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COMPLETENESS OF THE SET OF SUB-σ-ALGEBRAS

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ABSTRACT. A new metric is introduced on the set of all sub-σ-algebras of a complete probability space from functional analysis point of view. In this note, we will show that the resulting metric space is complete.

KEY WORDS AND PHRASES. Conditional expectation, metric, Sidǎk operator, Banach lattice.

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1. INTRODUCTION.

Let (Ω, Σ, P) be a complete probability space and $S(\Sigma)$ the set of all sub-σ-algebras of Σ. In paper Wang [1], a metric d was introduced on $S(\Sigma)$ to investigate the convergence rate of conditional expectations. Some topological properties of $(S(\Sigma), d)$ were discussed in the paper Taylor and Wang [2], where it have been shown that $(S(\Sigma), d)$ is compact if and only if Σ is purely atomic, $(S(\Sigma), d)$ is connected if and only if Σ has at most one atom. Moreover the important subset consisting of all continuous sub-σ-algebras was shown to be closed and nowhere dense. Some of these results were also adapted to the theory of von Nuemann algebra.

In this note, we show that $(S(\Sigma), d)$ is complete.

2. NOTATIONS AND PRELIMINARIES.

For any $\Sigma_0 \in S(\Sigma)$, $L^\infty(\Sigma_0) = L^\infty(\Omega, \Sigma_0, P)$ is a closed subspace of $L^\infty(\Omega, \Sigma, P)$ in a natural way, and also a subspace of $L^2(\Omega, \Sigma, P)$ since (Ω, Σ, P) is a probability space. Define $L_1^\infty(\Sigma_0) = \{f \in L^\infty(\Sigma_0) : \|f\|_\infty \leq 1\}$. This is the unit ball in $L^\infty(\Sigma_0)$, and is closed in $L^2(\Omega, \Sigma, P)$.

For $\Sigma_1, \Sigma_2 \in S(\Sigma)$, define

$$d(\Sigma_1, \Sigma_2) = \max\{\sup_{f \in L_1^\infty(\Sigma_1)} \inf_{g \in L_1^\infty(\Sigma_2)} \|f - g\|_2, \sup_{g \in L_1^\infty(\Sigma_2)} \inf_{f \in L_1^\infty(\Sigma_1)} \|f - g\|_2\}.$$

It is easy to check that d defines a metric on $S(\Sigma)$.

For any $\Sigma_0 \in S(\Sigma)$, let e^{Σ_0} denote the orthogonal projection of $L^2(\Omega, \Sigma, P)$ onto $L^2(\Omega, \Sigma_0, P)$. Let E^{Σ_0} denote the restriction of e^{Σ_0} to $L^\infty(\Omega, \Sigma, P)$. It is well known that E^{Σ_0} and e^{Σ_0} are restrictions of the conditional expectation mapping of $L^1(\Omega, \Sigma, P)$ onto $L^1(\Omega, \Sigma_0, P)$.

The following results were obtained in Wang [1].

THEOREM 1. Let $\Sigma_1, \Sigma_2 \in S(\Sigma)$, then $\sup_{f \in L_1^\infty(\Sigma)} \|E^{\Sigma_1}(f) - E^{\Sigma_2}(f)\|_2 \leq 2\sqrt{d(\Sigma_1, \Sigma_2)}$.

THEOREM 2. Let $\Sigma_n, n = 1, 2, \ldots$ be an arbitrary sequence in $S(\Sigma)$, then $E^{\Sigma_n}(f)$ L^p-converge to $E^{\Sigma_\infty}(f)$ uniformly on $L^p(\Sigma)$ for all $1 \leq p < \infty$ if and only if $\lim_{n\to\infty} d(\Sigma_n, \Sigma_\infty) = 0$.

For any $\Sigma_1, \Sigma_2 \in S(\Sigma)$, let

$$d'(\Sigma_1, \Sigma_2) = \sup_{A \in \Sigma_1} \inf_{B \in \Sigma_2} P(A \Delta B) \bigvee \sup_{B \in \Sigma_2} \inf_{A \in \Sigma_1} P(A \Delta B)$$

This metric was used by Boylan [3] to investigate the convergence rate of conditional expectations. He also showed that $(S(\Sigma), d')$ is complete.

In Taylor and Wang [2], we showed that

$$d'(\Sigma_1, \Sigma_2) \leq d(\Sigma_1, \Sigma_2) \leq 2\sqrt{2d'(\Sigma_1, \Sigma_2)(1 - d'(\Sigma_1, \Sigma_2))}$$

for any $\Sigma_1, \Sigma_2 \in S(\Sigma)$. A counterexample was also provided to argue that the right hand inequality cannot be improved to a form $d \leq kd'$ for some positive constant k.

From this inequality and Boylan's result, $(S(\Sigma), d)$ is complete. In this note, we give a direct and clear proof of this fact.

3. COMPLETENESS OF $(S(\Sigma), d)$

To prove the main theorem, we need a lemma.

LEMMA *Let f_n, f, g_n, g be in $L^2(\Sigma)$. If $\|f_n - f\|_2 \to 0$ and $\|g_n - g\|_2 \to 0$ as $n \to \infty$, then $\|f_n \vee g_n - f \vee g\|_2 \to 0$ as $n \to \infty$.*

The proof follows from the inequalities

$$\|f_n \vee g_n - f_n \vee g\|_2 \leq \|g_n - g\|_2, \|f_n \vee g - f \vee g\|_2 \leq \|f_n - f\|_2.$$

THEOREM 3. The metric space $(S(\Sigma), d)$ is complete.

PROOF: Let $\Sigma_n, n = 1, 2, \ldots$ be a sequence in $S(\Sigma)$ such that $\lim_{n,m\to\infty} d(\Sigma_n, \Sigma_m) = 0$. By theorem 1, $\lim_{n,m\to\infty} \|E^{\Sigma_n}(f) - E^{\Sigma_m}(f)\|_2 = 0$ for any $f \in L_1^\infty(\Sigma)$. For any $f \in L^\infty(\Sigma)$, $(\frac{f}{\|f\|_\infty}) \in L_1^\infty(\Sigma)$. Since $L_1^\infty(\Sigma)$ is a closed subset of $L^2(\Sigma)$, there is a limit function $f_\infty \in L^\infty(\Sigma)$ such that $\lim_{n\to\infty} E^{\Sigma_n}(f) = f_\infty$ and f_∞ is unique up to a set of probability zero.

Define an operator T on $L^\infty(\Sigma)$ by

$$T(f) = f_\infty = \lim_{n\to\infty} E^{\Sigma_n}(f), \quad f \in L^\infty(\Sigma)$$

and define $M = \{T(f) : f \in L^\infty(\Sigma)\}$. Then

1. $T : L^\infty(\Sigma) \to L^\infty(\Sigma)$ is linear, constant-preserving and positive;

2. T is a Sidǎk operator, since

$$\begin{aligned} T(f)\bigvee T(g) &= \lim_{n\to\infty} E^{\Sigma_n}(T(f))\bigvee \lim_{n\to\infty} E^{\Sigma_n}(T(g)) \quad \text{by 3} \\ &= \lim_{n\to\infty} E^{\Sigma_n}(T(f))\bigvee E^{\Sigma_n}(T(g)) \quad \text{by lemma} \\ &= \lim_{n\to\infty} E^{\Sigma_n}(T(f)\bigvee T(g)) \\ &= T(T(f)\bigvee T(g)); \end{aligned}$$

3. $M = \{f : f = T(f), f \in L^\infty(\Sigma)\}$, so M is closed and contains constants;

4. M is a Banach lattice. Since $L^\infty(\Sigma)$ is a Banach lattice, for any f and g in M, $f \vee g = T(f) \vee T(g) = T(T(f) \vee T(g)) \in M$;

5. let $f_n, n = 1, 2, \ldots$ be an increasing sequence of positive functions in M converging to $f \in L^\infty(\Sigma)$, then since $|f - f_n| \le |f - f_1|$ for any n and $|f - f_1|^2$ is integrable, by Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty} \int_\Omega |f - f_n|^2 dP = 0$, so $\lim_{n\to\infty} \|f - f_n\|_2 = 0$. But $\|f - T(f)\|_2 \le \|f - f_n\|_2 + \|T(f - f_n)\|_2 \le 2\|f - f_n\|_2$, hence $f = T(f) \in M$.

By theorem 2.2.5 of Rao [4], $M = L^\infty(\Sigma_0)$ for some sub-σ-algebra $\Sigma_0 \in S(\Sigma)$, so $T(L^\infty(\Sigma)) = L^\infty(\Sigma_0)$.

For any n, m and $f \in L_1^\infty(\Sigma_n)$, by theorem 1,

$$\|f - E^{\Sigma_m}(f)\|_2 = \|E^{\Sigma_n}(f) - E^{\Sigma_m}(f)\|_2 \le 2\sqrt{d(\Sigma_n, \Sigma_m)}$$

For any $\epsilon > 0$, choose N be such that $n, m \ge N$ implies that $d(\Sigma_n, \Sigma_m) < \epsilon^2/6$. For $n \ge N$ and $f \in L_1^\infty(\Sigma_n)$,

$$\begin{aligned} \|f - T(f)\|_2 &= \lim_{m\to\infty} \|E^{\Sigma_n}(f) - E^{\Sigma_m}(f)\|_2 \\ &\le sup \lim_{m\to\infty} 2\sqrt{d(\Sigma_n, \Sigma_m)} \\ &\le 2\sqrt{\epsilon^2/6} < \epsilon \end{aligned}$$

For these ϵ and N and $g \in L_1^\infty(\Sigma_0)$, for $n \ge N$,

$$\|g - E^{\Sigma_n}(g)\|_2 = \|T(g) - E^{\Sigma_n}(g)\|_2 = \lim_{m\to\infty} \|E^{\Sigma_m}(g) - E^{\Sigma_n}(g)\|_2 < \epsilon$$

Thus, $\lim_{n\to\infty} d(\Sigma_n, \Sigma_0) = 0$.

Now, this implies by theorem 2 that $E^{\Sigma_0}(f) = \lim_{n\to\infty} E^{\Sigma_n}(f) = T(f)$ in L^2-norm for all $f \in L_1^\infty(\Sigma)$, thus $E^{\Sigma_0}(f) = T(f)$ for all $f \in L_1^\infty(\Sigma)$, that is $T = E^{\Sigma_0}$.

Therefore, any Cauchy sequence $\{\Sigma_n, n = 1, 2, \ldots\}$ in $S(\Sigma)$ has a limit $\Sigma_0 \in S(\Sigma)$, and $(S(\Sigma), d)$ is complete.

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