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COMPARISONS BETWEEN DIFFERENT SPECTRA OF AN ELEMENT IN A BANACH ALGEBRA

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ABSTRACT. In this paper we study the relationships among the spectra of the cosets of an element of a Banach algebra in some quotient algebras. We also characterize the spectrum of any $\boldsymbol{a} \in M$ (where M is an ideal of a Banach algebra with identity and moreover has an identity) in the whole algebra in terms of the spectrum of $\boldsymbol{a}$ in M.

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1. INTRODUCTION.

Let L be a Banach algebra (that is, a linear associative algebra over either the real field or complex field, endowed with a complete norm such that $\| \boldsymbol{ab} \| \leq \| \boldsymbol{a} \| \; \| \boldsymbol{b} \|$ for any $\boldsymbol{a}, \boldsymbol{b} \in L$ and $\| \boldsymbol{e} \| = 1$ if L has an identity $\boldsymbol{e}$). We recall that the vector space $\widetilde{L} = L \times K$ (where K denotes the scalar field) is a Banach algebra (with identity, whether L has an identity or not) with respect to the product defined by

$$(\boldsymbol{a}, \alpha)(\boldsymbol{b}, \beta) = (\boldsymbol{ab} + \beta \boldsymbol{a} + \alpha \boldsymbol{b}, \alpha\beta) \text{ for any } (\boldsymbol{a}, \alpha), (\boldsymbol{b}, \beta) \in \widetilde{L}$$

and the norm defined by $\| (\boldsymbol{a}, \alpha) \| = \| \boldsymbol{a} \| + |\alpha|$ for any $(\boldsymbol{a}, \alpha) \in \widetilde{L}$. The identity element of $\widetilde{L}$ is $(\mathbf{0}, 1)$ (where $\mathbf{0}$ denotes the null element of L). Henceforth we shall identify the closed two-sided ideal $\{(\boldsymbol{a}, 0) : \boldsymbol{a} \in L\}$ of $\widetilde{L}$ with L.

Now let L be a complex Banach algebra with identity $\boldsymbol{e}$. For any $\boldsymbol{a} \in L$, let $\sigma(\boldsymbol{a})$ denote the spectrum of $\boldsymbol{a}$ with respect to L (where some ambiguity may arise, we shall use the symbol $\sigma^L(\boldsymbol{a})$ instead of $\sigma(\boldsymbol{a})$). Recently Seddighin ([2]) has proved that $\sigma^{\widetilde{L}}((\boldsymbol{a}, \alpha)) \subset \sigma^L(\boldsymbol{a} + \alpha \boldsymbol{e}) \cup \{\alpha\}$ for any $\boldsymbol{a} \in L$ and for any $\alpha \in \mathbf{C}$. Actually, in this paper we show how also the opposite inclusion can be proved, so that the equality $\sigma^{\widetilde{L}}((\boldsymbol{a}, \alpha)) = \sigma^L(\boldsymbol{a} + \alpha \boldsymbol{e}) \cup \{\alpha\}$ holds (Corollary 6). We derive the equality above from the more general result (Proposition 5) mentioned in the second part of the abstract.

By an ideal of L we shall always mean a two-sided ideal. Let $\mathbb{J}_L$ denote the set of all proper closed ideals of L. For any $\boldsymbol{a} \in L$ and for any $J \in \mathbb{J}_L$, we denote the spectrum of the coset of $\boldsymbol{a}$ in the quotient algebra L/J by $\sigma_J(\boldsymbol{a})$. We remark that $J_1 \subset J_2$ implies $\sigma_{J2}(\boldsymbol{a}) \subset \sigma_{J1}(\boldsymbol{a})$. Moreover, we have $\sigma_{\{\mathbf{0}\}}(\boldsymbol{a}) = \sigma(\boldsymbol{a})$. We are also concerned here with the relationships among the spectra of

$a \in L$ in different quotient algebras. In particular, we show that

$$\sigma_{(\cap_{1 \le k \le n} J_k)}(a) = \cup_{1 \le k \le n} \sigma_{J_k}(a)$$

(Proposition 1 and following remarks) and $\sigma_{(\cup_{J \in C} J)^-}(a) = \cap_{J \in C} \sigma_J(a)$ (where the symbol "$-$" denotes closure) if C is a chain of $\mathbb{J}_L$ (Proposition 4).

2. RESULTS.

PROPOSITION 1. Let L be a complex Banach algebra with identity, let $J_1, J_2 \in \mathbb{J}_L$ and let $a \in L$. Then $\sigma_{J_1 \cap J_2}(a) = \sigma_{J_1}(a) \cup \sigma_{J_2}(a)$.

PROOF. Let e denote the identity of L. Since $J_1 \cap J_2 \subset J_k$ for any $k = 1,2$, it follows that $\sigma_{J_1}(a) \cup \sigma_{J_2}(a) \subset \sigma_{J_1 \cap J_2}(a)$.

Now we prove that $\sigma_{J_1 \cap J_2}(a) \subset \sigma_{J_1}(a) \cup \sigma_{J_2}(a)$.

Let $\lambda \in (\mathbb{C} \backslash \sigma_{J_1}(a)) \cap (\mathbb{C} \backslash \sigma_{J_2}(a))$. Then for any $k = 1,2$ there exist $b_k \in L$ and $u_k, v_k \in J_k$ such that $b_k(\lambda e - a) = e + u_k$ and $(\lambda e - a)b_k = e + v_k$. Consequently,

$$(b_1 - u_1 b_2)(\lambda e - a) = e + u_1 - u_1(e + u_2) = e - u_1 u_2$$

and

$$(\lambda e - a)(b_1 - b_2 v_1) = e + v_1 - (e + v_2)v_1 = e - v_2 v_1 \ .$$

Since $u_k, v_k \in J_k$ for any $k = 1,2$, $u_1 u_2$ and $v_2 v_1$ belong to $J_1 \cap J_2$. Hence $\lambda e - a$ is both left and right invertible modulo $J_1 \cap J_2$, which implies that $\lambda e - a$ is invertible modulo $J_1 \cap J_2$. Hence $\sigma_{J_1 \cap J_2}(a) \subset \sigma_{J_1}(a) \cup \sigma_{J_2}(a)$.

We remark that from Proposition 1 it follows that $\sigma_{(\cap_{1 \le k \le n} J_k)}(a) = \cup_{1 \le k \le n} \sigma_{J_k}(a)$ for any $a \in L$ if $J_1, \ldots, J_n \in \mathbb{J}_L$.

Now let S be an infinite subset of $\mathbb{J}_L$. We remark that the inclusion $(\cup_{J \in S} \sigma_J(a))^- \subset \sigma_{(\cap_{J \in S} J)}(a)$ holds. The following example shows how the opposite inclusion may not hold.

EXAMPLE 2. Let B denote the unit ball of the complex plane, and let L denote the Banach algebra of all complex-valued functions which are continuous on B^- and holomorphic in B. For any $n \in \mathbb{N}$, let $J_n \in \mathbb{J}_L$ be defined by $J_n = \{f \in L \colon f(q_n) = 0\}$ (where $\{q_k\}_{k \in \mathbb{N}} \subset B$ has cluster points in B and is not dense in B). We remark that $\sigma_{J_n}(f) = \{f(q_n)\}$ for any $f \in L$ and for any $n \in \mathbb{N}$. Moreover, we have $\cap_{n \in \mathbb{N}} J_n = \{f \in L \colon f(q_n) = 0 \text{ for any } n \in \mathbb{N}\} = \{0\}$, as $f^{-1}(0) \cap B$ is a discrete set for any $f \in L \backslash \{0\}$. Thus, if $a \in L$ is defined by $a(x) = x$ for any $x \in B^-$, it follows that $\sigma_{(\cap_{n \in \mathbb{N}} J_n)}(a) = \sigma(a) = a(B^-) = B^- \subsetneq (\{q_n\}_{n \in \mathbb{N}})^- = (\cup_{n \in \mathbb{N}} \sigma_{J_n}(a))^-$.

COROLLARY 3. Let L be a complex Banach algebra with identity, let $M \in \mathbb{J}_L$ and let $a \in M$. Then $\sigma_{J \cap M}(a) = \sigma_J(a) \cup \{0\}$ for any $J \in \mathbb{J}_L$.

PROOF. Since $a \in M$, we have $\sigma_M(a) = \{0\}$. Now the result follows immediately from Proposition 1.

It is not difficult to give an example of strict inclusion $\sigma_J(a) \subsetneq \sigma_{J \cap M}(a)$. Let L denote the Banach algebra $\mathbb{C}^2$ endowed with pointwise product. Then, if we set

$$M = \{(0,y) \colon y \in \mathbb{C}\}, \quad J = \{(x,0) \colon x \in \mathbb{C}\} \text{ and } a = (0,1),$$

we have that $a \in M$, $J \cap M = \{0\}$ and $\sigma_J(a) = \{1\} \subsetneq \{0,1\} = \sigma_{J \cap M}(a)$.

We remark that the maximal ideals of a Banach algebra L with identity are closed. Hence, if C is a chain of proper closed ideals of L, we have that $(\cup_{J \in C} J)^- \in \mathbb{J}_L$.

PROPOSITION 4. Let L be a complex Banach algebra with identity, and let C be a nonempty chain of proper closed ideals of L. Then

$$\sigma_{(\cup_{J \in C} J)^-}(a) = \cap_{J \in C} \sigma_J(a)$$

for any $a \in L$.

PROOF. Let e denote the identity of L, and let $a \in L$. Since $M \subset (\cup_{J \in C} J)^-$ for any $M \in C$, it follows that $\sigma_{(\cup_{J \in C} J)^-}(a) \subset \sigma_M(a)$ for any $M \in C$. Hence $\sigma_{(\cup_{J \in C} J)^-}(a) \subset \cap_{J \in C} \sigma_J(a)$.

Now we prove the opposite inclusion. We prove that

$$\mathbf{C} \backslash \sigma_{(\cup_{J \in C} J)^-}(a) \subset \mathbf{C} \backslash (\cap_{J \in C} \sigma_J(a)) .$$

Let $\lambda \in \mathbf{C} \backslash \sigma_{(\cup_{J \in C} J)^-}(a)$. Then there exist $b \in L$ and $x_1, x_2 \in (\cup_{J \in C} J)$ such that $b(\lambda e - a) = e + x_1$ and $(\lambda e - a)b = e + x_2$. Let $M \in C$ be such that there exist $y_1, y_2 \in M$ such that $\| x_j - y_j \| < 1$ for any $j = 1,2$. Then

$$\| b(\lambda e - a) - y_1 - e \| = \| x_1 - y_1 \| < 1 \text{ and } \| (\lambda e - a)b - y_2 - e \| = \| x_2 - y_2 \| < 1 .$$

Since every element of L whose distance from e is less than one is invertible, it follows that $b(\lambda e - a) - y_1$ and $(\lambda e - a)b - y_2$ are invertible in L. Hence there exist $c, d \in L$ such that

$$cb(\lambda e - a) - cy_1 = (\lambda e - a)bd - y_2 d = e .$$

Since $y_j \in M$ for any $j = 1,2$, it follows that $\lambda e - a$ is both left invertible and right invertible modulo M. Hence $\lambda e - a$ is invertible modulo M. Therefore,

$$\lambda \in \mathbf{C} \backslash \sigma_M(a) \subset \mathbf{C} \backslash (\cap_{J \in C} \sigma_J(a)) .$$

We have thus proved that $\mathbf{C} \backslash \sigma_{(\cup_{J \in C} J)^-}(a) \subset \mathbf{C} \backslash (\cap_{J \in C} \sigma_J(a))$. Hence

$$\sigma_{(\cup_{J \in C} J)^-}(a) = \cap_{J \in C} \sigma_J(a) .$$

We remark that, if L is a complex Banach algebra with identity e and M is a closed subalgebra of L, also endowed with an identity f, the two identities may not coincide. Moreover, the two identities are necessarily different if $M \in \mathfrak{J}_L$. Nevertheless, the inclusion $\sigma^L(a) \subset \sigma^M(a) \cup \{0\}$ holds for any $a \in M$ in view of [1], (1.6.12). Since $\sigma^L(a + \alpha e) = \sigma^L(a) + \alpha$ and $\sigma^M(a + \alpha f) = \sigma^M(a) + \alpha$ for any $\alpha \in C$, also the inclusion $\sigma^L(a + \alpha e) \subset \sigma^M(a + \alpha f) \cup \{\alpha\}$ holds for any $a \in M$ and for any $\alpha \in \mathbf{C}$. Thus, in particular, the inclusion $\sigma^{\widetilde{L}}((a, \alpha)) \subset \sigma^L(a + \alpha e) \cup \{\alpha\}$ for any $a \in L$ and for any $\alpha \in \mathbf{C}$ can be deduced.

PROPOSITION 5. Let L be a complex Banach algebra with identity e, and let M be a proper ideal of L, endowed with an identity f. Then $M \in \mathfrak{J}_L$ (which means that M is closed) and $\sigma^L(a + \alpha e) = \sigma^M(a + \alpha f) \cup \{\alpha\}$ (where we set $\sigma^M(0) = \emptyset$ if $M = \{0\}$) for any $a \in M$ and for any $\alpha \in \mathbf{C}$.

PROOF. Let $a \in M$. Since $\sigma^L(a + \alpha e) = \sigma^L(a) + \alpha$ and $\sigma^{M^-}(a + \alpha f) = \sigma^{M^-}(a) + \alpha$ for any $\alpha \in \mathbf{C}$, it is sufficient to prove that M is closed and $\sigma^L(a) = \sigma^M(a) \cup \{0\}$.

Since f is the identity of M, it follows that $f^2 = f$. Since the case $M = \{0\}$ is trivial, we can suppose $M \neq \{0\}$, which implies $f \neq 0$. Moreover, since M is a proper ideal of L, we have that

$f \neq e$. Hence f is a proper idempotent of L. Then from [1], (1.6.15) it follows that fLf is a closed subalgebra of L, with identity f, and in addition $\sigma^L(a) = \sigma^{fLf}(a) \cup \{0\}$.

Since $M \in \mathbb{J}_L$ and $f \in M$ it follows that $fLf \subset M$. Moreover, since f is the identity of M, we have that $M = fMf \subset fLf$.

We have thus proved that $M = fLf$. Consequently, $M \in \mathbb{J}_L$ and $\sigma^L(a) = \sigma^M(a) \cup \{0\}$.

The algebras L and M and the element $a \in M$ introduced in the remark after Corollary 3 provide an example of strict inclusion $\sigma^M(a) \subsetneq \sigma^L(a)$.

Now let the hypotheses of Proposition 5 hold. For any complex-valued function h, holomorphic on an open neighborhood Δ of $\sigma^L(a)$, let $h^L(a) \in L$ and $h^M(a) \in M$ be defined by

$$h^L(a) = \left(\frac{1}{2\pi i}\right)\int_{+\partial D} h(\lambda)R^L(\lambda, a)\, d\lambda \text{ and } h^M(a) = \left(\frac{1}{2\pi i}\right)\int_{+\partial D} h(\lambda)R^M(\lambda, a)\, d\lambda,$$

where $R^L(\lambda, a)$ (respectively, $R^M(\lambda, a)$) denotes the inverse of $\lambda e - a$ (respectively, $\lambda f - a$) in L (respectively, M), $+\partial D$ denotes the positively oriented boundary of D and D is an open bounded subset of $\mathbb{C}$ such that $\sigma^L(a) \subset D \subset D^- \subset \Delta$, D has a finite number of components and ∂D consists of a finite number of simple closed rectifiable curves, no two of which intersect. We recall that the two integrals above are well defined and do not depend on the choice of D. From the spectral mapping theorem (see [3], VII, 5.5) and from Proposition 5 it follows that $\sigma^L(h^L(a)) = \sigma^M(h^M(a)) \cup \{h(0)\}$.

We remark that, actually, the statement above is only seemingly more general than the one of Proposition 5. In fact, for any $\lambda \in \mathbb{C}\backslash\sigma^L(a)$, we have

$$\begin{aligned}(\lambda e - a)(R^M(\lambda, a) - f/\lambda + e/\lambda) &= \lambda R^M(\lambda, a) - f + e - aR^M(\lambda, a) + a/\lambda - a/\lambda \\ &= (\lambda f - a)R^M(\lambda, a) - f + e = e\,,\end{aligned}$$

which implies $R^L(\lambda, a) = R^M(\lambda, a) - f/\lambda + e/\lambda$. Hence $h^L(a) = h^M(a) - h(0)f + h(0)e$.

Since any Banach algebra A is a closed proper ideal of $\widetilde{A}$, the following result is a consequence of Proposition 5.

COROLLARY 6. Let L be a Banach algebra with identity e. Then

$$\sigma^{\widetilde{L}}((a, \alpha)) = \sigma^L(a + \alpha e) \cup \{\alpha\} \quad \text{for any } a \in L \quad \text{and for any } \alpha \in \mathbb{C}\,.$$

Hence the first inclusion proved in [2], Theorem 2.1 can be replaced by an equality.

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