Internat. J. Math. & Math. Sci.
VOL. 16 NO. 3 (1993) 485-492

AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED BOUNDED REGION: AN EXTENSION TO HIGHER DIMENSIONS

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(Received August 30, 1991 and in revised form September 3, 1992)

ABSTRACT. The basic problem in this paper is that of determining the geometry of an arbitrary multiply connected bounded region in R^3 together with the mixed boundary conditions, from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the negative Laplacian, using the asymptotic expansion of the spectral function $\Theta(t)=\sum_{j=1}^{\infty} exp(-t\lambda_j)$ as $t\to 0$.

KEY WORDS AND PHRASES. Inverse problem, Laplace's operator, eigenvalue problem and spectral function.
1991 AMS SUBJECT CLASSIFICATION CODES. 35K, 35P.

1. INTRODUCTION.

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ for the negative Laplacian $-\Delta_3=-\sum_{i=1}^{3}\left(\frac{\partial}{\partial x^i}\right)^2$ in the $(x^1,x^2,x^3)-$ space.

Let $\Omega\subseteq R^3$ be a simply connected bounded domain with a smooth bounding surface S. Consider the Dirichlet/Neumann problem

$$(\Delta_3+\lambda)u=0 \text{ in } \Omega, \tag{1.1}$$

$$u=0 \text{ or } \frac{\partial u}{\partial n}=0 \text{ on } S, \tag{1.2}$$

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to S. Denote its eigenvalues, counted according to multiplicity, by

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\le\cdots\to\infty \text{ as } j\to\infty. \tag{1.3}$$

The problem of determining the geometry of Ω has been discussed by Pleijel [4], McKean and Singer [3], Waechter [5], Gottlieb [1], Hsu [2] and Zayed [6-8, 11], using the asymptotic expansion of the spectral function

$$\Theta(t)=\sum_{j=1}^{\infty} exp(-t\lambda_j) \text{ as } t\to 0. \tag{1.4}$$

It has been shown that, in the case of Dirichlet boundary conditions (D.b.c)

$$\Theta(t)=\frac{V}{(4\pi t)^{3/2}}-\frac{|S|}{16\pi t}+\frac{1}{12\pi^{3/2}t^{\frac{1}{2}}}\int_S H\, dS+a_0+0(t^{\frac{1}{2}}) \text{ as } t\to 0, \tag{1.5}$$

while, in the case of Neumann boundary conditions (N.b.c.),

$$\Theta(t)=\frac{V}{(4\pi t)^{3/2}}+\frac{|S|}{16\pi t}+\frac{1}{12\pi^{3/2}t^{\frac{1}{2}}}\int_S H\, dS+a_0+0(t^{\frac{1}{2}}) \text{ as } t\to 0, \tag{1.6}$$

In these formulae, V and $|S|$ are respectively the volume and the surface area of Ω, while

$H=\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ is the mean curvature of S, where R_1 and R_2 are the principal radii of curvature. Furthermore, the constant term a_0 in (1.5) and (1.6) has the following forms:

$$a_0=\begin{cases}\frac{1}{512\pi}\int_S\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2 dS, \text{in the case of D.b.c. (see [5])},\\ \frac{7}{512\pi}\int_S\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2 dS, \text{ in the case of N.b.c. (see [2]).}\end{cases} \tag{1.7}$$

In terms of the mean curvature H and Gaussian curvature $N=\frac{1}{R_1R_2}$, (1.7) may be rewritten in the forms:

$$a_0=\begin{cases}\frac{1}{128\pi}\int_S(H^2-N)dS, \text{in the case of D.b.c.,}\\ \frac{7}{128\pi}\int_S(H^2-N)dS, \text{ in the case of N.b.c.}\end{cases} \tag{1.8}$$

The object of this paper is to discuss the following more general inverse problem: Let Ω be an arbitrary multiply connected bounded region in R^3 which is surrounded internally by simply connected bounded domains Ω_i with smooth bounding surfaces $S_i, i=1,2,\dots,m-1$, and externally by a simply connected bounded domain Ω_m with a smooth bounding surface S_m. Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

$$(\Delta_3+\lambda)u=0 \text{ in } \Omega, \tag{1.9}$$

together with one of the following mixed boundary conditions:

$$\frac{\partial u}{\partial n_i}=0 \text{ on } S_i, \quad i=1,\dots,k, \quad u=0 \text{ on } S_i, \quad i=k+1,\dots m, \tag{1.10}$$

or

$$u=0 \text{ on } S_i, \quad i=1,\dots,k, \quad \frac{\partial u}{\partial n_i}=0 \text{ on } S_i, \quad i=k+1,\dots,m, \tag{1.11}$$

where $\frac{\partial}{\partial n_i}$ denote differentiations along the inward pointing normals to $S_i, i=1,\dots,m$. Determine the geometry of Ω from the asymptotic form of the spectral function $\Theta(t)$ for small positive t.

Note that problem (1.9)-(1.11) has been investigated recently by Zayed [11] in the special case when Ω is an arbitrary doubly connected region (i.e., $m=2$).

2. STATEMENT OF RESULTS.

Suppose that the bounding surfaces $S_i(i=1,\dots,m)$ of the region Ω are given locally by infinitely differentiable functions $x^n=y^n(\sigma_i), n=1,2,3,$ of the parameters $\sigma_i^\alpha=$ constants, are lines of curvature, the first and second fundamental forms of $S_i(i=1,\dots,m)$ can be written respectively in the following forms:

$$\prod{}_{1i}(\sigma_i,\Delta\sigma_i)=a_{1i}(\sigma_i)(\Delta\sigma_i^1)^2+a_{2i}(\sigma_i)(\Delta\sigma_i^2)^2,$$

and

$$\prod{}_{2i}(\sigma_i,\Delta\sigma_i)=b_{1i}(\sigma_i)(\Delta\sigma_i^1)^2+b_{2i}(\sigma_i)(\Delta\sigma_i^2)^2.$$

In terms of the coefficients $a_{1i},a_{2i},b_{1i},b_{2i}$ the principal radii of curvatures for $S_i(i=1,\dots,m)$ are given by:

$$R_{1i}=a_{1i}/b_{1i} \text{ and } R_{2i}=a_{2i}/b_{2i}.$$

Consequently, the mean curvatures H_i and Gaussian curvatures N_i of the bounding surfaces $S_i(i=1,\dots,m)$ are defined by:

$$H_i=\frac{1}{2}\left(\frac{1}{R_{1i}}+\frac{1}{R_{2i}}\right) \text{ and } N_i=\frac{1}{R_{1i}R_{2i}}.$$

Let $|S_i|,(i=1,\dots,m)$ be the surface areas of the bounding surfaces S_i $(i=1,\dots m)$ respectively. Then, the results of problem (1.9)-(1.11) can be summarized in the following cases:

CASE 1. (N.b.c. on $S_i, i=1,\dots,k$ and D.b.c. on S_i, $i=k+1,\dots,m$)

$$\Theta(t)=\frac{V}{(4\pi t)^{3/2}}+\frac{1}{16\pi t}\left\{\sum_{i=1}^{k}|S_i|-\sum_{i=k+1}^{m}|S_i|\right\}+\frac{1}{12\pi^{3/2}t^{\frac{1}{2}}}\sum_{i=1}^{m}\int_{S_i}H_i dS_i$$

$$+\frac{1}{128\pi}\left\{7\sum_{i=1}^{k}\int_{S_i}(H_i^2-N_i)dS_i+\sum_{i=k+1}^{m}\int_{S_i}(H_i^2-N_i)dS_i\right\}$$

$$+ (t/\pi^3)^{\frac{1}{2}} \left\{ \frac{13}{1440} \sum_{i=1}^{k} \int_{S_i} H_i^3 dS_i - \frac{1}{315} \sum_{i=k+1}^{m} \int_{S_i} H_i^3 dS_i \right\}$$

$$+ 0(t) \text{ as } t \to 0. \tag{2.1}$$

CASE 2. (D.b.c. on $S_i, i = 1, ..., k$ and N.b.c. on $S_i, i = k+1, ...m$)

In this case, the asymptotic expansion of $\Theta(t)$ as $t \to 0$ follows directly from (2.1) with the interchanges $S_i, i = 1, ..., k \leftrightarrow S_i, i = k+1, ..., m$.

With reference to formulae (1.5), (1.6) and to the articles [1], [2], [7], [11], the asymptotic expansion (2.1) may be interpreted as follows:

(i) Ω is an arbitrary multiply connected bounded region in R^3 and we have the mixed boundary conditions (1.10) or (1.11) as indicated in the specifications of the two respective cases.

(ii) For the first five terms, Ω is an arbitrary multiply connected bounded region in R^3 of volume V.

In Case 1, the bounding surfaces $S_i, i = 1, ..., k$ are of surface areas $\sum_{i=1}^{k} |S_i|$, mean curvatures H_i and Gaussian curvature N_i together with Neumann boundary conditions, while the bounding surfaces $S_i, i = k+1, ..., m$ are of surface areas $\sum_{i=k+1}^{m} |S_i|$, mean curvatures H_i and Gaussian curvature N_i together with Dirichlet boundary conditions.

We close this section with the following remarks:

REMARK 2.1. On setting $k = 0$ in (2.1) with the usual definition that $\sum_{i=1}^{0}$ is zero, we obtain the result of D.b.c. on $S_i, i = 1, ..., m$.

REMARK 2.2. On setting $k = m$ in (2.1) with the usual definition that $\sum_{i=m+1}^{m}$ is zero, we obtain the result of N.b.c. on $S_i, i = 1, ..., m$.

3. FORMULATION OF THE MATHEMATICAL PROBLEM.

In analogy with the two-dimensional problem (see [9, 10]), it is easy to show that $\Theta(t)$ associated with problem (1.9)-(1.11) is given by:

$$\Theta(t) = \iiint_{\Omega} G(\underset{\sim}{x}, \underset{\sim}{x}; t) d\underset{\sim}{x}, \tag{3.1}$$

where $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ is Green's function for the heat equation

$$\left(\Delta_3 - \frac{\partial}{\partial t} \right) u = 0, \tag{3.2}$$

subject to the mixed boundary conditions (1.10) or (1.11) and the initial condition

$$\lim_{t \to 0} G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = \delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2), \tag{3.3}$$

where $\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$ is the Dirac delta function located at the source point, $\underset{\sim}{x}_2$.

Let us write

$$G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) + \chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t), \tag{3.4}$$

where

$$G_0(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) = (4\pi t)^{-3/2} exp\left\{ - \frac{|\underset{\sim}{x}_1 - \underset{\sim}{x}_2|^2}{4t} \right\}, \tag{3.5}$$

is the "fundamental solution" of the heat equation (3.2) while $\chi(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ is the "regular solution" chosen so that $G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t)$ satisfies the mixed boundary conditions (1.10) or (1.11).

On setting $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{x}$ we find that

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + K(t), \tag{3.6}$$

where

$$K(t) = \iiint_{\Omega} \chi(\underset{\sim}{x}, \underset{\sim}{x}; t) d\underset{\sim}{x}. \tag{3.7}$$

In what follows, we shall use Laplace transforms with respect to t, and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\underset{\sim}{x}_1, \underset{\sim}{x}_2; s^2) = \int_0^{+\infty} e^{-s^2 t} G(\underset{\sim}{x}_1, \underset{\sim}{x}_2; t) dt. \tag{3.8}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ satisfies the membrane equation

$$(\Delta_3 - s^2)\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2) = -\delta(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) \text{ in } \Omega, \tag{3.9}$$

together with the mixed boundary conditions (1.10) or (1.11).

The asymptotic expansion of $K(t)$ as $t \to 0$, may then be deduced directly from the asymptotic expansion of $\bar{K}(s^2)$ as $s \to \infty$, where

$$\bar{K}(s^2) = \iiint_{\Omega} \bar{\chi}(\underset{\sim}{x},\underset{\sim}{x};s^2)d\underset{\sim}{x}. \tag{3.10}$$

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [7] that the membrane equation (3.9) has the fundamental solution

$$\bar{G}_0(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2) = \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{x}_2})}{4\pi r_{\underset{\sim}{x}_1\underset{\sim}{x}_2}}, \tag{4.1}$$

where $r_{\underset{\sim}{x}_1,\underset{\sim}{x}_2} = |\underset{\sim}{x}_1 - \underset{\sim}{x}_2|$ is the distance between the points $\underset{\sim}{x}_1 = (\underset{\sim}{x}_1^1,\underset{\sim}{x}_1^2,\underset{\sim}{x}_1^3)$ and $\underset{\sim}{x}_2 = (\underset{\sim}{x}_2^1,\underset{\sim}{x}_2^2,\underset{\sim}{x}_2^3)$ of the domain Ω. The existence of the solution (4.1) enables us to construct integral equations for $\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ satisfying the mixed boundary conditions (1.10) or (1.11). Therefore, in Case 1, Green's theorem gives:

$$\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2) = \frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{x}_2})}{4\pi r_{\underset{\sim}{x}_1\underset{\sim}{x}_2}} + \frac{1}{2\pi}\sum_{i=1}^{k}\int_{S_i}\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{y};s^2)\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}x_2}}\right]d\underset{\sim}{y}$$

$$+\frac{1}{2\pi}\sum_{i=k+1}^{m}\int_{S_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{y};s^2)\right]\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}dy. \tag{4.2}$$

On applying the iteration method (see [7], [9], [11]) to the integral equation (4.2), we obtain the Green's function $\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ which has the regular part:

$$\bar{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2) = \frac{1}{8\pi^2}\sum_{i=1}^{k}\int_{S_i}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}\right]d\underset{\sim}{y}$$

$$+\frac{1}{8\pi^2}\sum_{i=k+1}^{m}\int_{S_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\right]\frac{\exp(-sr_{\underset{\sim}{y}\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}\underset{\sim}{x}_2}}d\underset{\sim}{y}$$

$$+\frac{1}{8\pi^2}\sum_{i=1}^{k}\int_{S_i}\int_{S_i}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}M_i(\underset{\sim}{y},\underset{\sim}{y}')\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}\left[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\right]d\underset{\sim}{y}\,d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\sum_{i=k+1}^{m}\int_{S_i}\int_{S_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}\left[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\right]M_i^*(\underset{\sim}{y},\underset{\sim}{y}')\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}d\underset{\sim}{y}\,d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\sum_{i=1}^{k}\int_{S_i}\left\{\sum_{i=k+1}^{m}\int_{S_i}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}\right]L_i(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\right\}\times$$

$$\times\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}\left[\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}\right]d\underset{\sim}{y}'$$

$$+\frac{1}{8\pi^2}\sum_{i=k+1}^{m}\int_{S_i}\left\{\sum_{i=1}^{k}\int_{S_i}\frac{\exp(-sr_{\underset{\sim}{x}_1\underset{\sim}{y}})}{r_{\underset{\sim}{x}_1\underset{\sim}{y}}}L_i^*(\underset{\sim}{y},\underset{\sim}{y}')d\underset{\sim}{y}\right\}\frac{\exp(-sr_{\underset{\sim}{y}'\underset{\sim}{x}_2})}{r_{\underset{\sim}{y}'\underset{\sim}{x}_2}}d\underset{\sim}{y}', \tag{4.3}$$

where

$$M_i(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}K_i^{(\nu)}(\underset{\sim}{y},\underset{\sim}{y}'),\tag{4.4}$$

$$M_i^*(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}{}^*K_i^{(\nu)}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.5}$$

$$L_i(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}K_{-i}^{(\nu)}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.6}$$

$$L_i^*(\underset{\sim}{y},\underset{\sim}{y}')=\sum_{\nu=0}^{\infty}{}^*K_{-i}^{(\nu)}(\underset{\sim}{y}',\underset{\sim}{y}),\tag{4.7}$$

$$K_i(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{exp\left(-sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right)}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}\right],\tag{4.8}$$

$$*K_i(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\frac{\partial}{\partial n_{i\underset{\sim}{y}'}}\left[\frac{exp\left(-sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right)}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}\right],\tag{4.9}$$

$$K_{-1}(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\frac{exp\left(-sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right)}{r_{\underset{\sim}{y}\underset{\sim}{y}'}},\tag{4.10}$$

and

$$*K_{-i}(\underset{\sim}{y}',\underset{\sim}{y})=\frac{1}{2\pi}\frac{\partial^2}{\partial n_{i\underset{\sim}{y}}\partial n_{i\underset{\sim}{y}'}}\left[\frac{exp\left(-sr_{\underset{\sim}{y}\underset{\sim}{y}'}\right)}{r_{\underset{\sim}{y}\underset{\sim}{y}'}}\right],\tag{4.11}$$

In the same way, we can show that in Case 2, the Green's function $\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ has a regular part of the same form (4.3) with the interchanges $S_i, i=1,\dots,k \leftrightarrow S_i, i=k+1,\dots,m$.

On the basis of (4.3) the function $\bar{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ will be estimated for $s\to\infty$. The case when $\underset{\sim}{x}$ and $\underset{\sim}{x}_2$ lie in the neighborhood of the bounding surfaces S_i, $i=1,\dots,m$ of Ω is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOOD OF $S_i, i=1,\dots,m$.

Let $h_i>0(i=1,\dots,m)$ be sufficiently small. Let $n_i(i=1,\dots,m)$ be the minimum distances from a point $\underset{\sim}{x}=(x^1,x^2,x^3)$ of the domain Ω to its bounding surfaces $S_i(i=1,\dots,m)$ respectively. Let $\underset{\sim}{n}_i(\sigma_i)(i=1,\dots,m)$ denote the inward drawn unit normals to $S_i(i=1,\dots,m)$ respectively. We note that the coordinates in the neighborhood of $S_i(i=k+1,\dots,m)$ are in the same form as in Section 5.1 of [11] with the interchanges $\sigma_2^1\leftrightarrow\sigma_i^1$, $\sigma_2^2\leftrightarrow\sigma_i^2$, $n_2\leftrightarrow n_i$, $h_2\leftrightarrow h_i$, $I_2\leftrightarrow I_i$, $\mathfrak{D}(I_2)\leftrightarrow\mathfrak{D}(I_i)$ and $\delta_2\leftrightarrow\delta_1$, $(i=k+1,\dots,m)$. Thus we have the same formulae (5.1.1)-(5.1.6) of Section 5.1 in [11] with the interchanges $n_2\leftrightarrow n_i$, $\underset{\sim}{n}_2(\sigma_2)\leftrightarrow\underset{\sim}{n}_i(\sigma_i)$, $II_1\leftrightarrow II_{1i}$, $II_2\leftrightarrow II_{2i}$, $H_1\leftrightarrow H_i$ and $N_1\leftrightarrow N_i$ $(i=k+1,\dots,m)$.

Similarly, the coordinates in the neighborhood of $S_i,(i=1,\dots,k)$ are similar to those obtained in Section 5.2 of [11] with the interchanges $\sigma_1^1\leftrightarrow\sigma_i^1$, $\sigma_1^2\leftrightarrow\sigma_i^2$, $n_1\leftrightarrow n_i$, $h_1\leftrightarrow h_i$, $I_1\leftrightarrow I_i$, $\mathfrak{D}(I_1)\leftrightarrow\mathfrak{D}(I_i)$ and $\delta_1\leftrightarrow\delta_i$, $(i=1,\dots,k)$. Thus, we have the same formulae (5.2.1)-(5.2.5) of Section 5.1 in [11] with the interchanges $n_2\leftrightarrow n_i$, $\underset{\sim}{n}_2(\sigma_2)\leftrightarrow\underset{\sim}{n}_i(\sigma_i)$, $II_1^*\leftrightarrow II_{1i}$, $II_2^*\leftrightarrow II_{2i}$, $H_1^*\leftrightarrow H_i$ and $N_1^*\leftrightarrow N_i$, $(i=k+1,\cdots m)$

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions

$$\frac{exp\left(-sr_{\underset{\sim}{x}\underset{\sim}{y}}\right)}{r_{\underset{\sim}{x}\underset{\sim}{y}}},\ \frac{\partial}{\partial n_{i\underset{\sim}{y}}}\left[\frac{exp\left(-sr_{\underset{\sim}{x}\underset{\sim}{y}}\right)}{r_{\underset{\sim}{x}\underset{\sim}{y}}}\right],\ i=1,\dots,m,\tag{6.1}$$

when the distance between $\underset{\sim}{x}$ and $\underset{\sim}{y}$ is small are very similar to those obtained in Section 6 of [11]. Consequently, the local behavior of the kernels

$$K_s(\underset{\sim}{y}',\underset{\sim}{y}),{}^*K_{-s}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{6.2}$$

$$*K_s(\underset{\sim}{y}',\underset{\sim}{y}),K_{-s}(\underset{\sim}{y}',\underset{\sim}{y}), \tag{6.3}$$

when the distance between $\underset{\sim}{y}$ and $\underset{\sim}{y}'$ is small, follows directly from the local expansions of the functions (6.1).

DEFINITION 1. If $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ are points in the half-part $\xi^3 > 0$, then we define

$$\widehat{\rho}_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2 + (\xi_1^3 + \xi_2^3)^2}.$$

An $e^{\lambda}(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s)$-function is defined for points $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\xi}_2$ belong to sufficiently small domains $\mathfrak{N}(I_i)$ except when $\underset{\sim}{\xi}_1 = \underset{\sim}{\xi}_2 \in I_i, (i = 1,\dots,m)$ and λ is called the degree of this function. For every positive integer Λ, it has the local expansion (see [11]):

$$e^{\lambda}(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s) = \Sigma^* f(\xi_1^1,\xi_1^2)(\xi_1^3)^{P_1}(\xi_2^3)^{P_2}\left(\frac{\partial}{\partial \xi_1^1}\right)^{\ell_1}\left(\frac{\partial}{\partial \xi_1^2}\right)^{\ell_2}\left(\frac{\partial}{\partial \xi_1^3}\right)^{\ell_3}\frac{exp(-s\widehat{\rho}_{12})}{\widehat{\rho}_{12}} + R^{\Lambda}(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s),$$

where Σ^* denotes a sum of a finite number of terms in which $f(\xi^1,\xi^2)$ are infinitely differentiable functions. In this expansion P_1, P_2, ℓ_1, ℓ_2, ℓ_3 are integers, where $P_1 \geq 0$, $P_2 \geq 0$, $\ell_1 \geq 0$, $\ell_2 \geq 0$, $\lambda = min(P_1 + P_2 - q)$, $q = \ell_1 + \ell_2 + \ell_3$ and the minimum is taken over all terms which occur in the summation Σ^*. The remainder $R^{\Lambda}(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s)$ has continuous derivatives of order $d \leq \Lambda$ satisfying

$$D^d R^{\Lambda}(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_2;s) = 0\left[s^{-\Lambda}exp(-As\widehat{\rho}_{12})\right] \text{ as } s \to \infty,$$

where A is a positive constant.

Thus, using methods similar to those obtained in Section 7 of [11], we can show that the functions (6.1) are e^{λ}-functions with degrees $\lambda = -1, -2$ respectively. Consequently, the functions (6.2) are e^{λ}-functions with degrees $\lambda = 0, -1$ while the functions (6.3) are e^{λ}-functions with degrees $\lambda = 0, 1$ respectively.

DEFINITION 2. If $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are points in large domains $\Omega + S_i$, then we define

$$\widehat{r}_{12} = \min_{\underset{\sim}{y}}\left(r_{\underset{\sim}{x}_1\underset{\sim}{y}} + r_{\underset{\sim}{x}_2\underset{\sim}{y}}\right) \text{ if } \underset{\sim}{y} \in S_i, i = 1,\dots,k,$$

and

$$\widehat{R}_{12} = \min_{\underset{\sim}{y}}\left(r_{\underset{\sim}{x}_1\underset{\sim}{y}} + r_{\underset{\sim}{x}_2\underset{\sim}{y}}\right) \text{ if } \underset{\sim}{y} \in S_i, i = k+1,\dots m.$$

An $E^{\lambda}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s)$-function is defined and infinitely differentiable with respect to $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ when these points belong to large domains $\Omega + S_i$ except when $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 \in S_i, i = 1,\dots,m$. Thus, the E^{λ}-function has a similar local expansion of the e^{λ}-function (see [7], [11]).

With the help of Section 8 in [11], it is easily seen that formula (4.3) is an $E^{-2}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s)$-function and consequently

$$\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2) = \sum_{i=1}^{k} 0\left\{\widehat{r}_{12}^{\,-2}exp(-A_i s\widehat{r}_{12})\right\} + \sum_{i=k+1}^{m} 0\left\{\widehat{R}_{12}^{\,-2}exp(-A_i s\widehat{R}_{12})\right\}, \tag{6.4}$$

which is valid for $s \to \infty$, where $A_i (i = 1,\dots,m)$ are positive constants. Formula (6.4) shows that $\bar{G}(\underset{\sim}{x}_1,\underset{\sim}{x}_2,s^2)$ is exponentially small for $s \to \infty$.

With reference to Sections 7 and 9 in [11], if the e^{λ}-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of Section 7 in [11], we obtain the following local behavior of $\bar{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ as $s \to \infty$ which is valid when $\widehat{r}_{12}$ and $\widehat{R}_{12}$ are small:

$$\bar{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)=\sum_{i=1}^{m}\bar{\chi}_i(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2), \tag{6.5}$$

where, if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to sufficiently small domains $\mathfrak{D}(I_i)$, $i=1,\dots,m$, then

$$\bar{\chi}_i(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)=-\frac{exp(-s\hat{\rho}_{12})}{8\pi\hat{\rho}_{12}}+0\left\{\frac{exp(-A_is\hat{\rho}_{12})}{\hat{\rho}_{12}}\right\} \text{ as } s\to\infty. \tag{6.6}$$

When $\hat{r}_{12}\geq\delta_i>0, i=1,\dots,k$ and $\hat{R}_{12}\geq\delta_i>0, i=k+1,\dots,m$, the function $\bar{\chi}(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)$ is of order $0\{exp(-sN_0)\}$ as $s\to\infty, N_0>0$. Thus, since $\lim_{r_{12}\to 0}\frac{\hat{r}_{12}}{\hat{\rho}_{12}}=\lim_{R_{12}\to 0}\frac{\hat{R}_{12}}{\hat{\rho}_{12}}=1$ (see [11]), then the local behavior of the formula (4.3) has the form (6.5), where if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to large domains $\Omega+S_i, i=1,\dots,k$, we get

$$\bar{\chi}_i(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)=-\frac{exp(-s\hat{r}_{12})}{8\pi\hat{r}_{12}}+0\left\{\frac{exp(-A_is\hat{r}_{12})}{\hat{r}_{12}}\right\} \text{ as } s\to\infty, \tag{6.7}$$

while, if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ belong to large domains $\Omega+S_i, i=k+1,\dots,m$, we get:

$$\bar{\chi}_i(\underset{\sim}{x}_1,\underset{\sim}{x}_2;s^2)=-\frac{exp(-s\hat{R}_{12})}{8\pi\hat{R}_{12}}+0\left\{\frac{exp(-A_is\hat{R}_{12})}{\hat{R}_{12}}\right\} \text{ as } s\to\infty. \tag{6.8}$$

7. CONSTRUCTION OF RESULTS.

Since for $\xi^3\geq h_i>0, i=1,\dots,m$ the functions $\bar{\chi}_i(\underset{\sim}{x},\underset{\sim}{x};s^2)$ are of orders $0(e^{-2A_ish_i})$, the integral over Ω of the function $\bar{\chi}(\underset{\sim}{x},\underset{\sim}{x};s^2)$ can be approximated in the following way (see (3.10)):

$$\begin{aligned}\bar{K}(s^2)=&\sum_{i=k+1}^{m}\int_{S_i}\int_{\xi^3=0}^{h_i}\bar{\chi}_i(\underset{\sim}{x},\underset{\sim}{x};s^2)\{1-2\xi^3H_i+(\xi^3)^2N_i\}d\xi^3dS_i\\ &-\sum_{i=1}^{k}\int_{S_i}\int_{\xi^3=0}^{h_i}\bar{\chi}_i(\underset{\sim}{x},\underset{\sim}{x};s^2)\{1+2\xi^3H_i+(\xi^3)^2N_i\}d\xi^3dS_i\\ &+\sum_{i=1}^{m}0(e^{-2A_ish_i}) \text{ as } s\to\infty.\end{aligned} \tag{7.1}$$

If the e^λ-expansions of $\bar{\chi}_i(\underset{\sim}{x},\underset{\sim}{x};s^2)$ are introduced into (7.1) and with the help of formula (10.2) of Section 10 in [11], we deduce after inverting Laplace transforms, that

$$K(t)=\frac{a_1}{t}+\frac{a_2}{t^{1/2}}+a_3+a_4t^{1/2}+0(t) \text{ as } t\to 0, \tag{7.2}$$

where

$$a_1=\frac{1}{16\pi}\left\{\sum_{i=1}^{k}|S_i|-\sum_{i=k+1}^{m}|S_i|\right\}, a_2=\frac{1}{12\pi^{3/2}}\sum_{i=1}^{m}\int_{S_i}H_idS_i,$$

$$a_3=\frac{1}{128\pi}\left\{7\sum_{i=1}^{k}\int_{S_i}(H_i^2-N_i)dS_i+\sum_{i=k+1}^{m}\int_{S_i}(H_i^2-N_i)dS_i\right\},$$

and

$$a_4=\frac{1}{\pi^{3/2}}\left\{\frac{13}{1440}\sum_{i=1}^{k}\int_{S_i}H_i^3dS_i-\frac{1}{315}\sum_{i=k+1}^{m}\int_{S_i}H_i^3dS_i\right\}.$$

On inserting (7.2) into (3.6) we arrive at our result (2.1).

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