Internat. J. Math. & Math. Sci.
VOL. 16 NO. 4 (1993) 675-678



# QUASI-INCOMPLETE REGULAR LB-SPACE

**JAN KUCERA and KELLY McKENNON**

Department of Mathematics
Washington State University
Pullman, Washington 99164-3113

(Received October 13, 1992 and in revised form November 11, 1992)

ABSTRACT. A regular quasi-incomplete locally convex inductive limit of Banach spaces is constructed.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, quasi-completeness, LB-space.
1991 MATHEMATICS SUBJECT CLASSIFICATION CODE : Primary 46A13, Secondary 46M40.

## 1. INTRODUCTION.

Throughout the paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $E_n \to E_{n+1}, n \in N$. Their locally convex inductive limit is denoted by $indE_n$. If all spaces $E_n$ are Banach, resp. Fréchet, then we call $indE_n$ an LB-, resp. LF-space.

According to [3], [4 § 5.2], the space $indE_n$ is called: $\alpha$-regular if any set bounded in $indE_n$ is contained in some $E_n$,

$\beta$-regular if any set which is bounded in $indE_n$ and contained in some $E_m$ is then bounded in another $E_n$,

regular if its is simultaneously $\alpha$-and $\beta$-regular.

By Makarov's Theorem, [4; § 5.6], every Hausdorff quasi-complete LF-space is regular. It is natural to ask whether this theorem can be reversed for LB-spaces. By Raikov's Theorem, [4; § 4.3], every LB-space is quasi-complete iff it is complete. So in [5] Mujica asks: Is every regular LB-space complete? In [6], resp. [7], the authors constructed quasi-, resp. sequentially -, incomplete $\beta$-regular LB-spaces. They erroneously claimed that those spaces were regular. Here we partially correct that error by presenting an example of a regular quasi-incomplete LB-space. The question of existence of a sequentially-incomplete regular LB-space still remains open.

## 2. NOTATION AND AUXILIARY RESULTS.

Let $N = \{1,2,3,\cdots\}, R = (-\infty,\infty)$. Define an order on $N^N$ by $\alpha,\beta \in N^N, \alpha \le \beta \iff \alpha(n) \le \beta(n)$ for all $n \in N$. For each $\alpha \in N^N, x \in R^{N\times N}$, and $m,n \in N$, put

$$\Gamma(\alpha,x,m) = \sup\{|x_{ij}|; i,j \ge m, j > \alpha(i)\}, a(n)_{ij} = \begin{Bmatrix} j^{-1} & \text{if} & i < n \\ 1 & & i \ge n \end{Bmatrix}, (i,j) \in N \times N,$$

$$X_n = \left\{x \in R^{N\times N}; \|x\|_n = \sup\{a(n)_{ij}|x_{ij}|; i,j \in N\} < +\infty\right\},$$

$$Y_n = \left\{y \in R^{N\times N}; |||y|||_n = \Sigma\{(a(n)_{ij})^{-1}|y_{ij}|; i,j \in N\} < +\infty\right\},$$

$$E_n = \{x \in X_n; \lim_{m\to\infty} \Gamma(\alpha,x,m) = 0 \text{ for some } \alpha \in N^N\}.$$

For brevity we write $X = indX_n, Y = projY_n, E = indE_n$. Finally, we have an inner product $(x,y) \mapsto < x,y >= \Sigma\{x_{ij}y_{ij}, i,j \in N\}$ defined on $X_n \times Y_n, n \in N$, and on $X \times Y$.

LEMMA 1. For any sequence $\{\alpha_k; k \in N\} \subset N^N$ there exists $\alpha \in N^N$ such that $\liminf_{m\to\infty} \frac{\alpha(m)}{\alpha_k(m)} \geq 1$ for all $k \in N$.

PROOF. Put $\alpha(m) = \max\{\alpha_k(m); k \leq m\}, m \in N$. Then $\alpha = (\alpha(1), \alpha(2), \cdots)$ has the required property.

LEMMA 2. For each $n \in N$ :

(a) $X_n, Y_n$ are Banach spaces.

(b) $E_n$ is a closed subspace of $X_n$. Hence it is also a Banach space.

(c) $X_n \subset X_{n+1}, Y_n \supset Y_{n+1}$, and $E_n \subset E_{n+1}$, where all inclusions are continuous.

PROOF. (a) Each $X_n$, resp. $Y_n$, as a weighted $l^\infty-$, resp. $l^1-$space, is Banach.

(b) If $x_1, x_2 \in E_n$, there are $\alpha_1, \alpha_2 \in N^N$ such that $\lim_{m\to\infty} \Gamma(\alpha_i, x_i, m) = 0, i = 1,2$. Then we have $\lim_{m\to\infty} \Gamma(\alpha_1 + \alpha_2, x_1 + x_2, m) = 0$. Hence $x_1 + x_2 \in E_n$ and $E_n$ is a linear subspace of $X_n$.

Let $\{x(k); k \in N\}$ be a sequence in $E_n$ with a limit $x \in X_n$. For each $k \in N$ take $\alpha_k \in N^N$ for which $\lim_{m\to\infty} \Gamma(\alpha_k, x(k), m) = 0$. By Lemma 1, there is $\alpha \in N^N$ such that $\liminf_{m\to\infty} \frac{\alpha(m)}{\alpha_k(m)} \geq 1$ for any $k \in N$.

Given an arbitrary $\varepsilon > 0$, choose $k \in N$ so that $\|x - x(k)\|_n < \varepsilon$. For this particular $k$, take $m_1, m_2 \in N$ so that $\frac{\alpha(m)}{\alpha_k(m)} > \frac{1}{2}$ for any $m \geq m_1$, and $\Gamma(\alpha_k, x(k), m) < \varepsilon$ for any $m \geq m_2$. Finally, put $m_0 = \max\{m_1, m_2, n\}$. If $m \geq m_0$ then for $i,j \geq m, j > 2\alpha(i)$, we have $j > \alpha_k(i)$ which implies $|x(k)_{ij}| \leq \Gamma(\alpha_k, x(k), m)$. Moreover $a(n)_{ij} = 1$ since $i \geq n$. Hence $|x_{ij}| = a(n)_{ij}|x_{ij}| \leq a(n)_{ij}|x_{ij} - x(k)_{ij}| + a(n)_{ij}|x(k)_{ij}| \leq \|x - x(k)\|_n + \Gamma(\alpha_k, x(k), m) < \varepsilon + \varepsilon$. Thus $\Gamma(2\alpha, x, m) < 2\varepsilon$ and $x \in E_n$.

(c) For each $(i,j) \in N \times N$, we have $a(n+1)_{ij} \leq a(n)_{ij}$. Hence $\|x\|_{n+1} \leq \|x\|_n$ for any $x \in X_n$ and $|||y|||_n \leq |||y|||_{n+1}$ for any $y \in Y_{n+1}$.

LEMMA 3. For each $n \in N$, let $\varepsilon_n > 0, B_n = \{x \in E_n; \|x\|_n < \varepsilon_n\}$, and $V$ be the convex hull of $U\{B_n; n \in N\}$. Then the closure $\bar{V}$ of $V$ in $E$ is the same as the $\sigma(E,Y)-$closure of $V$.

PROOF. Let $E'$ be the dual space for $E$. From the duality theory we know that $\bar{V}$ is the same as the $\sigma(E,E')-$closure of $V$. Since $Y \subset E'$, we have $\sigma(E,Y) \subset \sigma(E,E')$. Thus it remains to show that if $v \in E$ is a $\sigma(E,Y)-$limit of a net $\alpha \mapsto v(\alpha) : A \to V$, then $v$ is in the $\sigma(E,E')-$closure of $V$.

For each $\alpha \in A$, there exists $m(\alpha) \in N$ such that $v(\alpha) = \Sigma\{\lambda(\alpha,p)b(\alpha,p); p = 1,2,\cdots,m(\alpha)\}$, where $\lambda(\alpha,p) > 0, \Sigma\{\lambda(\alpha,p); p = 1,2,\cdots,m(\alpha)\} = 1$, and $b(\alpha,p) \in B_{n(\alpha,p)}, 1 \leq n(\alpha,1) < n(\alpha,2) < \cdots < n(\alpha, m(\alpha))$. Take $(i,j) \in N \times N$. Let $r$ be the largest integer, less than or equal to $m(\alpha)$, for which $S_r = \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1,2,\cdots,r\} \leq |v_{ij}|$. Denote the signum function by $sgn$ and put

$$c(\alpha,p)_{ij} = \left\{ \begin{array}{lll} (sgnv_{ij})|b(\alpha,p)_{ij}|, & & p \leq r \\ {[\lambda(\alpha,r+1)]^{-1}}(sgnv_{ij})(|v_{ij}| - S_r), & \text{if} & p = r+1 \leq m(\alpha) \\ 0, & & r+1 < p \leq m(\alpha) \end{array} \right\}.$$

Then $|c(\alpha,p)_{ij}| \leq |b(\alpha,p)_{ij}|$ for each $p \leq m(\alpha)$ which implies $c(\alpha,p) \in B_{n(\alpha,p)}$ and $w(\alpha) = \Sigma\{\lambda(\alpha,p)c(\alpha,p); p = 1,2,\cdots,m(\alpha)\} \in V$. Moreover

(1) $|w(\alpha)_{ij}| \leq |v_{ij}|$,

(2) $|v_{ij} - w(\alpha)_{ij}| \leq |v_{ij} - v(\alpha)_{ij}|$.

To prove (1) and (2), we have to distinguish two cases:

(a) $r < m(\alpha)$. Then $|w(\alpha)_{ij}| \leq \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1,2,\cdots,r+1\} = |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = (sgnv_{ij})(v_{ij} - w(\alpha)_{ij}) = |v_{ij}| - \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1,2,\cdots,r+1\} = 0 \leq |v_{ij} - v(\alpha)_{ij}|$.

(b) $r = m(\alpha)$. Then $|w(\alpha)_{ij}| \leq \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1,2,\cdots,m(\alpha)\} \leq \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1,2,\cdots,r\} \leq |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = |v_{ij}| - \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1,2,\cdots,m(\alpha)\} \leq |v_{ij} - \Sigma\{\lambda(\alpha,p)b(\alpha,p)_{ij}; p = 1,2,\cdots,m(\alpha)\}| = |v_{ij} - v(\alpha)_{ij}|$.

The Banach space $c_0(N \times N)$ of double null sequences is contained in $E_1$ and the identity maps $x \mapsto x \mapsto x : c_0(N \times N) \to E_1 \to E$ are continuous. Hence the restriction of each $f \in E'$ to $c_0(N \times N)$ is continuous. It follows from the Riesz-Kakutani-Hewitt Representation Theorem that there exists a signed, regular, bounded, Borel measure $\mu$ on the discrete locally compact Hausdorff space $N \times N$ such that $f(x) = \int x d\mu, x \in c_0(N \times N)$.

Each $x \in E$ is a pointwise limit, as well as a limit in $E$, of a sequence $\{x(k) \in c_0(N \times N); k \in N\}$ satisfying $|x(k)_{ij}| \leq |x_{ij}|, i,j,k \in N$. Hence it follows from the Lebesgue Dominant Theorem that $f(x(k)) = \int x(k) d\mu \to \int x d\mu$. Since $f(x(k)) \to f(x)$, we have $f(x) = \int x d\mu, x \in E$.

The $\sigma(E,Y)-$convergence implies the pointwise convergence. Thus, according to (2), $w(\alpha) \to v$ pointwise. Then, by (1) and the Lebesgue Dominant Theorem, we have $f(w(\alpha)) = \int w(\alpha) d\mu \to \int v d\mu = f(v), f \in E'$, and $v$ is in the $\sigma(E,E')-$closure of $V$.

LEMMA 4. Let $\bar{V}$ be the same closed neighborhood of 0 in $E$ as in Lemma 3 and for each $\alpha \in N^N, (i,j) \in N \times N$,

$$x(\alpha)_{ij} = \left\{ \begin{array}{l} 1 \text{ if } j \leq \alpha(i) \text{ and } j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}. \tag{3}$$

Then $x(\alpha) \in E_1, \|x(\alpha)\|_1 = 1$, and there exists $\gamma \in N^N$ such that $x(\alpha) - x(\beta) \in \bar{V}$ for any $\alpha, \beta \geq \gamma$.

PROOF. Clearly $\|x(\alpha)\|_1 = 1$ and $\Gamma(\alpha, x(\alpha), m) = 0$ for any $\alpha \in N^N, m \in N$. Hence $\lim_{m\to\infty} \Gamma(\alpha, x(\alpha), m) = 0$ and the first statement holds.

Let $V_0 = \{y \in Y; |< y, x >| \leq 1, x \in V\}$. Then the polar $(V_0)^0$ in $E$ is the $\sigma(E,Y)-$closure of $V$ which, by the Lemma 3, equals $\bar{V}$. The polars $V^0$ and $\bar{V}^0$ in $(E', \sigma(E',E))$ are equal. Hence $V^0 = \bar{V}^0 = (V_0)^{00}$ which implies that $V_0$ is $\sigma(E',E)-$dense in $V^0$. Thus to prove that $x(\alpha) - x(\beta) \in \bar{V}$ holds, it suffices to show $|< y, x(\alpha) - x(\beta) >| \leq 1$ for all $y \in V_0$.

Choose $\gamma \in N^N$ so that $\gamma(n) > \max\{4^n, \varepsilon_n^{-2}\}, n \in N$, and an arbitrary $y \in V_0$. Denote by $|y|$ the element of $Y$ defined by $|y|_{ij} = |y_{ij}|, (i,j) \in N \times N$. Since $V$ is a balanced set, we have $|y| \in V_0$. For each $n \in N$, put

$$d(n)_{ij} = \left\{ \begin{array}{l} \sqrt{j} \text{ if } i = n, j > \gamma(n), j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}.$$

Then $\|d(n)\|_n \leq (\gamma(n))^{\frac{-1}{2}} < \varepsilon_n$. Hence $d(n) \in B_n$ and $|< |y|, d(n) >| \leq 1$. Finally, for $\alpha, \beta \geq \gamma$, we have $|< y, x(\alpha) - x(\beta) >| = |\Sigma\{y_{ij}(x(\alpha)_{ij} - x(\beta)_{ij}); (i,j) \in N \times N\} \leq \Sigma\{|y_{ij}(x(\alpha)_{ij} - x(\beta)_{ij})|; j > \gamma(i), i \in N\} \leq \Sigma\{|y_{i,2^k}|; 2^k > \gamma(i), i \in N\} = \Sigma\{(d(i)_{i,2^k})^{-1}|y_{i,2^k}|d(i)_{i,2^k}; 2^k > \gamma(i), i \in N\} \leq \Sigma\{(\gamma(i))^{\frac{-1}{2}}\Sigma\{|y_{i,2^k}|d(i)_{i,2^k}; 2^k > \gamma(i)\}; i \in N\} \leq \Sigma\{(\gamma(i))^{\frac{-1}{2}}|< |y|, d(i) >|; i \in N\} \leq \Sigma\{(\gamma(i))^{\frac{-1}{2}}; i \in N\} \leq \Sigma\{(4^i)^{\frac{-1}{2}}; i \in N\} = \Sigma\{2^{-i}; i \in N\} = 1$, Q.E.D.

## 3. MAIN RESULTS.

PROPOSITION 1. The net (3) is bounded in $E_1$ and Cauchy in $E$.

Proof follows from Lemma 4.

PROPOSITION 2. The net (3) does not converge in $E$.

PROOF. Assume $x(\alpha) \to x$ in $E$. For each $(i,j) \in N \times N$ the functional $z \mapsto z_{ij} : E \to R$ is continuous. It implies $x(\alpha)_{ij} \to x_{ij}$. Fix $(i,j) \in N \times N$ and choose $\gamma \in N^N$ so that $\gamma(i) \geq j$. Then for $\alpha \geq \gamma$, we have

$$x(\alpha)_{ij} = x(\gamma)_{ij} = \left\{ \begin{array}{l} 1 \text{ if } j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}.$$

Take $\alpha \in N^N$ and $m \in N$. Then for $i \geq m, 2^k > \alpha(i)$, we have $1 = x_{i,2^k} \leq \Gamma(\alpha, x, m)$. Hence $x \notin E_n$ for any $n \in N$.

PROPOSITION 3. The space $E$ is regular.

PROOF. Assume that $E$ is not regular. Then there exists a set $B$ bounded in $E$ such that for any $n \in N$ either $B$ is contained and not bounded in $E_n$ or $B \backslash E_n \neq \emptyset$.

Choose $x(1) \in B, x(1) \neq 0$, and $(i(1), j(1)) \in N \times N$ so that $x(1)_{i(1),j(1)} \neq 0$. Put $\varepsilon_1 = |x(1)_{i(1),j(1)}|$. Suppose that $x(k), i(k), j(k)$, and $\varepsilon_k, k = 1, 2, \cdots, n-1$, where $n > 1$, have been selected. Then there are two cases: Either $B \subset E_n$ and $B$ is not bounded in $E_n$ or there exists $x \in B \backslash E_n$. In the second case $\|x\|_n = +\infty$. Hence in either case there is $x(n) \in B$ such that $\|x(n)\|_n > n \cdot \max\{\varepsilon_k; k = 1, 2, \cdots, n-1\}$ and we can choose $(i(n), j(n)) \in N \times N$ so that

(4) $|a(n)_{i(n),j(n)} x(n)_{i(n),j(n)}| \geq n \cdot \max\{\varepsilon_k; k = 1, 2, \cdots, n-1\}$. Put

(5) $\varepsilon_n = \min\{\frac{1}{k} a(n)_{i(k),j(k)} |x(k)_{i(k),j(k)}|; k = 1, 2, \cdots, n\}$. Then

(6) $\varepsilon_p \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$ for any $p, r \in N$.

In fact, for $p \geq r$ the inequality (6) follows from (5) and for $p < r$ the inequality (4) implies $\varepsilon_p \leq \frac{1}{r} a(r)_{i(r),j(r)} |x(r)_{i(r),j(r)}| \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$.

Let a 0–neighborhood $V$ be the same as in Lemma 3. Since $B$ is bounded in $E$ there exists $r \in N$ such that $B \subset rV$. Hence also $x(r) \in rV$ and $x(r) = r\Sigma\{\lambda_p y(p); p = 1, 2, \cdots, s\}$, where $\lambda_p \geq 0, \Sigma\{\lambda_p; p = 1, 2, \cdots, s\} = 1, y(p) \in B_p$. By (6), we have $a(p)_{i(r),j(r)} |y(p)_{i(r),j(r)}| \leq \|y(p)\|_p < \varepsilon_p \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$, which implies $|y(p)_{i(r),j(r)}| < \frac{1}{r} |x(r)_{i(r),j(r)}|, p = 1, 2, \cdots, s$. Hence $|x(r)_{i(r),j(r)}| = |r\Sigma\{\lambda_p y(p)_{i(r),j(r)}; p = 1, 2, \cdots, s\}| \leq r\Sigma\{|\lambda_p y(p)_{i(r),j(r)}|; p = 1, 2, \cdots, s\} < \Sigma\{|\lambda_p x(r)_{i(r),j(r)}|; p = 1, 2, \cdots, s\} = |x(r)_{i(r),j(r)}|$, a contradiction.

By combining all three Propositions we get:

THEOREM. The space $ind E_n$ is a regular LB-space which is not quasi-complete.

## REFERENCES


1. KOTHE, G., Topological vector spaces I, Springer 1969.
2. SHAEFER, H. Topological vector spaces, Springer 1971.
3. MAKAROV, B.M. Pathological properties of inductive limits of Banach spaces, Uspekhi Mat. Nauk 18 (1963), 171-178.
4. FLORET, K. Lokalkonvexe Sequenzen mit kompakten Abbildungen, J. reine and angewandte Math., Band 247, 1971, 155-195.
5. MUJICA, J. Functional analysis, holomorphy and approximation theory II, North Holland 1984.
6. KUCERA, J., McKENNON, K. Completeness of regular inductive limits, Int. J. Math. & Math. Sci., Vol. 12, No. 3, 1989, 425-428.
7. KUCERA, J., McKENNON, K. Example of a sequentially incomplete regular inductive limit of Banach spaces, Int. J. Math. & Math. Sci., Vol. 13, No. 4, 1990, 817-820.