Internat. J. Math. & Math. Sci.
VOL. 16 NO. 3 (1993) 493-502

THICKNESS IN TOPOLOGICAL TRANSFORMATION SEMIGROUPS

TYLER HAYNES

Mathematics Department
Saginaw Valley State University
University Center, Michigan 48710-0001

(Received June 27, 1989 and in revised form February 5, 1993)

ABSTRACT. This article deals with thickness in topological transformation semigroups (τ-semigroups). Thickness is used to establish conditions guaranteeing an invariant mean on a function space defined on a τ-semigroup if there exists an invariant mean on its functions restricted to a sub-τ-semigroup of the original τ-semigroup. We sketch earlier results, then give many equivalent conditions for thickness on τ-semigroups, and finally present theorems giving conditions for an invariant mean to exist on a function space.

KEY WORDS AND PHRASES. Thickness, topological transformation semigroup, transformation semigroup, invariant mean
1980 AMS SUBJECT CLASSIFICATION CODES. 22A20, 43A07

1. Left-Thickness in Semigroups

Mitchell introduced the concept of left-thickness in a semigroup [Mitchell, 1965]: a subset T of semigroup S is *left-thick* in S $\leftrightarrow \forall$ finite $U \subseteq S$, $\exists t \in S$: $Ut \subseteq T$.

Any left ideal of a semigroup is left-thick, but not conversely. The complete relationship between left ideals and left-thick subsets is this: Let $\beta(S)$ be the Stone-Čech compactification of semigroup S endowed with the discrete topology, and let $T \subseteq S$. Then T is left-thick in S $\leftrightarrow$ the closure of T in $\beta(S)$ contains a left ideal of $\beta(S)$ [Wilde & Witz, 1967, lemma 5.1]. (See Theorem 4.3.g *infra* for a more general formulation of this result.)

It can be shown that in the definition t can be taken in T or U can be a singleton.

Let B(S) = the set of all bounded complex- or real-valued functions on semigroup S. For any $s \in S$ and $f \in B(S)$, $T_s f$ denotes the function in B(S) defined by $T_s f(t) = f(st)$ ($\forall t \in S$).

A *mean* on B(S) is a member of the dual space $B(S)^*$ of B(S) which satisfies $\mu(1) = 1 = \|\mu\|$. Mean μ is *invariant* $\leftrightarrow \mu(T_s f) = \mu f$ ($\forall s \in S, f \in B(S)$).

The importance of left-thickness for our subject is because of this theorem [Mitchell, 1965, theorem 9].

Theorem. *Let T be a left-thick subsemigroup of semigroup S. Then B(S) has a left-invariant mean $\leftrightarrow$ B(T) has a left-invariant mean.*

H. D. Junghenn generalized Mitchell's concept of left-thickness [Junghenn, 1979, p. 38]. First it is necessary to define more terms.

Subspace F of B(S) is *left-translation invariant* $\leftrightarrow T_s f \in F$ ($\forall s \in S, f \in F$). Let $\mu \in F^*$, the dual space of F; define $T_\mu f$ ($\forall f \in F$) by $T_\mu f(s) = \mu(T_s f)$ ($\forall s \in S$). Then T_μ: $F \to B(S)$. F is *left-introverted* $\leftrightarrow$ $T_\mu(F) \subseteq F$ ($\forall \mu \in F^*$).

Definition. Let S be a semigroup; $F \subseteq B(S)$ be a left-translation invariant, left-introverted, norm-closed subalgebra containing the constant functions; $T \subseteq S$ be non-empty;
$F(T) = \{g \in F \mid \chi_T \le g \le 1\}$. Then
T is *F-left thick* in S $\leftrightarrow \forall \epsilon > 0, g \in F(T)$, and finite $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ $\exists s \in S$: $g(s_i s) > 1 - \epsilon$ ($i = 1, \dots, n$)

If $\chi_T \in F$, then Junghenn's definition of F-left thickness reduces to Mitchell's definition of left-thickness: let $g = \chi_T$, then for $0 < \epsilon < 1$, $1 - \epsilon < g(s_i s) = \chi_T(s_i s) \rightarrow s_i s \in T$ ($i = 1, \dots, n$).

Junghenn generalizes Mitchell's theorem thus:

Theorem. *If T is a left-thick subsemigroup of S, then F has a left-invariant mean* ⇔ $F|_T$ *has a left-invariant mean.*

2. *Transformation Semigroups*

Thickness can be defined in the more general setting of a transformation semigroup. This section defines such semigroups and other necessary terms.

Definition 2.1. A *transformation semigroup* is a system $\langle S,X,\pi\rangle$ consisting of a semigroup S, a set X, and a mapping π: $S\times X \to X$ which satisfies

1. $\pi(s,\pi(t,x)) = \pi(st,x)$ $(\forall s,t\in S, x\in X)$;
2. $\pi(e,x) = x$ $(\forall x\in X)$ whenever S has two-sided identity e.

If $\pi(s,x)=sx$ expresses the image of (s,x) under π, then condition (1) becomes $s(tx)=(st)x$ and condition (2) becomes $ex=x$.

The abbreviated notion $\langle S,X\rangle$ will denote a transformation semigroup whenever the meaning of π is clear or whenever π is generic.

$\langle T,Y\rangle$ is a *subtransformation semigroup* of $\langle S,X\rangle$ ⇔ T is a subsemigroup of S, $Y\subseteq X$, and $TY\subseteq Y$.

Definition 2.2. Let semigroup S and set X both be endowed with Hausdorff topologies. Transformation semigroup $\langle S,X,\pi\rangle$ is a *topological transformation semigroup*, or *τ-semigroup* ⇔ π is separately continuous in the variables s and x.

Again, a τ-semigroup will be denoted briefly by $\langle S,X\rangle$.

Let C(X) denote the set of continuous and bounded complex- or real-valued functions on X.

Definition 2.3. Let $\langle S,X\rangle$ be a τ-semigroup. T_sf denotes, for any $s\in S$ and $f\in C(X)$, the function in C(X) defined by $T_sf(x)=f(sx)$ $(\forall x\in X)$. If F is a linear subspace of C(X), then F is *S-invariant* ⇔ $T_sf\in F$ $(\forall s\in S, f\in F)$. Notation: $T_S = \{T_s|s\in S\}$ and $T_sF = \{T_sf|f\in F\}$.

Observe that $T_tT_s = T_{st}$ $(\forall s,t\in S)$.

Definition 2.4. Let $\langle S,X\rangle$ be a τ-semigroup; F be a linear space $\subseteq C(X)$ which is norm-closed, conjugate-closed, S-invariant, and contains the constant functions; $G\subseteq C(S)$ a linear space, and let $\mu\in F^*$. Define $T_\mu f$ $(\forall f\in F)$ by $T_\mu f(s) = \mu(T_sf)$ $(\forall s\in S)$. Then T_μ: $F\to B(S)$. F is *G-introverted* ⇔ $T_\mu(F)\subseteq G$ $(\forall \mu\in F^*)$.

In the preceding definition F^* may be replaced by $C(X)^*$ since every functional in F^* can be extended to a functional in $C(X)^*$. Also it can be shown that F^* can be replaced by M(F), the set of all means on F.

Definition 2.5. Let F be G-introverted, $\mu\in F^*$, and $\lambda\in G^*$. The *evolution product* of λ and μ, denoted $\lambda\mu$, is defined by $\lambda\mu f = \lambda(T_\mu f)$ $(\forall f\in F)$.

Note that $\lambda\mu\in F^*$ and that if G is norm-closed, conjugate-closed, and contains the constant functions, then $\lambda\in M(G)$ and $\mu\in M(F)$ imply $\lambda\mu\in M(F)$.

A *mean* on $F\subseteq C(X)$ is defined in the same way as a mean on B(S) was defined in section 1. If F is an algebra under pointwise multiplication, then mean μ is *multiplicative* ⇔ $\mu(fg) = \mu(f)\mu(g)$ $(\forall f,g\in F)$.

Let M(F) = set of all means on F, and MM(F) = set of all multiplicative means on F. M(F) and MM(F) are both w*-compact, being closed subsets of the unit ball in F^*.

Mean $\mu\in M(F)$ is *invariant* ⇔ $\mu(T_sf) = \mu(f)$ $(\forall f\in F, s\in S)$. Note that μ is invariant ⇔ $e(s)T_\mu = T_\mu$ $(\forall s\in S)$.

An *evaluation* at $x\in X$ is defined by $e(x)f = f(x)$ $(\forall f\in F)$; clearly an evaluation is a mean. A *finite mean* on F is a convex combination of evaluations.

A mean is multiplicative if and only if it is the w*-limit of evaluations.

A special case of transformation semigroup is furnished by letting X = S and $\pi = \lambda(\cdot)$ where λ_s: S→S is defined for any fixed s∈S by $\lambda_s(t) = st$ (∀t∈S). If G⊆C(S) is a linear space, then $L_s g(t) = g(st)$ (∀s,t∈S,g∈G); also, λ,μ∈M(G) → λμ∈M(G). If F⊆C(X) is a linear space then $L_s T_\mu = T_\mu T_s$ (∀s∈S,μ∈M(F)). Mean μ∈M(G) is *left-invariant* ↔ $\mu(L_s g) = \mu(g)$ (∀g∈G).

3. *Thickness in Transformation Semigroups*

Junghenn's generalization of F-left thickness carries over in a straightforward way to transformation semigroups. The corresponding concept is defined in Definition 3.1, and a plethora of alternative characterizations is given by Theorem 3.3.

Assumptions:

⟨S,X⟩ is a transformation semigroup;

G⊆C(S) is a subalgebra;

F⊆C(X) is an algebra which is norm-closed, S-invariant, G-introverted, and contains the constant functions;

Y⊆X.

Notation:

$F(Y) = \{g\in F \mid \chi_Y \le g \le 1\} = \{g\in F \mid 0\le g\le 1,\ g\equiv 1 \text{ on } Y\}$

$Z(Y) = \{g\in F \mid g\equiv 0 \text{ on } Y\}$.

Definition 3.1. Y is *F,S-thick* in X ↔ ∀ε>0,g∈F(Y), and finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$, ∃x∈X: $g(s_k x) > 1-\epsilon$ (k=1,...,n).

Remark 3.2. If X = S and the action is left multiplication, then the definition is identical to Junghenn's.

Relative to Theorem 3.3 b,h,i,j *infra* it is necessary to recall that a norm-closed subalgebra F of C(X) is also a closed lattice, so that, in particular, f∈F → |f|∈F [Simmons, p. 159, lemma].

Theorem 3.3. The following statements are equivalent:

a. Y is F,S-thick in X;

b. ∀ε>0, finite $D = \{g_1,g_2,\ldots,g_m\}\subseteq F(Y)$, finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$ ∃x∈X: $\inf\{g_i(s_k x) \mid g_i\in D, s_k\in U\} > 1-\epsilon$;

c. ∀ε>0, finite $D = \{g_1,g_2,\ldots,g_m\}\subseteq F(Y)$, finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$

$$\exists x\in X:\ \frac{1}{n}\sum_{k=1}^{n} g_i(s_k x) > 1-\epsilon\ (i=1,\ldots,m) \text{ and } \frac{1}{m}\sum_{i=1}^{m} g_i(s_k x) > 1-\epsilon\ (k=1,\ldots,n);$$

d. ∃λ∈MM(F), ∀s∈S,g∈F(y): $\lambda(T_s g) = 1$ and $\lambda(g) = 1$;

e. ∃μ∈M(F), ∀s∈S,g∈F(Y): $\mu(T_s g) = 1$ and $\mu(g) = 1$;

f. ∃μ∈M(F), ∀ν∈M(G),g∈F(Y): $\nu\mu(g) = 1$;

g. Cle(Y) contains a compact MM(G)-invariant set;

h. ∀ε>0,g∈Z(Y), finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$ ∃x∈X: $|g(s_k x)| < \epsilon$ (k=1,...,n);

i. ∀ε>0, finite $D = \{g_1,g_2,\ldots,g_m\}\subseteq Z(Y)$, finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$; ∃x∈X: $\sup\{|g_j(s_k x)| \mid g_j\in D, s_k\in U\} < \epsilon$;

j. ∀ε>0, finite $D = \{g_1,g_2,\ldots,g_m\}\subseteq Z(Y)$, finite $U = \{s_1,s_2,\ldots,s_n\}\subseteq S$;

$$\exists x\in X:\ \frac{1}{n}\sum_{k=1}^{n} |g_i(s_k x)| < \epsilon\ (i=1,\ldots,m) \text{ and } \frac{1}{m}\sum_{i=1}^{m} |g_i(s_k x)| < \epsilon\ (k=1,\ldots,n);$$

k. ∃λ∈MM(F), ∀s∈S,g∈Z(Y): $\lambda(T_s g) = 0$ and $\lambda(g) = 0$;

l. ∃μ∈M(F), ∀s∈S,g∈Z(Y): $\mu(T_s g) = 0$ and $\mu(g) = 0$;

m. ∃μ∈M(F), ∀ν∈M(G),g∈Z(Y): $\nu\mu(g) = 0$.

PROOF: $a \rightarrow b$: $f(x) = \inf\{g_i(x)|g_i\epsilon D\}$ is in $F(Y)$ because $0\le g_i\le 1$, $g_i\equiv 1$ on Y $(i=1,\dots,m)$. By (a) $\exists x\epsilon X$: $f(s_kx)>1-\epsilon$ $(k=1,\dots,n)$. Because U is finite, $\inf\{f(s_kx)|s_k\epsilon U\}>1-\epsilon$.

$$b \rightarrow c:\ \inf\{g_i(s_kx)|g_i\epsilon D, s_k\epsilon U\}>1-\epsilon \rightarrow \sum_{k=1}^{n} g_i(s_kx)\ge n\,[\inf\{g_i(s_kx)\}]>n(1-\epsilon)$$

and $\sum_{i=1}^{m} g_i(s_kx)\ge m\,[\inf\{g_i(s_kx)\}]>m(1-\epsilon)$.

$c \rightarrow d$: For each (ϵ,U,D) in (c) choose $x = x(\epsilon,U,D)$ so that $\frac{1}{n}\sum_{k=1}^{n} g(s_kx)$ $>1-\frac{1}{n}\epsilon$ $(\forall g\epsilon D)$. Let $r\epsilon U, g\epsilon D$. Then $g(s_kx)\le 1$ $(k=1,\dots,n) \rightarrow \sum_{s_k\neq r} g(s_kx)\le n-1 = -\sum_{s_k\neq r} g(s_kx)$ $\ge -n+1 \rightarrow g(rx) = \sum_{k=1}^{n} g(s_kx) - \sum_{s_k\neq r} g(s_kx)>1-\epsilon$. Define $(\epsilon,U,D) \le (\epsilon',U',D') \rightarrow$ $\epsilon\ge\epsilon', U\subseteq U', D\subseteq D'$. The net $\langle e(x(\epsilon,U,D))\rangle \subset MM(F)$ has a subnet $\langle e(x_m)\rangle$ which w*-converges to some $\lambda'\epsilon MM(F)$, since $MM(F)$ is compact. For $\delta>0$ and $(\epsilon,U,D) \ge (\delta,\{s\},\{g\})$ it follows that $1-\delta\le 1-\epsilon<g(sx(\epsilon,U,D)) = e(x(\epsilon,U,D))\,T_sg$ by the earlier inequality. Therefore, $1-\delta\le \lim_m [e(x_m)(T_sg)] = [\lim_m e(x_m)]\,(T_sg) = \lambda'(T_sg)$. Since δ was arbitrary, $1\le\lambda' T_sg$. Because $0\le g\le 1$, $T_sg\le 1$, and so $\lambda'(T_sg)\le 1$. Thus, the first part of (d) is proven. Let $v\epsilon MM(G)$; then $\lambda=v\lambda'\epsilon MM(F)$ and $(T_{\lambda'}T_sg)(t) = \lambda'\,[T_tT_sg] = \lambda'(T_{st}g) = 1 \rightarrow \lambda(T_sg) = v\lambda'(T_sg) =$ $v[T_{\lambda'}T_sg] = v1 = 1$; also $v\lambda'(g) = v[T_{\lambda'}g] = v1 = 1$.

$d \rightarrow e$: $MM(F)\subseteq M(F)$.

$e \rightarrow f$: Let $v\epsilon M(G)$ and μ be as in (e), so that $(T_\mu g)(s) = (\mu T_sg) = 1$; then $v\mu(g) = v(T_\mu g) = v(1) = 1$.

$f \rightarrow a$: We prove (not (a)) $\rightarrow$ (not (f)). Suppose $\exists\epsilon>0$, $h\epsilon F(Y), U =$ $\{s_1,s_2,\dots,s_n\}\subseteq S$ such that $\forall x\epsilon X$, $\exists s_x\epsilon U$: $h(s_xx)\le 1-\epsilon$. Define $v = \frac{1}{n}\sum_{k=1}^{n} e(s_k)$. Then $(\forall x\epsilon X)$

$[ve(x)]h = \frac{1}{n}\sum_{k=1}^{n} h(s_kx)\le 1-\epsilon/n$ because $0\le h\le 1$ and, for some $s_k = s_x$, $h(s_kx)\le 1-\epsilon$. This inequality, valid for all evaluations $e(x)$, also holds for all finite means, and so for all limits $\mu\epsilon M(F)$ of finite means: $v\mu(h)\le 1-\frac{\epsilon}{n}$. Therefore (f) is impossible.

$d \rightarrow g$: Choose $\lambda\epsilon MM(F)$ as in (d). $MM(G)\lambda$ is then an $MM(G)$-invariant set. Since $Cl[e(Y)]$ is closed, it suffices to show that $e(s)\lambda\epsilon Cl[e(Y)]$ for $\forall s\epsilon S$. Suppose that $\exists s_0$: $e(s_0)\lambda\notin Cl[e(Y)]$. Then, since $MM(F)$ is compact Hausdorff and so completely regular, $\exists h\epsilon C(MM(F))$: $0\le h\le 1$, $h(e(s_0)\lambda) = 0$, and $h(Cl[e(Y)]) = 1$. $g = h\circ e\epsilon F(Y)$ because for $y\epsilon Y$ $g(y) = h(e(y)) = 1$. Then $\lambda(T_{s_0}g) = [e(s_0)\lambda]g = h(e(s_0)\lambda) = 0$, contradicting (d).

$g \rightarrow d$: Let I be an $MM(G)$-invariant set $\subseteq Cl(e(Y))$. If $\lambda\epsilon I$, then $e(s)\lambda\epsilon I\subseteq Cl(e(Y))$ $(\forall s\epsilon S)$. Therefore, $\lambda(T_sg) = [e(s)\lambda]\,g = 1$ $(\forall g\epsilon F(Y))$. Clearly $\lambda(g) = 1$ $(\forall g\epsilon F(Y))$.

a $\to$ h: Assume Y is F,S-thick in X. Let $\epsilon>0$, $g\in Z(Y)$, finite $U\subseteq S$. If $g\equiv 0$, result is trivial; hence, assume that $g\neq 0$. Then $1-\frac{1}{||g||}|g|\in F(Y)$. Consequently, $\exists x\in X$:

$$1-\frac{1}{||g||}|g(s_kx)| \geq 1-\frac{\epsilon}{||g||}, \text{ whence } |g(s_kx)|<\epsilon \ (k=1,\dots,n).$$

h $\to$ a: Assume (h). Let $\epsilon>0$, $g\in F(Y)$, finite $U\subseteq S$. Then $1-g\in Z(Y)$. Therefore, $\exists x\in X: |1-g(s_kx)|<\epsilon \to -\epsilon<1-g(s_kx)<\epsilon \to -g(s_kx)< -1+\epsilon \to g(s_kx)> 1-\epsilon$ $(k=1,\dots,n)$.

h $\to$ i: $\sup\{|g_j|\,|g_j\in D\}\in Z(Y)$, because $g_j\equiv 0$ on Y $(j=1,\dots,m)$.

i $\to$ k: For each (ϵ,U,D) in (i) choose $x = x(\epsilon,U,D)$. Define $(\epsilon,U,D)\leq(\epsilon',U',D') \to \epsilon\geq\epsilon', U\subseteq U', D\subseteq D'$. The net $\langle e(x(\epsilon,U,D))\rangle\subseteq MM(F)$ has a subnet $\langle e(x_m)\rangle$ which converges to some $\lambda\in MM(F)$ since MM(F) is compact. Let $\delta>0$. If $(\epsilon,U,D) \geq (\delta,\{s\},\{g\})$, then $\delta\geq\epsilon>\sup\{|g_j(s_kx(\epsilon,U,D))|\,|g_j\in D, s_k\in U\}\geq|g(sx(\epsilon,U,D))|$. Ergo $\delta\geq \lim_m[e(x_m)|T_sg|] = [\lim_m e(x_m)]|T_sg| = \lambda|T_sg|$. Since δ was arbitrary, the first part of (k) is proven. The second part is shown in the same manner as the second part of (c) $\to$ (d).

i $\to$ j: Trivial.

j $\to$ i: In the first part of (j), replace ϵ by $\frac{\epsilon}{n}$: $\frac{\epsilon}{n} > \frac{1}{n}\sum_{k=1}^{n}|g_j(s_kx)|$ $(j=1,\dots,n) \to$

$$\epsilon > \sum_{k=1}^{n}|g_j(s_kx)| > \sup\{|g_j(s_kx)|\,|g_j\in D, s_k\in U\}.$$

k $\to$ l, l $\to$ m: Trivial.

m $\to$ h: We show (not (h)) $\Rightarrow$ (not (m)). Suppose $\exists\epsilon>0$, $h\in Z(Y)$, finite $U\subseteq S$ such that $\forall x\in X, \exists s_x\in U$: $|h(s_xx)|\geq\epsilon$. Define $v=\frac{1}{n}\sum_{k=1}^{n}e(s_k)$. Then $\forall x\in X$: $[v(e(x))]|h| =$

$$\frac{1}{n}\sum_{k=1}^{n}|h(s_kx)|\geq\epsilon/n,$$ because $|h| \geq 0$ and for some $s_k = s_x$, $|h(s_kx)|\geq\epsilon$. Hence, replacing e(x) by any finite mean, then for any $\mu\in M(F)$, $v\mu|h|\geq\epsilon/n$. Therefore (m) is impossible. QED

Remark 3.4. Parts d., e., k., and l., of Theorem 3.3 suggest that S behaves with regard to thickness as though it contained an identity. In fact, if S^1 denotes the semigroup S with a discrete identity 1 adjoined, then Y is F,S-thick in X $\to$ Y is F,S^1-thick in X where S^1 acts on X in the natural way.

Corollary 3.5. If the characteristic function $\chi_Y\in F$, then the following statements are equivalent:

a. Y is F,S-thick in X;
b. $\forall$ finite $U = \{s_1,s_2,\dots,s_n\}\subseteq S, \exists x\in X$: $s_kx\in Y$ $(k=1,\dots,n)$;
c. $\forall$ finite $U = \{s_1,s_2,\dots,s_n\}\subseteq S, \exists y\in Y$: $s_ky\in Y$ $(k=1,\dots,n)$;
d. The family $\{s^{-1}Y|s\in S\}$ has the finite intersection property;
e. $\bigcap_{s\in S}\mathrm{Cl}\, e(s^{-1}Y) \neq\emptyset$ where $e(s^{-1}Y) = \{e(x)|sx\in Y\}$.

PROOF: e $\to$ a: Let $\mu\in\bigcap_{s\in S}\mathrm{Cl}\, e(s^{-1})Y$; also let $s\in S$, $g\in F(Y)$. Then $\mu\in\mathrm{Cl}\, e(s^{-1}Y)$, so $\exists$ net $\langle x_n\rangle$ such that $\mu=w^*-\lim e(x_n)$ and $sx_n\in Y$ $(\forall n)$; whence $\mu T_sg = [w^*-\lim_n e(x_n)]\,T_sg = \lim_n[g(sx_n)] = \lim 1_n = 1$. Now let $\lambda\in M(G)$. Then $\lambda\mu\in M(F)$ and $\lambda\mu T_sg = \lambda[T_\mu(T_sg)] =$

$\lambda[L_sT_\mu g] = \lambda[L_s l] = 1$; also $\lambda\mu(g) = \lambda[T_\mu g] = \lambda[\mu T_{(\bullet)}g] = \lambda[1] = 1$. Therefore by 3.3.e Y is F,S-thick. QED

Results for transformation semigroups comparable to the theorems of section 1 can be generalized in the same way as in [Junghenn 1979, p. 40, theorem 2].

Theorem 3.6. Let $\langle S,X\rangle$ be a transformation semigroup;
$\langle T,Y\rangle$ be a subtransformation semigroup of $\langle S,X\rangle$; and
$F\subseteq B(X)$ be a translation invariant, conjugate-closed, norm-closed subalgebra which contains the constant functions.

If F has invariant mean μ with respect to $\langle T,X\rangle$ such that $\inf\{\mu(g)\,|\,g\in F(Y)\}>0$, then $F|_Y$ has invariant mean with respect to $\langle T,Y\rangle$.

PROOF: X is embedded in the compact set MM(F) by $e(\bullet)$, and $F\approx C(MM(F))$ by the Gelfand representation theorem. Also $Cl\ e(Y)\subseteq MM(F)$. By the Riesz representation theorem, the invariant mean μ defines a regular Borel probability measure $\hat\mu$ on MM(F) such that $\mu(f) = \int_{MM(F)}\hat f d\hat\mu\ (\forall f\in F)$. Invariance of μ is reflected in $\hat\mu$ as follows:

$$\int_{MM(F)}T_{e(t)}\hat f d\hat\mu = \int_{MM(F)}\widehat{T_t f}\, d\hat\mu = \mu(T_t f) = \mu(f) = \int_{MM(F)}\hat f d\hat\mu\ (\forall t\in T).$$

Since μ is regular, $\hat\mu(Cl\ e(Y)) = \inf\{\hat\mu(U)\,|\,U \text{ open}, Cl\ e(Y)\subseteq U\}$. Now let U be any open set such that $Cl\ e(Y)\subseteq U$. Because MM(F) is normal, by Urysohn's lemma, $\exists\,\hat g\in C(MM(F))\approx F$ such that $\hat g(Cl\ e(Y))\equiv 1$, $\hat g(U^c)\equiv 0$, and $0\le \hat g\le 1$; thus $\hat g\le\chi_U$ and g, the correlative of $\hat g$, is in F(Y). $\mu(g) = \int_{MM(F)}\hat g d\hat\mu \le \int_{MM(F)}\chi_U d\hat\mu = \hat\mu(U)$. Therefore by hypothesis $0 < \inf\{\mu(g)\,|\,g\in F(Y)\} \le \inf\{\hat\mu(U)\,|\,U \text{ open}, Cl\ e(Y)\subseteq U\} = \hat\mu(Cl\ e(Y))$. Ergo,

$$v(f) = \frac{1}{\hat\mu(Cl\ e(Y))}\int_{Cl\ e(Y)}\hat f d\hat\mu \text{ is a mean on F.}$$

Define v_0 on $F|_Y$ by $v_0(f|_Y) = v(f)$. v_0 is well-defined because $f|_Y \equiv g|_Y \Rightarrow f-g\in Z(Y) \Rightarrow \widehat{(f-g)} \equiv 0$ on $Cl\ e(Y) \Rightarrow 0 = v(f-g) = v(f)-v(g)$. Also $v_0\in M(F|_Y)$.

To show that v_0 is invariant it suffices to prove that $\int_{Cl\ e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{Cl\ e(Y)}\hat f d\hat\mu\ (\forall t\in T)$.

Fix $t\in T$. Define $E_1 = e(t)^{-1}(Cl\ e(Y))\backslash Cl\ e(Y)$, $E_n = e(t)^{-1}(E_{n-1})$ $(n\ge 2)$.
The E_n are pairwise disjoint: $\mu\in E_2 \Rightarrow e(t)\mu\in E_1 \Rightarrow e(t)\mu\notin Cl\ e(Y) \Rightarrow \mu\notin E_1$, so $E_1\cap E_2 = \emptyset$. Assume that E_m and E_n are pairwise disjoint $(1\le m<n)$. Then $\mu\in E_{n+1} \Rightarrow e(t)\mu\in E_n \Rightarrow e(t)\mu\notin E_m$ $(1\le m<n)$ $\Rightarrow \mu\notin e(t)^{-1}E_m = E_{m+1} = E_p$ $(2\le p=m+1<n+1)$, so $E_{n+1}\cap E_p = \emptyset$. Also $\mu\in E_{n+1} \Rightarrow e(t)^n\mu\in E_1$ (by induction) $\Rightarrow e(t)^n\mu\notin Cl\ e(Y)$, but $\mu\in E_1 \Rightarrow e(t)\mu\in Cl\ e(Y) \Rightarrow e(t)^n\mu\in CL\ e(Y)$ (by invariance of Y), so $E_{n+1}\cap E_1 = \emptyset$. The E_n are Borel sets since $\mu\to e(t)\mu$ is w*=continuous for $\forall\mu\in MM(F)$.

Because $(\forall n\ge 2)$ $T_{e(t)}\chi_{E_{n-1}}(\mu) = \chi_{E_{n-1}}(e(t)\mu) = \chi_{e(t)^{-1}E_{n-1}}(\mu)$, it follows that

$$\hat\mu(E_n) = \hat\mu(e(t)^{-1}E_{n-1}) = \int_{MM(F)}\chi_{e(t)^{-1}E_{n-1}}d\hat\mu = \int_{MM(F)}T_{e(t)}\chi_{E_{n-1}}d\hat\mu = \int_{MM(F)}\chi_{E_{n-1}}d\hat\mu = \hat\mu(E_{n-1}).$$

Therefore, $1\ge\hat\mu(E_1\cup E_2\cup\ldots\cup E_n) = \sum_{j=1}^{n}\hat\mu(E_j) = n\hat\mu(E_1)$. Since this holds for arbitrary n, $\hat\mu(E_1) = 0$.

Because Y is invariant, $e(T)Cl\ e(Y)\subseteq Cl\ e(Y)$, whence $Cl\ e(Y)\backslash e(t)^{-1}Cl\ e(Y) = \emptyset$. Since $Cl\ e(Y)\Delta e(t)^{-1}Cl\ e(Y) = [Cl\ e(Y)\backslash e(t)^{-1}Cl\ e(y)]\cup E_1 = E_1$, $\hat\mu[Cl\ e(Y)\Delta e(t)^{-1}Cl\ e(Y)] = 0$, so

$$\int_{Cl\ e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{e(t)^{-1}Cl\ e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{MM(F)}T_{e(t)}[\hat f\chi_{Cl\ e(Y)}]d\hat\mu = \int_{Cl\ e(Y)}\hat f d\hat\mu. \qquad \text{QED}$$

Theorem 3.7. Let ⟨S,X⟩ be a τ-semigroup;
⟨T,Y⟩ be a sub τ-semigroup of ⟨S,X⟩;
$F \subseteq B(X)$ be a translation invariant, norm-closed, G-introverted subalgebra which contains the constant functions.

1. If $F|_Y$ has an invariant mean with respect to ⟨T,Y⟩ and T is G-thick in S, then F has an invariant mean with respect to ⟨S,X⟩.
2. If G has a left-invariant mean and Y is F,S-thick in X, then $F|_Y$ has an invariant mean with respect to ⟨T,Y⟩.

PROOF: 1. Functional $\bar{\mu}$ in $F|_Y{}^*$ defines a functional μ in F^* by $\mu f = \bar{\mu} f|_Y$ $(\forall f \in F)$, thus $\mu T_t f = \bar{\mu} T_t f|_Y$ $(\forall f \in F, t \in T)$. Therefore, because F is G-introverted, $F|_Y$ is $G|_T$-introverted.

Relative to the algebra $F|_Y$ defined on ⟨T,Y⟩: Let $\bar{\mu}$ be an invariant mean of $F|_Y$; then $e(t)\bar{\mu} = \bar{\mu}(T_s \bullet) = \bar{\mu}$ $(\forall t \in T)$ where $e(t) \in MM(G|_T)$. Let $\bar{\lambda} \in \text{Cl } e(T) = MM(G|_T)$, and let $\langle e(t_\alpha)\rangle \subseteq e(T) \subseteq MM(G|_T)$ be a net such that $\bar{\lambda} = w^*-\lim e(t_\alpha)$. Ergo,

$\bar{\lambda}\bar{\mu} = [w^*-\lim_\alpha e(t_\alpha)]\bar{\mu} = \lim_\alpha [e(t_\alpha)\bar{\mu}] = \lim_\alpha \bar{\mu} = \bar{\mu}$. That is, $\bar{\lambda}\bar{\mu} = \bar{\mu}$ $(\forall \bar{\lambda} \in \text{Cl } e(T))$.

Relative to the algebra F defined on ⟨S,X⟩: ∃ left-ideal K of Cl e(S) in $\text{Cl } e(T) \subseteq MM(G)$ [Wilde & Witz, 1967, lemma 5.1]. Choose $\lambda_0 \in K$. Then $e(s)\lambda_0 \in K \subseteq \text{Cl } e(T) \subseteq MM(G)$ $(\forall s \in S)$.

Any $\lambda \in \text{Cl } e(T) \subseteq MM(G)$ gives rise to a $\bar{\lambda} \in \text{Cl } e(T) \subseteq MM(G|_T)$ in the following way: $\lambda = w^*-\lim_\alpha e(t_\alpha) \in MM(G)$. Now $\langle e(t_\alpha)\rangle$ is a net in $e(T) \subseteq MM(G|_T)$ so has a convergent subnet $\langle e(t_\beta)\rangle$ with $\bar{\lambda} = w^*-\lim e(t_\beta) \in MM(G|_T)$. $\bar{\lambda}$ may not be unique. For $\bar{\mu} \in F|_Y{}^*$ define $\mu \in F^*$ by $\mu f = \bar{\mu} f|_Y$ $(\forall f \in F)$ as we have done earlier. Then for all $f \in F$ $\bar{\lambda}\bar{\mu} f|_Y = \bar{\lambda}(T_{\bar{\mu}} f|_Y) =$

$\lim_\beta [e(t_\beta)T_{\bar{\mu}} f|_Y] = \lim_\beta [\bar{\mu} T_{t_\beta} f|_Y]$; also, $\lambda\mu f = \lambda(T_\mu f) = \lim_\alpha [\bar{\mu} T_{t_\alpha} f|_Y]$; ergo $\lambda\mu(f) =$

$\bar{\lambda}\bar{\mu}(f|_Y)$, regardless of the choice of $\bar{\lambda}$ which is associated with λ.

Finally, choose $\bar{\mu}$ to be an invariant mean of $F|_Y$, and define $\mu \in M(F)$ as before. Then $\lambda\mu(f) = \bar{\lambda}\bar{\mu}(f|_Y) = \bar{\mu}(f|_Y) = \mu(f)$, that is, $\lambda\mu = \mu$ $(\forall \lambda \in \text{Cl } e(T) \subseteq MM(G))$. In particular, $e(s)\lambda_0\mu = \mu$ $(\forall s \in S)$, so that $\lambda_0\mu$ is invariant.

2. Because Y is F,S-thick in X, then by Theorem 3.3.f $\exists \mu \in M(F)$ such that $\nu\mu(f) = 1$ $(\forall \nu \in M(G), f \in F(Y))$. Let ν be an invariant mean of G. Then $\nu\mu$ is an invariant mean of F such that $\nu\mu(f) = 1$ $(\forall f \in F(Y))$. By Theorem 3.6 $F|_Y$ has an invariant mean with respect to ⟨T,Y⟩. QED

In the preceding theorem the thickness condition on T in (1) implies the thickness condition on Y in (2) according to the following lemma:

Lemma 3.8. Let ⟨S,X⟩ be a τ-semigroup;
⟨T,Y⟩ be a sub τ-semigroup of ⟨S,X⟩;
$F \subseteq B(X)$ be a translation-invariant, norm-closed, G-introverted subalgebra which contains the constant functions.

If T is G-thick in S, then Y is F,S-thick in X.

PROOF: Let $f \in F(Y)$: $0 \le f \le 1$, $f \equiv 1$ on Y. Then $T_{e(y)} f \in F(T)$ $(\forall y \in Y)$. By Theorem 3.3.e applied to L(S,G) $\exists \mu \in M(G)$ such that $1 = \mu(L_s T_{e(y)} f) = \mu(T_{e(y)} T_s f) = \mu e(y) T_s f$ and $1 = \mu T_{e(y)} f = \mu e(y) f$. Then $\mu e(y) \in M(F)$ has the properties required by Theorem 3.3.e for Y to be F,S-thick. QED

Junghenn's theorem of section 1 is obtained from Theorem 3.7 and Lemma 3.8 by letting X = S, Y = T, and the action be left multiplication.

4. *Multiplicative Means and Thickness*

Several results connect multiplicative means with thickness. F is assumed to be an S-invariant, norm-closed algebra $\subseteq C(X)$ which contains the constant functions.

Theorem 4.1 If F has an invariant multiplicative mean, then for any finite partition $\{A_i\}_1^n$ of X $\exists k$ such that A_k is F,S-thick.

PROOF: Let $v \in MM(F)$ be invariant. v induces a regular Borel probability measure $\hat{v}$ defined on MM(F), and $\sum_1^n \hat{v}(Cl\ e(A_i)) \geq 1$. Because v is multiplicative, for each i $\hat{v}(Cl\ e(A_i)) = 0$ or $\hat{v}(Cl\ e(A_i)) = 1$. Hence, $\exists k$ such that $\hat{v}(Cl\ e(A_k)) = 1$. Therefore, $v(f) = 1$ $(\forall f \in F(A_k))$ because $\chi_{A_k} \leq f \leq 1 \rightarrow \chi_{Cl\ e(A_k)} \leq \hat{f} \leq 1$ and $1 = \hat{v}(Cl\ e(A_k)) = \int \chi_{Cl\ e(A_k)} d\hat{v} \leq \int \hat{f} d\hat{v} = v(f) \leq 1$. Then, by Theorem 3.3.d A_k is F,S-thick. QED

Definition 4.2 $K(f,s) = \{\mu \in MM(F) \mid \mu(T_s f - f) = 0\}$

Theorem 4.3. The following are equivalent:

a. F has an invariant multiplicative mean;

b. It is not the case that $MM(F) \subseteq \bigcup_{\substack{f \in F \\ s \in S}} K^c(f,s)$;

c. It is not the case that $\exists f_1,\ldots,f_n \in F$; $\exists s_1,\ldots,s_n \in S$: $MM(F) \subseteq \bigcup_{i=1}^n K^c(f_i,s_i)$;

d. $\forall f_1,\ldots,f_n \in F$; $\forall s_1,\ldots,s_n \in S$; $\forall \delta > 0$; $\exists x_\delta$: $e(x_\delta) \sum_{i=1}^n | T_{s_i} f_i - f_i | < \delta$;

e. $\forall f_1,\ldots,f_n \in F$; $\forall s_1,\ldots,s_n \in S$; $\forall \delta > 0$, $\exists x_\delta$: $T_{s_i} f_i(x_\delta) - f_i(x_\delta)| < \delta$ $(i = 1,\ldots,n)$;

f. $\forall f_1,\ldots,f_n \in F$; $\forall s_1,\ldots,s_n \in S$; $\exists \lambda \in MM(F)$: $\lambda | T_{s_i} f_i - f_i | = 0$ $(i = 1,\ldots,n)$;

g. $\forall f_1,\ldots,f_n \in F$; $\forall s_1,\ldots,s_n \in S$; $\exists \lambda \in MM(F)$: $\lambda(T_{s_i} f_i - f_i) = 0$ $(i = 1,\ldots,n)$;

h. $\forall \epsilon > 0$; $\forall f_1,\ldots,f_n \in F$; $\forall s_1,\ldots,s_n \in S$: $\exists c_1,\ldots,c_n \in C$; $\exists Y \subseteq X$: $|f_k - c_k| < \epsilon$ and $|T_{s_k} f_k - c_k| < \epsilon$ on Y $(k=1,\ldots,n)$ and Y is F,S-thick in X.

PROOF: a $\rightarrow$ b: F has an invariant multiplicative mean $\rightarrow \exists \lambda \in MM(F)$: $\lambda \in K(f,s)$ $(\forall f \in F, s \in S)$ $\rightarrow$ the $K^c(f,s)$ do not cover all of MM(F).

~b $\rightarrow$ ~c: MM(F) is compact and the $K^c(f,s)$ are open.

~c $\rightarrow$ ~d: Let $f_1,\ldots,f_n \in F$ and $s_1,\ldots,s_n \in S$ be as in the negation of (c). If for any $\delta > 0$ $\exists x_\delta \in X$ such that $e(x_\delta) \sum | T_{s_k} f_k - f_k | = \sum | T_{s_k} f_k(x_\delta) - f(x_\delta) | < \delta$, then the net $\langle e(x_\delta) \rangle_{\delta > 0} \subseteq MM(F)$ contains a convergent subnet $\langle e(x_{\delta_\alpha}) \rangle_{\alpha \in A}$ of $\langle e(x_\delta) \rangle$ and $w*-\lim_\alpha e(x_{\delta_\alpha}) = \lambda \in MM(F)$; thus, for any $\epsilon > 0$ $\exists \alpha_0 \in A$: $\alpha \geq \alpha_0 \rightarrow | \lambda \sum | T_{s_k} f_k - f_k | - e(x_{\delta_\alpha}) \sum | T_{s_k} f_k - f_k | | < \frac{\epsilon}{2}$. Let $\alpha_1 \in A$ be $\geq \alpha_0$ and such that $\delta_{\alpha_1} < \frac{\epsilon}{2}$, so that $e(x_{\delta_{\alpha_1}}) \sum | T_{s_k} f_k - f_k | < \frac{\epsilon}{2}$. Then $0 \leq \lambda \sum | T_{s_k} f_k - f_k | < e(x_{\delta_{\alpha_1}}) \cdot \sum | T_{s_k} f_k - f_k | + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, $\lambda \sum | T_{s_k} f_k - f_k | = 0 \rightarrow | T_{s_k} f_k - f_k | = 0$ $(\forall k) \rightarrow \lambda(T_{s_k} f_k - f_k) = 0$ $(\forall k)$. The last equation contradicts that $\lambda \in \bigcup_{1=1}^n K^c(f_i, s_i)$.

$\sim d \to \sim c$: Suppose that $\exists\, f_1,\dots,f_n \in F$ and $s_1,\dots,s_n \in S$ and $\delta>0$ such that $(\forall x)\ e(x) \sum |T_{s_k} f_k - f_k| \geq \delta$. Let $\lambda \in MM(F)$, so that $\lambda = w^*-\lim e(x_\nu)$ with $x_\nu \in X$ $(\forall \nu)$.

Then $\lambda \sum |T_{s_k} f_k - f_k| = w^* - \lim e(x_\nu) \sum |T_{s_k} f_k - f_k| \geq \delta \to \exists k^0$ such that

$\frac{\delta}{n} \leq \lambda |T_{s_{k^0}} f_{k^0} - f_{k^0}| = |\lambda(T_{s_{k^0}} f_{k^0} - f_{k^0})|$ $(\lambda|g| = |\lambda g|$ because λ is multiplicative)

$$\Rightarrow \lambda(T_{s_{k^0}} f_{k^0} - f_{k^0}) \neq 0 \to \lambda \notin K(f_{k^0}, s_{k^0}) \to \lambda \in K^c(f_{k^0}, s_{k^0}) \to \lambda \in \bigcup_{k=1}^{n} K^c(f_k, s_k).$$

$c \to f$: $\langle e(x_\delta)\rangle_{\delta>0}$ is a net in $MM(F)$ so has a convergent subnet $\langle e(x_{\delta_\alpha})\rangle_{\alpha\in A}$. Let λ denote the w^*-limit of $\langle e(x_{\delta_\alpha})\rangle$. Then by the same reasoning as in $\sim c \to \sim d$, $\exists \alpha_1 \in A$ such that $0 \leq \lambda|T_{s_k} f_k - f_k| < e(x_{\delta_{\alpha_1}})\ |T_{s_k} f_k - f_k| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary, $\lambda|T_{s_k} f_k - f_k| = 0$.

$f \to e$: Since $\lambda \in MM(F)$, $\lambda = w^*-\lim e(x_\nu)$ for some net $\langle e(x_\nu)\rangle$ with $x_\nu \in X$ $(\forall\nu)$. By the definition of w^*-convergence, for any $\delta>0\ \exists\, e(x_\delta) \in \langle e(x_\nu)\rangle$ such that $e(x_\delta)|T_{s_i} f_i - f_i| < \delta$ $(i=1,\dots,n)$.

$a \to h$: Assume (a) and let $f_1,\dots,f_n \in F$; $s_1,\dots,s_n \in S$; and $\epsilon > 0$.

Notation: $L(r_1,\dots,r_n) = f^{-1}(S_\epsilon(r_1)) \cap f_2^{-1}(S_\epsilon(r_2)) \cap \dots \cap f_n^{-1}(S_\epsilon(r_n)) \cap (T_{s_1} f_1)^{-1}(S_\epsilon(r_1)) \cap \dots \cap (T_{s_n} f_n)^{-1}(S_\epsilon(r_n))$ for $r_1,\dots,r_n \in \mathbf{C}$, where $S_\epsilon(r_k) = \{x \in \mathbf{C} \mid |x - r_k| < \epsilon\}$ $(k=1,\dots,n)$. If some $L(r_1,\dots,r_n)$ is F,S-thick, then it suffices for the Y of (h) with $r_1 = c_1,\dots,r_n = c_n$. Assume that no $L(r_1,\dots,r_n)$ is F,S-thick. A contradiction shall be deduced. For each non-empty $L(r_1,\dots,r_n)$ and for each $\lambda \in MM(F)$, $\exists s \in S$, $\exists g \in Z(L(r_1,\dots,r_n))$ such that $\lambda(T_s(g)) \neq 0$ by (k) of Theorem 4.3. In particular, if λ is invariant, then $\lambda(g) = \lambda(T_s(g)) \neq 0$. Let $\langle e(x_\nu)\rangle$ be a net in $MM(F)$ such that $\lambda = w^* - \lim_\nu e(x_\nu)$. Then for $i=1,\dots,n$, $\exists N_i$ such that $\nu \geq N_i \to |f_i(x_\nu) - \lambda f_i| < \epsilon$ and $|T_{s_i} f_i(x_\nu) - \lambda f_i| < \epsilon$; this entails that $\nu \geq N_1, N_2, \dots, N_n \to |f_i(x_\nu) - \lambda f_i| < \epsilon$ and $|T_{s_i} f_i(x_\nu) - \lambda f_i| < \epsilon$ $(i=1,\dots,n) \to x_\nu \in L(\lambda f_1,\dots,\lambda f_n)$. For $L(\lambda f_1,\dots,\lambda f_n)$, $\exists g \in Z(L(\lambda f_1,\dots,\lambda f_n))$ with $\lambda(g) \neq 0$, as previously noted, so $g(x_\nu) = 0$ for all $\nu \geq N_1, N_2, \dots, N_n$. Therefore, $\lambda(g) = \lim_\nu e(x_\nu) g = 0$, a contradiction.

$d \to e$, $e \to d$, $f \to g$, $h \to e$: Easy QED

REFERENCES

1. Berglund, John F., Junghenn, Hugo D., and Milnes, Paul. *Compact Right Topological Semigroups and Generalizations of Almost Periodicity*. Lecture Notes in Mathematics No. 663. Berlin: Springer-Verlag, 1978.

2. Junghenn, Hugo D. "Amenability of Function Spaces on Thick Sub-semigroups." *Proceedings of the American Mathematical Society*, Vol. 75, No. 1 (1979), 37–41.

3. Lau, Anthony To-Ming. "Extremely Amenable Algebras." *Pacific Journal of Mathematics*, Vol. 33, No. 2 (1970), 329–336.

4. ———. "Action of topological semigroups, invariant means, and fixed points." *Studia Mathematica*, Vol. 43 (1972), 139–156.

5. Mitchell, Theodore. "Constant Functions and Left Invariant Means on Semigroups." *Transactions of the American Mathematical Society*, Vol. 119 (1965 August), 244–261.

6. Sakai, Kokichi. "On amenable transformation semigroups."
   I. *Journal of Mathematics, Kyoto Univ.* 16–3 (1976), 555–595.
   II. *Ibid.*, 597–626.
   III. *Scientific Reports, Kagoshima Univ.*, No. 25 (1976 September), 31–51.

7. Simmons, George F. *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill Book Co., 1963.

8. Wilde, Carroll, and Witz, Klaus. "Invariant Means and the Stone-Čech Compactification." *Pacific Journal of Mathematics*, Vol. 21 (1967), 577–586.