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# THE MEET OF LOCALLY CONNECTED GROUP TOPOLOGIES

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ABSTRACT. The collection of all group topologies on a given group is a complete lattice partially ordered by inclusion. The purpose of this paper is to develop a convenient method of defining the meet of an arbitrary collection of group topologies and demonstrate that the meet of a collection of locally connected group topologies is also locally connected.

KEY WORDS AND PHRASES. Topological groups, quotient spaces, locally connected, translatable and semicontinuous topologies.

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## 1. INTRODUCTION.

There has been considerable interest, by Graev [1], Comfort [2] and others, in finding group topologies on a given group. An approach is to start with a topology on a given group and derive an associated group topology. In [3] Roelcke and Dierolf developed a method for finding a neighborhood basis for the finest group topology contained in the intersection of an arbitrary collection of topologies on a group. In general, however, this basis does not consist of open sets. In [4] Clark and Schneider studied the meet in the lattice of group topologies and developed a convenient method for finding a basis of open sets for the meet of a countable collection of group topologies on an abelian group.

In this paper we will develop a convenient method for finding an open neighborhood basis for the finest group topology contained in any topology on a group. This method will enable us to find a neighborhood basis for the meet of an arbitrary collection of group topologies and demonstrate that the meet of an arbitrary collection of locally connected group topologies is also locally connected.

## 2. PRELIMINARY RESULTS.

Let G be a group. Then it is well known that $\mathcal{F}$ is a fundamental system for G iff $\mathcal{F}$ is a collection of subsets of G each containing the identity and satisfying the following properties:

1) If $U, V \in \mathcal{F}$ then there exists $W \in \mathcal{F}$ such that $W \subset U \cap V$.
2) If $U \in \mathcal{F}$ and $a \in U$ then there exists $V \in \mathcal{F}$ such that $aV \subset U$.
3) If $U \in \mathcal{F}$ and $x \in G$ then there exists $V \in \mathcal{F}$ such that $xVx^{-1} \subset U$.
4) If $U \in \mathcal{F}$ then there exists $V \in \mathcal{F}$ such that $V^{-1} \subset U$.
5) If $U \in \mathcal{F}$ then there exists $V \in \mathcal{F}$ such that $VV \subset U$.

If $\mathcal{F}$ is a fundamental system then there exists a unique topology t on G such that t is a group topology having $\mathcal{F}$ as a basis at the identity.

DEFINITION 1. Let G be a group and $\mathcal{T}$ be a collection of subsets of G each containing the identity of G. Then $\mathcal{T}$ is a translatable system for G iff $\mathcal{T}$ satisfies properties 1, 2 and 3 above and by [5] $\mathcal{T}$ is a semifundamental system for G iff $\mathcal{T}$ satisfies properties 1 through 4 above.

In [5] Clay defined a semicontinuous topology and showed that a basis of open neighborhoods at the identity is a semifundamental system. She also showed that if $\mathcal{S}$ is a semifundamental system on a group G then there is a unique topology t on G such that t is a semicontinuous topology having $\mathcal{S}$ as a basis at the identity.

DEFINITION 2. A translatable group is a group G endowed with a topology t such that Ua and aU are elements of t for all $U \in t$ and $a \in G$. If (G,t) is a translatable group then we say that t is a translatable topology on G.

It is easy to see that if t is a translatable topology then a basis of open neighborhoods at the identity is a translatable system and if $\mathcal{T}$ is a translatable system for a group G then there is a unique topology t on G such that t is a translatable topology having $\mathcal{T}$ as a basis at the identity.

DEFINITION 3. Let G be a group and t any topology on G. The Graev operator assigns to the pair (G,t) the pair (G,g(t)) where g(t) is the finest group topology on G contained in t.

The collection of all group topologies on a given group is a complete lattice partially ordered by inclusion. The join of an arbitrary collection of group topologies $\{t_\alpha\}_{\alpha\in\Delta}$ has $\{V : V \in t_\alpha$ for some $\alpha \in \Delta\}$ as a subbasis and the meet is the join of $\{t : t \subset t_\alpha$ for all $\alpha\in\Delta$ and t is a group topology$\}$. Samuel pointed out in [6] that $\bigcap_{\alpha\in\Delta} t_\alpha$ may not be a group topology. Therefore $\bigwedge_{\alpha\in\Delta} t_\alpha \neq \bigcap_{\alpha\in\Delta} t_\alpha$ (in general) but $g(\bigcap_{\alpha\in\Delta} t_\alpha) \subset t_\alpha$ for all $\alpha\in\Delta$ and $g(\bigcap_{\alpha\in\Delta} t_\alpha)$ is a group topology. Therefore $g(\bigcap_{\alpha\in\Delta} t_\alpha) = \bigwedge_{\alpha\in\Delta} t_\alpha$.

If $G_i = G$ for all $i \in \mathbb{N}$ then we let $\bigoplus_{i=1}^\infty G$ denote $\bigoplus_{i=1}^\infty G_i$, $\prod_{i=1}^\infty G$ denote $\prod_{i=1}^\infty G_i$ and define $m_\infty : \bigoplus_{i=1}^\infty G \longrightarrow G$ by $m_\infty(\langle a_i\rangle) = a_1 a_2 a_3 \cdots a_n$ where $a_m = e$ for all $m > n$. Let T be a translatable topology on $\bigoplus_{i=1}^\infty G$, t be a translatable topology on G, and S be the topology inherited by $\bigoplus_{i=1}^\infty G$ as a subspace of $\prod_{i=1}^\infty G$ when $\prod_{i=1}^\infty G$ is endowed with the box topology with t on each factor. Then $q_\infty(T)$ shall denote the quotient topology on G induced by $m_\infty : (\bigoplus_{i=1}^\infty G, T) \longrightarrow G$ and $q_\infty[t]$ shall denote the quotient topology $q_\infty(S)$.

THEOREM 1. If T is a translatable topology on $\bigoplus_{i=1}^\infty G$ then the quotient map $m_\infty : (\bigoplus_{i=1}^\infty G, T) \longrightarrow (G, q_\infty(T))$ is open.

PROOF: Let $V \in T$ and $\langle a_i\rangle \in m_\infty^{-1}(m_\infty(V))$ then there exists $\langle b_i\rangle \in V$ such that $m_\infty(\langle b_i\rangle) = m_\infty(\langle a_i\rangle)$. Since $\langle a_i\rangle$ and $\langle b_i\rangle$ are in $\bigoplus_{i=1}^\infty G$ there exists m such that $a_n = e$ and $b_n = e$ for all $n > m$ and $b_1 b_2 b_3 \cdots b_m = a_1 a_2 a_3 \cdots a_m$.

Let $h_1 = b_1^{-1} a_1$ and $h_i = b_i b_{i+1} b_{i+2} \cdots b_m a_m^{-1} a_{m-1}^{-1} a_{m-2}^{-1} \cdots a_i^{-1}$ for all $i=3,4,5,\cdots m$. Let $u = \langle e, h_1^{-1}, h_3^{-1}, h_4^{-1}, \cdots, h_{m-1}^{-1}, h_m^{-1}, e, e, e, \cdots\rangle$ and $w = \langle h_1, h_3, h_4, \cdots, h_m, e, e, e, \cdots\rangle$ then $u, w \in \bigoplus_{i=1}^\infty G$ and $\langle a_i\rangle = u\langle b_i\rangle w \in uVw \in T$ since T is translatable.

Let $\langle x_i\rangle \in uVw$. Then there exists $\langle y_i\rangle \in V$ such that $\langle x_i\rangle = u\langle y_i\rangle w$ and there exists $n_y$ such that $y_i = e$ for all $i > n_y$. Let $k = \max\{m, n_y\}$ then:

$$m_\infty(\langle x_i\rangle) = x_1 x_2 x_3 \ldots x_k = (y_1 h_1)(h_1^{-1} y_2 h_3)(h_3^{-1} y_3 h_4) \cdots (h_{m-1}^{-1} y_{m-1} h_m)(h_m^{-1} y_m) y_{m+1} y_{m+1} \cdots y_k$$
$$= y_1 y_2 y_3 \cdots y_k \in m_\infty(V) \blacksquare \tag{2.1}$$

COROLLARY 1. Let $n \in \mathbb{N}$ and define $m_n : \prod_{i=1}^n G \longrightarrow G$ by $m_n(a_1, a_2, a_3, \ldots a_n) = a_1 a_2 a_3 \cdots a_n$. Let T be a translatable topology on $\prod_{i=1}^n G$ then the quotient map $m_n : (\prod_{i=1}^n G, T) \longrightarrow G$ is open.

## 3. THE GRAEV OPERATOR.

Let $n \in \{0,1,2,\dots\}$, $P_n$ be the set of all permutations on $\{n+1, n+2, n+3, \dots\}$ and $\sigma_i$ denote $\sigma^{-1}(i)$ for all $i \in \mathbb{N}$ and $\sigma \in P_n$. Suppose s is a semicontinuous topology on G and $\mathcal{S}$ is a semifundamental system for s. Let $<A_i>$ be a sequence of elements of $\mathcal{S}$ and:

$$P(<A_i>,n) = \bigcup_{\substack{\sigma \in P_n \\ j \in \mathbb{N}}} (A_1 \times A_2 \times A_3 \times \cdots \times A_n \times A_{\sigma_{n+1}} \times A_{\sigma_{n+2}} \times \cdots \times A_{\sigma_{n+j}} \times e \times e \dots) \tag{3.1}$$

Then $\{P(<A_i>,n): A_i \in \mathcal{S}$ and $n \in \{0,1,2,\dots\}\}$ is a translatable system for a topology, $C_s$, on $\oplus_{i=1}^{\infty} G$. (Introduced by Bradd Clark, $C_s$ is known as the cross topology due to the "shape" of the basic neighborhoods of the identity element.) Now if we let $C(<A_i>) = \bigcup_{\sigma \in P_0} [\bigcup_{n \in \mathbb{N}} (A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} \cdots A_{\sigma_n})]$ then $\mathcal{C}(s) = \{C(<A_i>): A_i \in \mathcal{S}\}$ is a neighborhood basis of the identity for $q_\infty(C_s)$. This follows from the fact that the quotient map $m_\infty: (\oplus_{i=1}^{\infty} G, C_s) \longrightarrow G$ is open.

THEOREM 2. $q_\infty(C_s) = g(s)$.

PROOF: One can easily see that $\mathcal{C}(s)$ satisfies properties 1, 3 and 4 of a fundamental system. Therefore let us consider properties 2 and 5. Let $U = C(<U_i>)$ be an element of $\mathcal{C}(s)$ and $a \in U$ then:

(2) There exists $n \in \mathbb{N}$ and $\delta \in P_0$ such that $a \in U_{\delta_1} U_{\delta_2} U_{\delta_3} \cdots U_{\delta_n}$. Therefore we can find $a_i \in U_{\delta_i}$ for $i = 1,2,3,\dots,n$ such that $a = a_1 a_2 a_3 \dots a_n$ and we can find $V_i \in \mathcal{S}$ such that $V_i a_i \subset U_{\delta_i}$ for all $i = 1,2,3,\dots,n$. Let $b_i = a_i a_{i+1} a_{i+2} \cdots a_n$ then there exists $U'_i \in \mathcal{S}$ and $W \in \mathcal{S}$ such that $b_i U'_i \subset V_i b_i$ for all $i = 1,2,3,\dots,n$ and $W \subset \bigcap_{i=1}^{n} U'_i$. Now $U_{\delta_1} U_{\delta_2} U_{\delta_3} \cdots U_{\delta_n} \supset aW^n$ and we can find, for all $i \in \mathbb{N}$, $W_i \in \mathcal{S}$ such that $W_i \subset U_{\delta_{n+1}} \cap W^n$. Therefore $a[C(<W_i>)] \subset U$.

(5) There exists $V_i \in \mathcal{S}$ for $i = 1,3,5,\dots$ such that $V_i \subset U_i \cap U_{i+1}$. Let:

$$<A_i> = <U_1, U_3, U_5, \dots>, \quad <B_i> = <U_2, U_4, U_6, \dots> \text{ and } V = C(<V_i>). \tag{3.2}$$

Then $V^2 \subset [C(<A_i>)][C(<B_i>)] \subset U$.

Therefore $q_\infty(C_s)$ is a group topology and since s is translatable $q_\infty(C_s) \subset s$. Let $U \in g(s)$ such that $e \in U$ and suppose $\mathcal{S}$ is the set of all open neighborhoods of e. Then there exists $V_1 \in g(s)$ such that $e \in V_1$ and $(V_1)^3 \subset U$. We can also find $V_i \in g(s)$ such that $e \in V_i$ and $(V_i)^3 \subset V_{i-1}$ for all $i = 2,3,4,\dots$.

Let $\sigma \in P_0$ and $k_{m,n} = \min\{\sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots, \sigma_n\}$ ($\sigma_i$ denotes $\sigma^{-1}(i)$ for all $i \in \mathbb{N}$). Then $V_{\sigma_1} V_{\sigma_2} \subset (V_{k_{1,2}})^2 \subset (V_{k_{1,2}})^3 \subset U$, $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \subset (V_{k_{1,3}})^3 \subset U$ and $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} V_{\sigma_4} \subset (V_{k_{1,2}})^2 (V_{k_{3,4}})^2 \subset (V_{k_{1,4}})^3 \subset U$.

Suppose then that $V_{\sigma_1} V_{\sigma_2} V_{\sigma_3} \cdots V_{\sigma_i} \subset (V_{k_{1,i}})^3 \subset U$ for all $i \le n-1$ and $\sigma \in P_0$. Let $\sigma_j = k_{1,n}$ then:

$$V_{\sigma_1}V_{\sigma_2}V_{\sigma_3}\cdots V_{\sigma_n} = V_{\sigma_1}V_{\sigma_2}V_{\sigma_3}\cdots V_{\sigma_{j-1}}V_{\sigma_j}V_{\sigma_{j+1}}\cdots V_{\sigma_n} \quad (3.3)$$

$$\subset (V_{k_{1,j-1}})^3 V_{k_{1,n}} (V_{k_{j+1,n}})^3 \subset (V_{k_{1,n}})^3 \subset U. \quad (3.4)$$

Therefore $C(\langle V_i\rangle) \subset U$ and $g(s) \subset q_\infty(C_s) \subset s$. Therefore $g(s) = q_\infty(C_s)$ since $q_\infty(C_s)$ is a group topology ■

If G is abelian then $\mathscr{C}(s) = \{ \bigcup_{n\in\mathbb{N}}(V_1V_2V_3\ldots V_n) \mid V_i \in \mathscr{S} \}$ and by a proof similar to the one above we can prove the following corollary.

COROLLARY 2 If G is abelian then $g(s) = q_\infty[s]$.

Suppose then that we have an arbitrary collection of group topologies $\{t_\alpha\}_{\alpha\in\Delta}$ on a given group. Since $\bigcap_{\alpha\in\Delta} t_\alpha$ is semicontinuous we have that $\mathscr{C}(\bigcap_{\alpha\in\Delta} t_\alpha)$ is a fundamental system for $\bigwedge_{\alpha\in\Delta} t_\alpha$.

In [7] Clark and Schneider defined S(t) to be the finest semicontinuous topology on G contained in t and gave a concrete description of S(t). Since g(t) is semicontinuous g(S(t)) = g(t). Therefore we can also use Theorem 2 to prove the following corollary.

COROLLARY 3. If t is any topology on a group G and $C_{S(t)}$ is Clark's Cross topology with respect to S(t) then $q_\infty(C_{S(t)}) = g(t)$.

## 4. LOCAL CONNECTEDNESS.

Suppose A and B are connected subsets of a translatable group G then aB is connected for all $a \in A$. Let $x \in B$ then $Ax \subset AB$ and Ax is connected. Therefore AB $= Ax \cup AB = Ax \cup [\bigcup_{a\in A}(aB)]$ is connected since $ax \in Ax \cap aB$ for all $a \in A$. Therefore we have the following theorem.

THEOREM 3. Suppose t is a locally connected and translatable topology on G then $q_\infty[t]$ is also locally connected.

In a similar fashion we can prove the following corollary.

COROLLARY 4. If t is a locally connected and translatable topology on G and $C_t$ is Clark's Cross topology with respect to t then $q_\infty(C_t)$ is also locally connected.

Suppose then that $\{t_\alpha\}_{\alpha\in\Delta}$ is a collection of locally connected group topologies on G then $\bigcap_{\alpha\in\Delta} t_\alpha$ is also locally connected and therefore $g(\bigcap_{\alpha\in\Delta} t_\alpha) = \bigwedge_{\alpha\in\Delta} t_\alpha$ is also locally connected.

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