi}\Lambda\to\Gamma$ the *bundle projection*. We also call Γ and Λ the *base* and the *fibre* of $\Gamma\times^{\phi}\Lambda$, respectively. Note that the map p^{ϕ} maps vertices to vertices but an image of an edge can be either an edge or a vertex. If Λ is the complement $\overline{K_n}$ of the complete graph K_n of n vertices, then every Λ-bundle over Γ is an n-fold covering graph of Γ

Let Γ_{ω} and Λ_{μ} be two weighted graphs and let $\phi \in C^1(\Gamma;Aut(\Lambda))$. We define the *product of μ and ω with respect to ϕ, $\omega\times^{\phi}\mu$,* as follows:

(1) For each vertex (u,v) of $V(\Gamma\times^{\phi}\Lambda)$, $(\omega\times^{\phi}\mu)(u,v) = \omega(u)+\mu(v)$.

(2) For each edge $e = (u_1,v_1)(u_2,v_2)$ of $E(\Gamma\times^{\phi}\Lambda)$,

$$(\omega\times^{\phi}\mu)(e) = \begin{cases} \omega(u_1u_2) & \text{if } u_1u_2 \in D(\Gamma) \text{ and } v_2 = \phi(u_1u_2)v_1 \\ \mu(v_1v_2) & \text{if } u_1 = u_2 \text{ and } v_1v_2 \in E(\Gamma). \end{cases}$$

We call the weighted graph $(\Gamma\times^{\phi}\Lambda)_{\omega\times^{\phi}\mu}$ the *Λ_{μ}-bundle over Γ_{ω} associated with ϕ*. Briefly, we call it a *weighted graph bundle.*

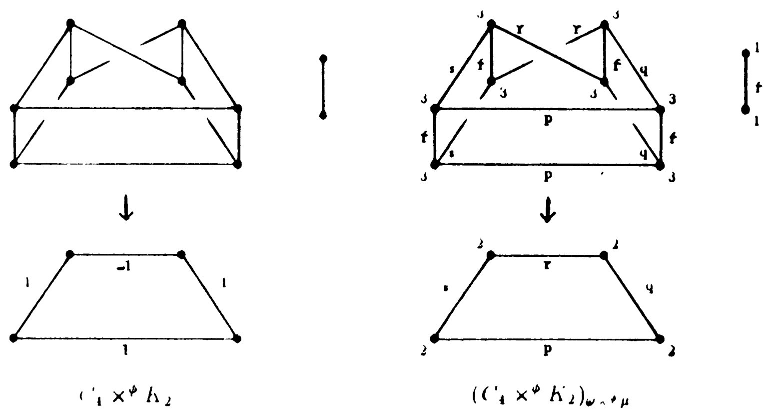

FIGURE 1. The graphs $C_4\times^{\phi}K_2$ and $(C_4\times^{\phi}K_2)_{\omega\times^{\phi}\mu}$.

3. CHARACTERISTIC POLYNOMIALS.

In this section, we give a computation for the characteristic polynomial of a weighted graph bundle $\Gamma\times^{\phi}\Lambda$, where Λ is either complete graph K_2 of two vertices or its complement $\overline{K_2}$, and study their related topics. Note that $Aut(K_2) = Aut(\overline{K_2}) = Z_2$.

For a given graph Γ with weight function ω and for a $\phi \in C^1(\Gamma;Z_2)$, we define a new weight function ω^{ϕ} on Γ as follows:

(1) For $e \in E(\Gamma)$,

$$\omega^{\phi}(e) = \begin{cases} \omega(e) & \text{if } \phi(e) = 1 \\ -\omega(e) & \text{if } \phi(e) = -1 \end{cases}$$

(2) For $v \in V(\Gamma)$, $\omega^{\phi}(v) = w(v)$.

A subgraph of Γ is called an *elementary configuration* if its components are either complete graph K_1 or K_2 or a cycle $C_m (m \geq 3)$. We denote by E_k the set of all elementary configurations of Γ having k vertices. In [3], the characteristic polynomial of a weighted graph Γ_ω is given as follows:

$$P(\Gamma_\omega; \lambda) = \sum_{k=0}^{n} a_k(\Gamma_\omega)\lambda^{n-k},$$

where

$$a_k(\Gamma_\omega) = \sum_{S \in E_k} (-1)^{\kappa(S)} 2^{|C(S)|} \prod_{u \in I_v(s)} \omega(u) \prod_{e \in I_E(S)} \omega(e)^2 \prod_{e \in C(S)} \omega(e).$$

In the above equation, symbols have the following meaning: $\kappa(S)$ is the number of components of S, $C(S)$ the set of all cycles, $C_m (m \geq 3)$, in S, and $I_v(S)(I_E(S))$ is the set of all isolated vertices (edges) in S. Moreover, the product over empty index set is defined to be 1.

For a fixed voltage assignment $\phi \in C^1(\Gamma; Z_2)$, we denote by $E_{\phi-1}$ the set of edges of Γ such that $\phi(e) = -1$, i.e., $E_{\phi-1} = \{e \in E(\Gamma) : \phi(e) = -1\}$. Let $\Gamma(E_{\phi-1})$ be the edge subgraph of Γ induced by $E_{\phi-1}$ having weight zero in vertices. If Γ_ω is a weighted graph, then the weight function of its subgraph S is the restriction of ω on S.

THEOREM 1. Let $\overline{K_2}$ be a constant weighted graph, say $\mu(v) = c$ for $v \in \overline{K}_2$. Then, for each $\phi \in C^1(\Gamma; Z_2)$, we have

$$P((\Gamma \times^{\phi} \overline{K_2})_{\omega \times \phi_c}; \lambda) = P(\Gamma_\omega; \lambda - c) P(\Gamma_{\omega^\phi}; \lambda - c).$$

PROOF. Let $A(\Gamma_\omega)$ be the adjacency matrix of Γ_ω and let $A(\Gamma_{\omega^\phi})$ the adjacency matrix of Γ_{ω^ϕ}. Then we have

$$A(\Gamma_\omega) = A\big((\Gamma \backslash (E_{\phi-1}))_\omega\big) + A(\Gamma(E_{\phi-1})_\omega),$$
$$A(\Gamma_{\omega^\phi}) = A\big((\Gamma \backslash (E_{\phi-1}))_\omega\big) - A(\Gamma(E_{\phi-1})_\omega).$$

Let $V(\Gamma \times^{\phi} \overline{K}_2) = \{(u_1, 1), \cdots, (u_n, 1), (u_1, -1), \cdots, (u_n, -1)\}$. If is not difficult to show that

$$A\big((\Gamma \times^{\phi} \overline{K_2})_{\omega \times \phi_c}\big) = \left(A(\Gamma_\omega) - A(\Gamma(E_{\phi-1})_\omega) + \begin{bmatrix} c & & 0 \\ & c & \\ & & \ddots \\ 0 & & c \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ (A(\Gamma(E_{\phi-1})_\omega)) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let M be a regular matrix of order 2 satisfying

$$M^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Put

$$X = A(\Gamma_\omega) - A(\Gamma(E_{\phi-1})_\omega) + \begin{bmatrix} c & & 0 \\ & c & \\ & & \ddots \\ 0 & & c \end{bmatrix}$$

$$Y = A(\Gamma(E_{\phi-1})_\omega).$$

Then

$$(I \otimes M^{-1})\, A\left((\Gamma \times^{\phi} \overline{K_2})_{\omega \times \phi_c}\right)(I \otimes M)$$

$$= \begin{bmatrix} X+Y & 0 \\ 0 & X-Y \end{bmatrix}$$

$$= \begin{bmatrix} A(\Gamma_\omega) + \begin{bmatrix} c & & 0 \\ & c \ddots & \\ 0 & & c \end{bmatrix} & 0 \\ 0 & A(\Gamma_{\omega^\phi}) + \begin{bmatrix} c & & 0 \\ & c \ddots & \\ 0 & & c \end{bmatrix} \end{bmatrix}.$$

Since $|(I \otimes M^{-1})(I \otimes M)| = 1$ and

$$\left|\left[\lambda I - A\left((\Gamma \times^{\phi} \overline{K_2})_{\omega \times \phi_c}\right)\right]\right| = \left|\left[\lambda I - (I \otimes M^{-1})\, A\left((\Gamma \times^{\phi} \overline{K_2})_{\omega \times \phi_c}\right)(I \otimes M)\right]\right|,$$

we have our theorem. □

THEOREM 2. Let $K_{2_\mu} = (K_2, \mu)$ be a weighted graph having constant weight on vertices. Then, for each $\phi \in C(\Gamma; Z_2)$, we have

$$P\left((\Gamma \times^{\phi} K_2)_{\omega \times \phi_\mu}; \lambda\right) = P(\Gamma_\omega; \lambda - c_v - c_e) P(\Gamma_{\omega^\phi}; \lambda - c_v + c_e),$$

where $c_v = \mu(v_1) = \mu(v_2)$ for the vertices v_1, v_2 and $c_e = \mu(e)$ for the edge e in K_2.

PROOF. Clearly, we have

$$A\left((\Gamma \times^{\phi} K_2)_{\omega \times \phi_\mu}\right) = \left(A(\Gamma_\omega) - A\left(\Gamma(E_{\phi-1})_\omega\right) + \begin{bmatrix} c_v & & 0 \\ & c_v \ddots & \\ 0 & & c_v \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \left(A\left(\Gamma(E_{\phi-1})_\omega\right) + \begin{bmatrix} c_e & & 0 \\ & c_e \ddots & \\ 0 & & c_e \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $c_v = \mu(v_1) = \mu(v_2)$ and $c_e = \mu(e)$ for the edge e in K_2. Let M be a regular matrix of order 2 satisfying

$$M^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$(I \otimes M^{-1})\, A\left((\Gamma \times^{\phi} K_2)_{\omega \times \phi_\mu}\right)(I \otimes M)$$

$$= \begin{bmatrix} X + Y + \begin{bmatrix} c_e & & 0 \\ & c_e \ddots & \\ 0 & & c_e \end{bmatrix} & 0 \\ 0 & X - Y + \begin{bmatrix} c_e & & 0 \\ & c_e \ddots & \\ 0 & & c_e \end{bmatrix} \end{bmatrix}.$$

$$= \begin{bmatrix} A(\Gamma_\omega)+Z_1 & 0 \\ 0 & A(\Gamma_{\omega^\phi})+Z_2 \end{bmatrix}$$

where X and Y are the same matrices as in the proof of Theorem 1 and for $i=1,2$,

$$Z_i = \begin{bmatrix} c_v+(-1)^{i-1}c_e & & 0 \\ & c_v+(-1)^{i-1}c_e & \\ & \ddots & \\ 0 & & c_v+(-1)^{i-1}c_e \end{bmatrix}.$$

Using method similar to the proof of Theorem 1, we have our theorem. □

Note that for any $\phi \in C^1(\Gamma; Aut(\Lambda))$, the Laplacian function of $\Gamma \times^\phi \Lambda$ is the product of Laplacian functions of Γ and Λ with respect to ϕ. Clearly, the Laplacian function of the $\overline{K_2}$ is the zero function; and the Laplacian function of the K_2 has value 1 and -1 for each of its vertices and its edge, respectively. We shall denote the Laplacian function of a graph by $\mathcal{L}$ if it makes no confusion. Then Theorem 1 and Theorem 2 give the following corollary.

COROLLARY 1. For any $\phi \in C^1(\Gamma; Z_2)$,

(1) $P\big((\Gamma \times^\phi \overline{K_2})_{\mathcal{L}}; \lambda\big) = P(\Gamma_{\mathcal{L}}; \lambda) P(\Gamma_{\mathcal{L}^\phi}; \lambda)$.

(2) $P\big((\Gamma \times^\phi K_2)_{\mathcal{L}}; \lambda\big) = P(\Gamma_{\mathcal{L}}; \lambda) P(\Gamma_{\mathcal{L}^\phi}; \lambda - 2)$. □

Now, we consider another invariant of weighted graphs called the *signature*. Since $A(\Gamma_\omega)$ is symmetric, $A(\Gamma_\omega)$ can be diagonalized through congruence over $\boldsymbol{R}$. Let d_+ denote the number of positive diagonal entries, and d_- the number of negative diagonal entries. The *signature* of a weighted graph (Γ_ω) is defined by $\sigma(A(\Gamma_\omega)) = d_+ - d_-$ and is denoted by $\sigma(\Gamma_\omega)$. It is an invariant for weighted 2-isomorphic graphs (see [7]).

From now on, we will consider the weight function on $\overline{K_2}$ as zero function and the weight function μ on K_2 as the map defined by $\mu(v)=0$ for each $v \in V(K_2)$ and $\mu(e)=c_e$ for the edge e of K_2. Then we can compute the signature of a double covering of Γ.

COROLLARY 2. $\sigma\big((\Gamma \times^\phi \overline{K_2})_{\omega \times \phi_0}\big) = \sigma(\Gamma_\omega) + \sigma(\Gamma_{\omega^\phi})$ for $\phi \in C^1(\Gamma; Z_2)$. □

For convenience, we adapt the following notations. For a real number c, a weighted graph Γ_η and an eigenvalue λ of Γ_η,

$$P(c)_\eta^- = \{\lambda < 0 : \lambda + c > 0\},$$

$$P(c)_\eta^+ = \{\lambda > 0 : \lambda + c > 0\},$$

$$Z(c)_\eta = \{\lambda \neq 0 : \lambda + c = 0\},$$

$$N(c)_\eta^- = \{\lambda < 0 : \lambda + c < 0\},$$

$$N(c)_\eta^+ = \{\lambda > 0 : \lambda + c < 0\}.$$

We also denote the multiplicity of λ by $m_\eta(\lambda)$.

By using the above notations and Theorem 2, we get the signature of a K_2-bundle over Γ.

COROLLARY 3. For $\phi \in C^1(\Gamma; Z_2)$,

(1) if $c_e \geq 0$, then

$$\begin{aligned}\sigma\big((\Gamma \times^\phi K_2)_{\omega \times \phi_\mu}\big) = \sigma(\Gamma_\omega) + \sigma(\Gamma_{\omega^\phi}) \\ + \Big(2\textstyle\sum_{\lambda \in P(c_e)_\omega^-} m_\omega(\lambda) + m_\omega(0) + \sum_{\lambda \in Z(c_e)_\omega} m_\omega(\lambda)\Big) \\ - \Big(2\textstyle\sum_{\lambda \in N(-c_e)_{\omega^\phi}^+} m_{\omega^\phi}(\lambda) + m_{\omega^\phi}(0) + \sum_{\lambda \in Z(-c_e)_{\omega^\phi}} m_{\omega^\phi}(\lambda)\Big).\end{aligned}$$

(2) if $c_e < 0$, then

$$\sigma\Big((\Gamma \times^{\phi} K_2)_{\omega \times \phi_\mu}\Big) = \sigma(\Gamma_\omega) + \sigma(\Gamma_{\omega^\phi})$$
$$-\Big(2\textstyle\sum_{\lambda \in N(c_e)^+_\omega} m_\omega(\lambda) + m_\omega(0) + \sum_{\lambda \in Z(c_e)_\omega} m_\omega(\lambda)\Big)$$
$$+\Big(2\textstyle\sum_{\lambda \in P(-c_e)^-_{\omega^\phi}} m_{\omega^\phi}(\lambda) + m_{\omega^\phi}(0) + \sum_{\lambda \in Z(-c_e)_{\omega^\phi}} m_{\omega^\phi}(\lambda)\Big).$$

□

REMARK. Though the results in this section stated only for a simple graph, it remains true for any graph.

4. APPLICATIONS TO LINKS.

In a signed graph Γ_ω, an edge e of Γ is said to be *positive* if $\omega(e) = 1$ and *negative* otherwise. For a signed graph Γ_ω, we define a new weight function $\widetilde{\omega}$ of Γ by $\widetilde{\omega}(e) = \omega(e)$ for any edge $e \in \Gamma_\omega$ and $\widetilde{\omega}(u_i) = \sum_{j=1, i \neq j}^{n} a_{ij}$, where a_{ij} is the number of positive edges minus the number of negative edges which have two end vertices u_i and u_j. Given a knot or link L in $\boldsymbol{R}^3$, we project it into $\boldsymbol{R}^2$ so that each crossing point has proper double crossing. The image of L is called a *link (or knot) diagram* of L, and we do not distinguish between a diagram and the image of L.

We may assume without loss of generality that a link diagram $\widetilde{L}$ of L intersects itself transversely and has only finitely many crossings. The link diagram $\widetilde{L}$ divides $\boldsymbol{R}^2$ into finitely many domains, which will be classified as shaded or unshaded. No two shaded or unshaded domains have an edge in common. We now construct a signed planar graph Γ_ω from $\widetilde{L}$ as follows: take a point v_i from each unshaded domain D_i. These points form the set of vertices $V(\Gamma_\omega)$ of Γ_ω. If the boundaries of D_i and D_j intersect k-times, say, crossing at $c_{\ell_1}, c_{\ell_2}, \cdots, c_{\ell_k}$, then we form multiple edges $e_{\ell_1}, e_{\ell_2}, \cdots, e_{\ell_k}$ on $\boldsymbol{R}^2$ with common end vertices v_i and v_j, where each edge e_{ℓ_m} passes through a crossing c_{ℓ_m}, for $m = 1, 2, \cdots, k$. To define the weight of an edge, first, we define the index $\epsilon(c)$ to each crossing c of the link diagram as in Figure 2. To each edge of Γ passes through exactly one crossing, say c, of $\widetilde{L}$, the weight $\omega(e)$ will be defined as $\omega(e) = \epsilon(c)$. (See Figure 3).

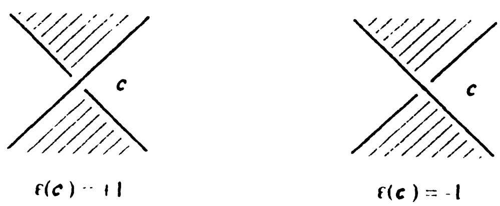

FIGURE 2. The index $\epsilon(c)$.

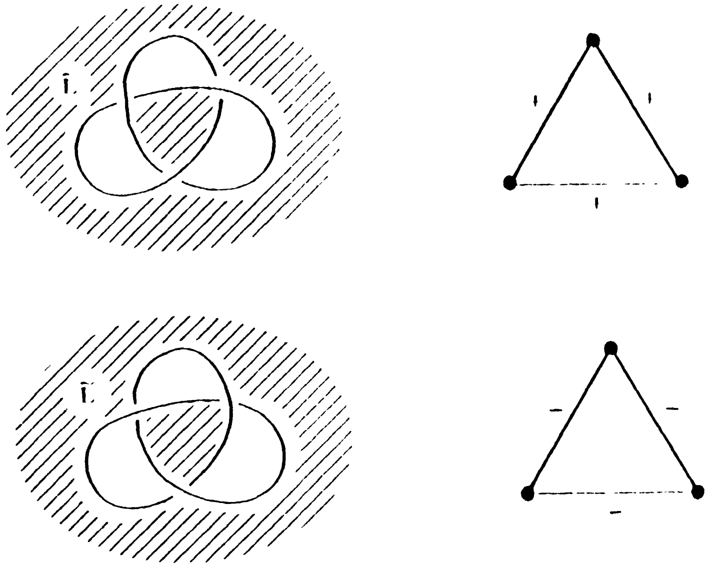

FIGURE 3. The correspondence between $\widetilde{L}$ and $\Gamma_\omega(\widetilde{L})$.

The resulting signed planar is called the *graph of a link with respect to* $\widetilde{L}$ and is denoted by $\Gamma_\omega(\widetilde{L})$. The signed planar graph $\Gamma_\omega(\widetilde{L})$ depends not only on $\widetilde{L}$ but also on shading. Conversely, given a signed planar graph Γ_ϑ, one can construct uniquely the link diagram $L(\widetilde{L}_\vartheta)$ of a link so that $\Gamma_\omega(L(\widetilde{\Gamma}_\vartheta)) = \Gamma_\vartheta$.

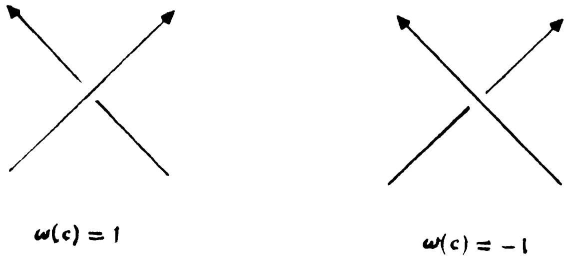

FIGURE 4. The index $\omega(c)$.

Suppose that we are given an oriented link L. The orientation of L induces the orientation of a diagram $\widetilde{L}$. We then define the second index $\omega(c)$, called the *twist or writhe* at each crossing c as show in Figure 4. We now need the third index $\eta_\rho(c)$ at crossing c. Let $\widetilde{L}$ be an oriented diagram and ρ shading on $\widetilde{L}$. Let $\eta_\rho(c) = \omega(c)\delta_{e(c)\omega(c)}$, where δ denotes Kronecker's delta. We define $\eta_\rho(\widetilde{L}) = \sum \eta_\rho(c)$, where the summation runs over all crossing in $\widetilde{L}$. The index $\eta_\rho(\widetilde{L})$ depends not only on the shading ρ but also on the orientation of $\widetilde{L}$. The following Lemma can be found in ([7], [4]).

LEMMA 1. The signature $\sigma(L)$ of a link L is $\sigma(L) = \sigma(\Gamma(\widetilde{L})) - \eta_\rho(\widetilde{L})$. □

Let $\widetilde{L}_1$ and $\widetilde{L}_2$ be link diagrams of L_1 and L_2, respectively. The link L_2 is called a *double covering* of the link L_1 if $\Gamma_\omega(\widetilde{L}_2)$ is a double covering of $\Gamma_\omega(\widetilde{L}_1)$ as weighted graphs and it can be extended to a branched covering on $\mathbf{R}^2$. Let ϕ be a voltage assignment in $C^1(\Gamma_\omega(\widetilde{L}); Z_2)$ such that $\phi(e) = -1$ for some edge e and $\phi(e) = 1$ otherwise, then $\Gamma_\omega(\widetilde{L}) \times^\phi \overline{K_2}$ is a planar double covering of $\Gamma_\omega(\widetilde{L})$ of which the corresponding link is a double covering of L.

Therefore, one can construct the double covering link diagram $\widetilde{L}(\Gamma_\omega(\widetilde{L}) \times^\phi \overline{K_2})$ of $\widetilde{L}$. Moreover, we can give an orientation on $\widetilde{L}(\Gamma_\omega(\widetilde{L}) \times^\phi \overline{K_2})$ so that the covering map from $\widetilde{L}$ to $\widetilde{L}(\Gamma_\omega(\widetilde{L}) \times^\phi \overline{K_2})$ preserves the orientation. We have $\eta_\rho(\widetilde{L}(\Gamma_\omega(\widetilde{L}) \times^\phi \overline{K_2})) = 2\eta_\rho(\widetilde{L})$ (see Figure 5).

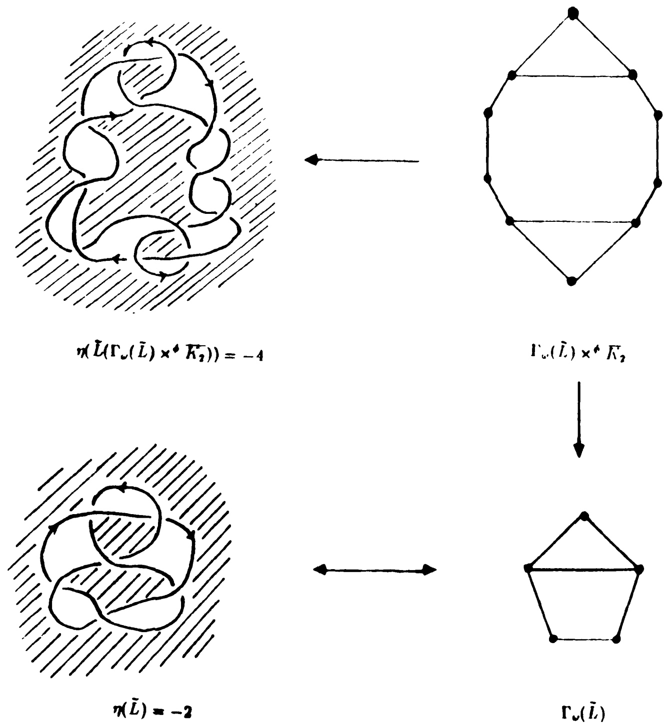

FIGURE 5. Covering graph and covering link.

Therefore, by using Lemma 1 and Corollary 2, we get the following theorem.

THEOREM 3. For any oriented link diagram $\widetilde{L}$,

$$\sigma(\widetilde{L}\,(\Gamma_\omega(\widetilde{L}\,) \times^\phi \overline{K_2})) = \sigma(\Gamma_\omega(\widetilde{L}\,)) + \sigma(\Gamma_{\omega^\phi}(\widetilde{L}\,)) - 2\eta_\rho(\widetilde{L}\,)$$

for each $\phi \in C^1(\Gamma; Z_2)$ such that $\phi(e) = -1$ for some edge $e \in \Gamma_\omega(\widetilde{L}\,)$ and $\phi(e) = 1$ otherwise.

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