Internat. J. Math. & Math. Sci.
VOL. 16 NO. 4 (1993) 669-674

A COMMON FIXED POINT THEOREM OF MEIR AND KEELER TYPE

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(Received June 10, 1992 and in revised form September 22, 1992))

ABSTRACT. In this paper, we introduce the concept of compatible mappings of type (A) on a metric space, which is equivalent to the concept of compatible mappings under some conditions, and give a common fixed point theorem of Meir and Keeler type. Our result extends, generalized and improves some results of Meir-Keeler, Park-Bae, Park-Rhoades, Pant and Rao-Rao, etc.

1991 AMS SUBJECT CLASSIFICATION CODE. 54H25.
KEY WORDS AND PHRASES. Common fixed points, compatible mappings of type (A), generalized $(\varepsilon,\delta)-\{S,T\}$-contractions, and $\{S,T\}$-iterations.

1. INTRODUCTION.

In [6], Jungck proved a common fixed point theorem of commuting mappings on a metric space. Since then, he and many authors extended, generalized and unified this theorem in many ways ([2], [4], [5], [7]-[10], [15]-[20], [22], [23]). For example, Sessa ([22]) introduced the concept of weakly commuting mappings, which is a generalization of the concept of commuting mappings, and he and others proved some fixed point theorems for weakly commuting mappings ([20]-[23]).

Recently, Jungck ([8]) proposed a generalization of the concept of weakly commuting mappings, which is called compatible mappings, and he generalized some fixed point theorems of Meir-Keeler type, especially, a theorem of Park-Bae ([16]), and in [11], Jungck, Murthy and Cho introduced the concept of compatible mappings of type (A) on metric spaces and obtained some fixed point theorems for these mappings.

On the other hand, in [14], Meir and Keeler established a fixed point theorem for a self-mapping f of a metric space (X,d) satisfying the following condition:

For every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\varepsilon \le d(x,y) < \varepsilon+\delta \text{ implies } d(fx,fy)<\varepsilon. \tag{1.1}$$

In [13], Maiti and Pal also proved a fixed point theorem for a self-mapping f of a metric space (X,d) satisfying the following condition, which is a generalization of (1.1):

For every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\varepsilon \le max\{d(x,y),d(x,fx),d(y,fy)\} < \varepsilon+\delta \text{ implies } d(fx,fy)<\varepsilon. \tag{1.2}$$

In [17] and [18], Park-Rhoades and Rao-Rao proved some fixed point theorems for self-mappings f and g of a metric space (X,d) satisfying the following condition, respectively, which is a generalization of (1.2):

For every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\varepsilon \le max\{d(fx,fy),d(fx,gx),d(fy,gy),\tfrac{1}{2}(d(fx,gy)+d(fy,gx))\} < \varepsilon+\delta \text{ implies } d(gx,gy)<\varepsilon \tag{1.3}$$

Many other fixed point theorems of Meir-Keeler type are given in [1], [3], [8], [12], [15], [16], [19] and [21].

In this paper, we introduce the concept of compatible mappings of type (A), which is equivalent to the concept of compatible mappings under some conditions, and give a common fixed point theorem for compatible mappings of type (A). which extends, generalizes and improves some common fixed point theorems of Meir-Keeler type.

2. COMPATIBLE MAPPINGS OF TYPE (A).

In this section, we show that two pairs of compatible mappings and compatible mappings of type (A) are equivalent under some conditions and give several properties of compatible mappings of type (A) for our main results. Throughout this paper, (X,d) denotes a metric space.

DEFINITION 2.1. Let $S,T:(X,d)\to(X,d)$ be mappings. S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(ST(x_n),TS(x_n))=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n)=\lim_{n\to\infty} T(x_n)=t$ for some t in X.

DEFINITION 2.2. Let $S,T:(X,d)\to(X,d)$ be mappings. S and T are said to be *compatible* of type (A) if

$$\lim_{n\to\infty} d(TS(x_n),SS(x_n))=0 \text{ and } \lim_{n\to\infty} d(ST(x_n),TT(x_n))=0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n)=\lim_{n\to\infty} T(x_n)=t$ for some t in X.

In [11], the following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

PROPOSITION 2.1. Let $S,T:(X,d)\to(X,d)$ be continuous mappings. If S and T are compatible, then they are compatible of type (A).

PROPOSITION 2.2. Let $S,T:(X,d)\to(X,d)$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.

The following is a direct consequence of Propositions 2.1 and 2.2:

PROPOSITION 2.3. Let $S,T:(X,d)\to(X,d)$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A)

The following examples show that Proposition 2.3 is not true if S and T are discontinuous in some point of X.

EXAMPLE 2.1. Let $X=\mathbf{R}$, the set of real numbers, with the usual metric $d(x,y)=|x-y|$. Define $S,T:(X,d)\to(X,d)$ as follows:

$$S(x)=\begin{cases}\frac{1}{x} & \text{if } x\neq 0,\\ 1 & \text{if } x=0\end{cases} \qquad \text{and} \qquad T(x)=\begin{cases}\frac{1}{x^2} & \text{if } x\neq 0,\\ 2 & \text{if } x=0.\end{cases}$$

Then S and T are not continuous at $t=0$. Consider a sequence $\{x_n\}$ in X defined by $x_n=n^2, n=1,2,\cdots$. Then we have, as $n\to\infty$,

$$S(x_n)=\frac{1}{n^2}\to t=0,$$

$$T(x_n)=\frac{1}{n^4}\to t=0,$$

and

$$\lim_{n\to\infty} d(ST(x_n),TS(x_n))=\lim_{n\to\infty} d(n^4,n^4)=0$$

but

$$\lim_{n\to\infty} d(ST(x_n),TT(x_n))=\lim_{n\to\infty} d(n^8,n^4)=\lim_{n\to\infty}|n^8-n^4|=\infty$$

and

$$\lim_{n\to\infty} d(SS(x_n),TS(x_n))=\lim_{n\to\infty} d(n^2,n^4)=\lim_{n\to\infty}|n^2-n^4|=\infty.$$

Therefore, S and T are compatible but are not compatible of type (A).

EXAMPLE 2.2. Let $X=[0,1]$ with the usual metric $d(x,y)=|x-y|$.
Define $S,T:[0,1]\to[0,1]$ by

$$S(x)=\begin{cases} x & \text{if } x\in[0,\frac{1}{2}), \\ 1 & \text{if } x\in[\frac{1}{2},1] \end{cases} \quad \text{and} \quad T(x)=\begin{cases} 1-x & \text{if } x\in[0,\frac{1}{2}), \\ 1 & \text{if } x\in[\frac{1}{2},1]. \end{cases}$$

Then S and T are not continuous at $t=\frac{1}{2}$. Now, we assert that S and T are not compatible but are compatible of type (A). To see this, suppose that $\{x_n\}\subseteq[0,1]$ and that $T(x_n),S(x_n)\to t$. By definition of S and T, $t\in\{\frac{1}{2},1\}$. Since S and T agree on $[\frac{1}{2},1]$, we need only consider $t=\frac{1}{2}$. So we can suppose that $x_n\to\frac{1}{2}$ and that $x_n<\frac{1}{2}$ for all n. Then $T(x_n)=1-x_n\to\frac{1}{2}$ from the right and $S(x_n)=x_n\to\frac{1}{2}$ from the left. Thus, since $1-x_n>\frac{1}{2}$, for all n

$$ST(x_n)=S(1-x_n)=1$$

and, since $x_n<\frac{1}{2}$,

$$TS(x_n)=T(x_n)=1-x_n\to\tfrac{1}{2}.$$

Consequently,

$$d(ST(x_n),TS(x_n))\to\tfrac{1}{2}$$

but

$$d(ST(x_n),TT(x_n))=|ST(x_n)-TT(x_n)|=|1-T(1-x_n)|=|1-1|\to 0$$

and

$$d(TS(x_n),SS(x_n))=|TS(x_n)-SS(x_n)|=|(1-x_n)-x_n|=|1-2x_n|\to 0$$

as $x_n\to\frac{1}{2}$. Therefore, S and T are compatible mappings of type (A) but are not compatible.

Next, we give several properties of compatible of type (A) for our main theorems ([11]):

PROPOSITION 2.4. Let $S,T:(X,d)\to(X,d)$ be mappings. If S and T are compatible of type (A) and $S(t)=T(t)$ for some $t\in X$, then $ST(t)=TT(t)=TS(t)=SS(t)$.

PROPOSITION 2.5. Let $S,T:(X,d)\to(X,d)$ be mappings. Let S and T be compatible of type (A) and let $S(x_n)$, $T(x_n)\to t$ for some $t\in X$. Then we have the following:

(1) $\lim_{n\to\infty}TS(x_n)=S(t)$ if S is continuous at t.

(2) $ST(t)=TS(t)$ and $S(t)=T(t)$ if S and T are continuous at t.

PROOF. Immediate, from Proposition 2.2 and Proposition 2.2 (2) of [8].

3. A COMMON FIXED POINT THEOREM.

Before stating and proving our main theorem, we give some definitions and lemmas:

DEFINITION 3.1 ([8]). Let A,B,S and T be mappings of a metric space (X,d) into itself such that $A(X)\subset T(X)$ and $B(X)\subset S(X)$. For $x_o\in X$, any sequence $\{y_n\}$ defined by

$$\left.\begin{aligned} y_{2n-1}&=Tx_{2n-1}=Ax_{2n-2}, \\ y_{2n}&=Sx_{2n}=Bx_{2n-1} \end{aligned}\right\} \tag{3.1}$$

for $n=1,2,\cdots$, is called an $\{S,T\}$-iteration of x_o under A and B.
Note that Definition 3.1 assures us that $\{S,T\}$-iterations will exist since $A(X)\subset T(X)$ and $B(X)\subset S(X)$, although the sequence $\{y_n\}$ certainly need not be unique.

DEFINITION 3.2. Let A,B,S and T be mappings of a metric space (X,d) into itself. The pair $\{A,B\}$ is called a generalized $(\varepsilon,\delta)-\{S,T\}$-contraction if

$$A(X)\subset T(X) \text{ and } B(X)\subset S(X), \tag{3.2}$$

there exists a function $\delta:(0,\infty)\to(0,\infty)$ such that, for any $\varepsilon>0$ and $\delta(\varepsilon)<\varepsilon$,

$$\varepsilon\le max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{1}{2}(d(Sx,By)+d(Ty,Ax))\}<\delta(\varepsilon) \tag{3.3}$$

implies $d(Ax,By) < \varepsilon$ for all $x,y \in X$.

For our main theorem, first we give the following:

LEMMA 3.1. Let S and T be mappings of a metric space (X,d) into itself and the pair $\{A,B\}$ be a generalized (ε,δ)-$\{S,T\}$-contraction. If $x_o \in X$ and $\{y_n\}$ is an $\{S,T\}$-iteration of x_o under A and B, then we have the following:

(1) for every $\varepsilon > 0$, $\varepsilon \leq d(y_p,y_q) < \delta(\varepsilon)$ implies $d(y_{p+1},y_{q+1}) < \varepsilon$, where p and q are of opposite parity.

(2) $\lim_{n\to\infty} d(y_n,y_{n+1}) = 0$.

(3) $\{y_n\}$ is a Cauchy sequence in X.

PROOF. (1) Since the pair $\{A,B\}$ is a generalized $(\varepsilon,\delta)-\{S,T\}$-contraction, for every $\varepsilon > 0$,

$$\varepsilon \leq max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{1}{2}(d(Sx,By)+d(Ty,Ax))\} < \delta(\varepsilon)$$

implies $d(Ax,By) < \varepsilon$ for all $x,y \in X$.

Suppose that $\varepsilon \leq d(y_p,y_q) < \delta(\varepsilon)$. Putting $p = 2n$ and $q = 2m-1$ in the above inequality, we have

$$d(y_{p+1},y_{q+1}) = d(Y_{2n+1},y_{2m}) = d(Ax_{2n},Bx_{2m-1})$$

and

$$\varepsilon \leq d(y_p,y_q) = d(y_{2n},y_{2m-1}) = d(Sx_{2n},Tx_{2m-1})$$

$$\leq max\{d\{Sx_{2n},Tx_{2m-1}),d(Sx_{2n},Ax_{2n}),d(Tx_{2m-1},Bx_{2m-1}),\tfrac{1}{2}(d(Sx_{2n},Bx_{2m-1})+d(Tx_{2m-1},Ax_{2n}))\},$$

$$< \delta(\varepsilon),$$

which implies that

$$d(y_{p+1},y_{q+1}) = d(Ax_{2n},Bx_{2m-1}) < \varepsilon.$$

(2) For $x_o \in X$, by (3.3), we have

$$d(y_{2n},y_{2n+1}) = d(Ax_n,Bx_{2n-1})$$

$$< max\{d(Sx_{2n},Tx_{2n-1}),d(Sx_{2n},Ax_{2n}),d(Tx_{2n-1},Bx_{2n-1}),\tfrac{1}{2}(d(Sx_{2n},Bx_{2n-1})+d(Tx_{2n-1},Ax_{2n}))\}$$

$$= max\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),\tfrac{1}{2}(d(y_{2n},y_{2n})+d(y_{2n-1},y_{2n+1}))\}$$

$$= d(y_{2n-1},y_{2n}).$$

Similarly, we have $d(y_{2n+1},y_{2n+2}) < d(y_{2n},y_{2n+1})$.

Thus the sequence $\{d(y_n,y_{n+1})\}$ is non-increasing and converges to the greatest lower bound $t = 0$ of its range $t \geq 0$. In fact, otherwise, (1) implies that $d(y_{m+1},y_{m+2}) < t$ whenever $t \leq d(y_m,y_{m+1}) < \delta(t)$. But since $\{d(y_m,y_{m+1})\}$ converges to t, there exists a k such that $d(y_k,y_{k+1}) < \delta(t)$ and so $d(y_{k+1},y_{k+2}) < t$, which contradicts the designation of t. Therefore, we have $\lim_{n\to\infty} d(y_n,y_{n+1}) = 0$.

The proof of (3) follows from the lines of the proof of Lemma 3.1 (c) ([8]). This completes the proof.

Now we are ready to prove our main theorem:

THEOREM 3.2. Let A,B,S and T be mappings of a complete metric space (X,d) into itself satisfying the conditions (3.2),

(3.4) one of $A,B,S,$ and T is continuous,

(3.5) the pairs A,S and B,T are compatible of type (A) on X, (3.6) the pair $\{A,B\}$ is a generalized (ϵ,δ)-$\{S,T\}$-contraction such that δ is lower semi-continuous.

Then A,B,S and T have a unique common fixed point in X.

PROOF. By Lemma 3.1 (3), the $\{S,T\}$-iteration of x_o under A and B, $\{y_n\}$, is a Cauchy sequence in X. Since (X,d) is complete, $\{y_n\}$ converges to a point z in X. Since $\{Ax_{2n}\}$, $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ are subsequences of $\{y_n\}$, they also converge to z.

Suppose that S is continuous. Then we have $SSx_{2n}, SAx_{2n} \to Sz$ as $n \to \infty$. Since A and S are compatible of type (A), by Proposition 2.5 (1), $ASx_{2n} \to Sz$ as $n \to \infty$. Now, we claim that $Sz = z$. Suppose $Sz \neq z$ and let $M(x,y) = max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{1}{2}(d(Sx,By)+d(Ty,Ax))\}$. If $M_n = M(Sx_{2n}, x_{2n-1})$, then we have $M_n \to d(Sz,z) \neq 0$ as $n \to \infty$. Let $\epsilon = d(Sz,z)$ and remember that $\delta(\epsilon) > \epsilon$ by definition. Since $\delta:(0,\infty)\to(0,\infty)$ is lower semi-continuous, there exists an $\alpha \in (0,\epsilon)$ such that $\delta(t) > \epsilon$ for $t \in (\epsilon-\alpha, \epsilon+\alpha)$. Choose $t_o \in (\epsilon-\alpha,\epsilon)$. Then we have $0 < t_o < \epsilon < \delta(t_o)$. But since $M_n \to \epsilon$ as $n \to \infty$, there exists an integer n_o such that $M_n \in (t_o, \delta(t_o))$ for $n \geq n_o$. Therefore, by (3.3), we have

$$d(ASx_{2n}, Bx_{2n-1}) \leq t_o < \epsilon \text{ for } n \geq n_o.$$

But $d(ASx_{2n}, Bx_{2n-1}) \to d(Sz,z)$ as $n \to \infty$ and so we have $d(Sz,z) = \epsilon \leq t_o < \epsilon$, which is a contradiction. Thus we have $Sz = z$.

We also claim $Az = z$. Suppose not and let $d(Az,z) = \epsilon'$ and $M_n' = M(z, x_{2n-1})$. Then we have $M_n' \to \epsilon'$ as $n \to \infty$. Now duplicate the argument using the lower semi-continuity of δ to produce the contradiction $d(Az,z) < \epsilon'$. Thus we obtain $Sz = Az = z$. Since $A(X) \subset T(X)$, there exists a point $w \in X$ such that $z = Sz = Az = Tw$. Further, we claim that $Bw = z$. If $Bw \neq z$, then we have

$$\begin{aligned} d(z,Bw) &= d(Az,Bw) \\ &< max\{d(Sz,Tw), d(Sz,Az), d(Tw,Bw), \tfrac{1}{2}(d(Sz,Bw)+d(Tw,Az))\} \\ &= d(z,Bw), \end{aligned}$$

which is a contradiction. Hence $Bw = z = Tw$. Since B and T are compatible of type (A), by Proposition 2.4, $Bz = BTw = TTw = Tz$, that is, $Bz = Tz$.

Finally, we shall prove that $Bz = z$. If $Bz \neq z$, then we have

$$\begin{aligned} d(z,Bz) &= d(Az,Bz) \\ &< max\{d(Sz,Tz), d(Sz,Az), d(Tz,Bz), \tfrac{1}{2}(d(Sz,Bz)+d(Tz,Az))\} \\ &= d(z,Bz), \end{aligned}$$

which is a contradiction and so $Bz = z$. Thus, z is a common fixed point of A,B,S and T. The uniqueness of the common fixed point z follows easily from (3.6).

Similarly, we can also complete the proof when A or B or T is continuous. This completes the proof.

REMARK. Theorem 3.2 extends, generalizes and improves some results of Chung [3], Jungck [8], Maiti-Pal [13], Meir-Keeler [14], Pant [15], Park-Bae [16], Park-Rhoades [17], Rao-Rao [18], Rhoades [19], etc.

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