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REALCOMPACTIFICATION AND REPLETENESS OF WALLMAN SPACES

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ABSTRACT. The extension of bounded lattice continuous functions on an arbitrary set X to the set of lattice regular zero-one measures on an algebra generated by a lattice (a Wallman-type space) is investigated.

Next the subset of lattice regular zero-one measures on an algebra generated by a lattice which integrates all lattice continuous functions on X is introduced and various properties of it are presented.

Finally conditions are established using repleteness criteria whereby the space of lattice regular zero-one measures on an algebra generated by a lattice which are countably additive (a Wallman-type space) is realcompact.

KEY WORDS AND PHRASES. Realcompact, repleteness, Wallman spaces, normal lattice, lattice continuous functions.
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1. INTRODUCTION.

Let X be an arbitrary set and $\underline{L}$ a lattice of subsets of X. $A(\underline{L})$ denotes the algebra generated by $\underline{L}$, and $M(\underline{L})$ those bounded finitely additive measures on $A(\underline{L})$, and $M_R(\underline{L})$ those elements of $M(\underline{L})$ which are $\underline{L}$-regular while $M_R^\sigma(\underline{L})$ denotes those elements of $M_R(\underline{L})$ which are countably additive. The zero-one valued members of the above are designated by $I(\underline{L})$, $I_R(\underline{L})$, and $I_R^\sigma(\underline{L})$ respectively. For $A \in A(\underline{L})$, $w(A) = \{u \in I_R(\underline{L}) \mid u(A) = 1\}$, $w(\underline{L}) = \{w(L) \mid L \in \underline{L}\}$, then $I_R(\underline{L})$ with the topology of closed sets $\tau w(\underline{L})$ of arbitrary intersections of sets of $w(\underline{L})$ is a compact, T_1 topological space. It is one of the Wallman type spaces. Assuming $\underline{L}$ is disjunctive then it is T_2 if and only if $\underline{L}$ is normal.

We begin by considering briefly, because of their importance, certain fundamental properties of normal lattices. Then we proceed to a consideration of $I_R(\underline{L})$, and the extension of bounded lattices continuous functions on X to $I_R(\underline{L})$. These results are generally known (see [8]) but we give somewhat shorter more direct proofs here.

We next consider the space $Q(\underline{L})$ of measures in $I_R(\underline{L})$ which integrate <u>all</u> lattice continuous functions on X, and show its relationship to $I_R^\sigma(\underline{L})$, and under suitable conditions, its relationship to the G_δ-closure of X in $I_R(\underline{L})$.

Finally, we consider the Wallman type space $I_R^\sigma(\underline{L})$, and the lattice $w_\sigma(\underline{L})$, where for $A \in A(\underline{L})$, $w_\sigma(A) = \{u \in I_R^\sigma(\underline{L}) \mid u(A) = 1\}$, and where $w_\sigma(\underline{L}) = \{w_\sigma(L) \mid L \in \underline{L}\}$. It is well-known that if $\underline{L}$ is disjunctive then $w_\sigma(\underline{L})$ is replete. We consider in this space the lattice of closed sets $\tau w_\sigma(\underline{L})$ and its associated lattice of zero sets, and investigate their repleteness - thus obtaining sufficient conditions for the space $I_R^\sigma(\underline{L})$ to be realcompact.

Our notations and terminology is consistent with [1, 3, 5, 6, 11]. However, the main definitions and notations used throughout the paper are presented for the reader's convenience in section 2(a). We note also that a number of results on normal lattices in section 2(b) are related to work of [4 , 9].

2.(a)BACKGROUND AND NOTATION.

Let X be an abstract set, and $\underline{L}$ the lattice of subsets of X. We assume that ϕ, $X \in \underline{L}$ for most of our results. First:

Lattice Terminology:

$A(\underline{L})$ is the algebra generated by $\underline{L}$.

$\sigma(\underline{L})$ is the σ-algebra generated by $\underline{L}$.

$\delta(\underline{L})$ is the lattice of all countable intersections of sets from $\underline{L}$. $\underline{L}$ is a delta lattice (δ-lattice) if $\delta(\underline{L}) = \underline{L}$.

$\tau(\underline{L})$ is the lattice of arbitrary intersections of sets of $\underline{L}$.

$\underline{L}$ is complemented if $L \in \underline{L} => L' \in \underline{L}$ (prime denotes complement), that is, $\underline{L}$ is an algebra.

$\underline{L}$ is separating, if for any two elements $x \neq y$ of X, there exists an element $L \in \underline{L}$ such that $x \in L$ and $y \notin L$.

$\underline{L}$ is T_2 if, for any two elements $x \neq y$ of X, there exists $A, B \in \underline{L}$ such that $X \in A'$ and $y \in B'$ and $A' \cap B' = \phi$.

$\underline{L}$ is disjunctive if for any $x \in X$ and $A \in \underline{L}$ such that $x \notin A$, there exists a $B \in \underline{L}$ such that $x \in B$ and $A \cap B = \phi$.

$\underline{L}$ is regular if for any $x \in X$, and $A \in \underline{L}$ such that $x \notin A$ there exist $B, C \in \underline{L}$ such that $x \in B'$, $A \subset C'$ and $B' \cap C' = \phi$.

$\underline{L}$ is normal if for all L_1, $L_2 \in \underline{L}$ such that $L_1 \cap L_2 = \phi$ there exists $\tilde{L}_1, \tilde{L}_2 \in \underline{L}$ such that $L_1 \subset \tilde{L}_1'$, $L_2 \subset \tilde{L}_2'$, and $\tilde{L}_1' \cap \tilde{L}_2' = \phi$.

$\underline{L}$ is compact if every covering of X by elements of $\underline{L}'$ has a finite subcovering.

$\underline{L}$ is countably compact if every countable covering of X by .elements of $\underline{L}'$ has a finite subcovering.

$\underline{L}$ is Lindelöf if every covering of X by elements of $\underline{L}'$ has a countable subcovering.

$\underline{L}$ is countably paracompact if whenever $A_n \downarrow \phi, A_n \in \underline{L}$ there exists $B_n \in \underline{L}$ such that $A_n \subset B_n'$ and $B_n' \downarrow \phi$.

$\underline{L}$ is complement generated if, for $L \in \underline{L}$ there exists $L_n \in \underline{L}$ such that $L = \bigcap_{n=1}^{\infty} L_n'$.

It is well known that if $\underline{L}$ is complement generated then $\underline{L}$ is countably paracompact.

Measure Terminology

We denote by $M(\underline{L})$ the finitely additive bounded measures on $A(\underline{L})$ (we may and do assume all elements of $M(\underline{L})$ are ≥ 0).

$u \in M(\underline{L})$ is $\underline{L}$-regular if for any $A \in A(\underline{L}), u(A) = sup\{u(L) \mid L \subset A, L \in \underline{L}\}$; (equivalently) $= inf\{u(L') \mid A \subset L', L \in \underline{L}\}$.

$u \in M(\underline{L})$ is σ-smooth on $\underline{L}$ if $L_n \in \underline{L}, n = 1,2,\ldots$ and $L_n \downarrow \phi => u(L_n) \to 0$

$u \in M(\underline{L})$ is σ-smooth on $A(\underline{L})$ if $A_n \in A(\underline{L})$, $n = 1,2,\ldots$ and $A_n \downarrow \phi => u(A_n) \to 0$. Note u is σ-smooth on $A(\underline{L})$ iff u is countably additive.

We will use the following notations.

$M_R(\underline{L}) =$ the set of $\underline{L}$-regular measures of $M(\underline{L})$.

$M_\sigma(\underline{L}) =$ the set of σ-smooth measures on $\underline{L}$ of $M(\underline{L})$.

$M^\sigma(\underline{L}) =$ the set of σ-smooth measures on $A(\underline{L})$ of $M(\underline{L})$.

$M_R{}^\sigma(\underline{L}) =$ the set of $\underline{L}$-regular measures of $M^\sigma(\underline{L})$.

Note that if $u \in M_R(\underline{L})$ and $u \in M_\sigma(\underline{L})$ then $u \in M_R{}^\sigma(\underline{L})$.
Also we denote by $I(\underline{L})$, $I_R(\underline{L})$, $I_\sigma(\underline{L})$, $I^\sigma(\underline{L})$, and $I_R{}^\sigma(\underline{L})$ the subsets of $M(\underline{L})$, $M_R(\underline{L})$, $M_\sigma(\underline{L})$, $M^\sigma(\underline{L})$, and $M_R{}^\sigma(\underline{L})$ consisting of zero-one valued measures.
Now for $u_1, u_2 \in I(\underline{L}), u_1 \le u_2(\underline{L})$ means $u_1(L) \le u_2(L)$ for $L \in \underline{L}$.
Let $J(\underline{L})$ denote those $u \in I(\underline{L})$ such that whenever $L_n \in \underline{L}, n = 1,2,\ldots$ and $\bigcap_{n=1}^{\infty} L_n \in \underline{L}$ then $u\left(\bigcap_{n=1}^{\infty} L_n\right) = \inf_n u(L_n)$.
Clearly, $I^\sigma(\underline{L}) \subset J(\underline{L}) \subset I_\sigma(\underline{L})$.
For $u \in M(\underline{L})$ the support of $u, S(u) = \cap\{L \in \underline{L} \mid u(L) = u(X)\}$. $\underline{L}$ is replete if for any $u \in I_R{}^\sigma(\underline{L})$, $u \neq 0$, $S(u) \neq \phi$.
Let $C(\underline{L})$ be the set of all real-valued $\underline{L}$-continuous functions defined on X, where $f: X \to R$ is called $\underline{L}$-continuous if $f^{-1}(E) \in \underline{L}$ for any closed set $E \subset R$. If X is a topological space, $C(X)$ denotes the continuous functions on X or equivalently we can write $C(X) = C(F)$ where F is the lattice of closed sets of X. $z(\underline{L})$ is the lattice of zero sets of functions in $C(\underline{L})$.
$C_b(\underline{L}) =$ set of all real valued bounded $\underline{L}$-continuous functions defined on X.
Next we define $w(A) = \{u \in I_R(\underline{L}) \mid u(A) = 1\}$ for $A \in A(\underline{L})$, and $w(\underline{L}) = \{w(L) \mid L \in \underline{L}\}$.
We have for A, $B \in A(\underline{L})$:

(1) $w(A \cup B) = w(A) \cup w(B)$

(2) $w(A \cap B) = w(A) \cap w(B)$

(3) $w(A)' = w(A')$

(4) $w(A(\underline{L})) = A(w(\underline{L}))$

(5) $A \subset B => w(A) \subset w(B)$

Note $w(\underline{L})$ is a lattice and if $\underline{L}$ is disjunctive then $w(A) = w(B)$ if and only if $A = B$.

The Wallman topology is obtained by taking $w(\underline{L})$ as a base for the closed sets of a topology on $I_R(\underline{L})$. $< I_R(\underline{L}), \tau w(\underline{L}) >$ is the general Wallman space associated with X and $\underline{L}$. Note we have $w(L) = \bar{L}$ for $L \in \underline{L}$ if $\underline{L}$ is separating and disjunctive. We also define: $w_\sigma(A) = \{u \in I_R{}^\sigma(\underline{L}) \mid u(A) = 1\}$ where $A \in A(\underline{L})$, and note $w(\underline{L}) \cap I_R{}^\sigma(\underline{L}) = w_\sigma(\underline{L})$.

We now consider two lattices. Let $\underline{L}_1$ and $\underline{L}_2$ denote lattices of subsets. of X where $\underline{L}_1 \subset \underline{L}_2$. $\underline{L}_1$ semi-separates $\underline{L}_2$ if $A \in \underline{L}_1$, $B \in \underline{L}_2$ and $A \cap B = \phi$ implies there exists $C \in \underline{L}_1$, $B \subset C$ and $A \cap C = \phi$. $\underline{L}_1$ separates $\underline{L}_2$ if $A, B \in \underline{L}_2$ and $A \cap B = \phi$ implies there exists $C, D \in \underline{L}_1$ such that $A \subset C, B \subset D$, and $C \cap D = \phi$. $\underline{L}_2$ is $\underline{L}_1$-countable paracompact if for every sequences $\{B_n\}$ of sets of $\underline{L}_2$, such that $B_n \downarrow \phi$ there exists $\{A_n \in \underline{L}_1\}$ such that $A_n{}' \downarrow \phi$ and $B_n \subset A_n{}'$.

$\underline{L}_2$ is $\underline{L}_1$- cb if given $B_n \downarrow \phi$, $B_n \in \underline{L}_2$ there exists $\{A_n\}, A_n \in \underline{L}_1$ such that $A_n \downarrow \phi$ and $B_n \subset A_n$. Clearly if $\underline{L}_1$ separates $\underline{L}_2$ then $\underline{L}_1$ semiseparates $\underline{L}_2$.

If $\nu \in M(\underline{L}_2)$ then by $\nu \mid A(\underline{L}_1)$ we mean ν restricted to $A(\underline{L}_1)$. We state the following well known results:

Let $\underline{L}_1 \subset \underline{L}_2$ be two lattices of subsets of X. If $\underline{L}_1$ semiseparates $\underline{L}_2$ then for $\nu \in M_R(\underline{L}_2)$, $u = \nu \mid A(\underline{L}_1) \in M_R(\underline{L}_1)$.

Suppose $\underline{L}_1 \subset \underline{L}_2$ are two lattices of subsets of X. Then if $u \in M_R(\underline{L}_1)$, u extends to $\nu \in M_R(\underline{L}_2)$. Moreover, the extension is unique if $\underline{L}_1$ separates $\underline{L}_2$.

We will frequently assume in the sequel that $\underline{L}_1 \subset \underline{L}_2$ and $\underline{L}_2$ is $\underline{L}_1$ countably paracompact or countably bounded, but we note that this is unnecessary in certain situations as the following facts listed below show:

(1) If $\underline{L}_2$ is $\underline{L}_1$ countably bounded and if $\underline{L}_1$ is countably paracompact (e.g., if $\underline{L}_1$ is complement generated) then $\underline{L}_2$ is $\underline{L}_1$ countably paracompact.

(2) If $\underline{L}_2$ is countably paracompact and if $\underline{L}_1$ separates $\underline{L}_2$ then $\underline{L}_2$ is $\underline{L}_1$ countably paracompact.

(3) Suppose $\underline{L}_2$ is $\underline{L}_1$ countably paracompact and $\underline{L}_1$ semiseparates $\underline{L}_2$ then $\underline{L}_2$ is $\underline{L}_1$ countably bounded.

(4) If $\underline{L}_2$ is countably paracompact and if $\underline{L}_1$ separates $\underline{L}_2$ then $\underline{L}_2$ is $\underline{L}_1$ countably bounded.

2.(b)NORMAL LATTICES AND MEASURES.

In this section we will consider a number of measure implications of normal lattices and other special lattices as well as converse implications. We first note:

THEOREM 2.1. Let $\underline{L}$ be a complemented generated lattice. The $u \in I_\sigma(\underline{L}')$ implies $u \in I_R{}^\sigma(\underline{L})$.

PROOF. Since $\underline{L}$ is complemented generated then $\underline{L}$ is countably paracompact and therefore $I_\sigma(\underline{L}') \subset I_\sigma(\underline{L})$. Therefore it suffices to show $u \in I_R(\underline{L})$, but this is easy for if $L \in \underline{L}$ then $L = \bigcap_{n=1}^{\infty} L_n', L_n \in \underline{L}$ all n, and we may assume that the $L_n' \downarrow \phi$. Now if $u(L) = 0$, and if all $u(L_n') = 1$ then $\bigcap_{n=1}^{\infty} L_n' \cap L' = \phi$ and $u(L_n' \cap L') = 1$ all n which is a contradiction since $u \in I_\sigma(\underline{L}')$. It follows that $u(L) = inf\{u(\tilde{L}') \mid L \subset \tilde{L}', \tilde{L} \in \underline{L}\}$ and this implies $u \in I_R(\underline{L})$.

REMARK. It is equally easy to show if $\underline{L}$ is complement generated and $u \in M_\sigma(\underline{L}')$ then $u \in M_R{}^\sigma(\underline{L})$.

THEOREM 2.2. Let $u \in J(\underline{L})$ and let $\underline{L}$ be a δ-lattice then $u\left(\bigcup_{i=1}^{\infty} L_i'\right) \leq \sum_{i=1}^{\infty} u(L_i')$ where all $L_i \in \underline{L}$.

PROOF. Suppose $u\left(\bigcup_{i=1}^{\infty} L_i'\right) = 1$ and $\sum_{i=1}^{\infty} u(L_i') = 0$. Now $\sum_{i=1}^{\infty} u(L_i') = 0$ implies $u(L_i') = 0$ all i and $\bigcap_{i=1}^{\infty} L_i = \left(\bigcup_{i=1}^{\infty} L_i'\right)'$ therefore $u\left(\bigcap_{i=1}^{\infty} L_i\right) = 0$ where obviously $\bigcap_{i=1}^{\infty} L_i \in \underline{L}$. Also $u\left(\bigcap_{i=1}^{\infty} L_i\right) = inf u(L_i)$ since $u \in J(\underline{L})$. So $u\left(\bigcap_{i=1}^{\infty} L_i\right) = 0$ implies there exists an i_0 such that $\bigcap_{i=1}^{\infty} L_i \subset L_{i_0}$ and $u(L_{i_0}) = 0$. Therefore $u(L_{i_0}') = 1$ which is a contradiction, therefore theorem is proved.

THEOREM 2.3. If $\underline{L}$ is normal and complement generated then $u \in J(\underline{L}) => u \in I_R{}^\sigma(\underline{L})$.

PROOF. Let $u \in J(\underline{L})$; we know that $u \leq \nu$ on $\underline{L}$ where $\nu \in I_R(\underline{L})$. This gives $\nu \leq u$ on $\underline{L}'$. Suppose $u \neq \nu$. Then there exists $L \in \underline{L}$ such that $u(L) = 0$, $\nu(L) = 1$. However, $L = \bigcap_{n=1}^{\infty} L_n'$ since $\underline{L}$ is complement generated so $L \subset L_n'$. Therefore $\nu(L) = 1 => \nu(L_n') = 1$ for all n which implies $u(L_n') = 1$ for all n as $\nu \leq u$ on $\underline{L}'$. Now $L = \cap L_n' => L \cap L_n = \phi$ therefore since $\underline{L}$ is normal there exists $A_n', B_n' \in \underline{L}'$ such that $L \subset A_n'$, $L_n \subset B_n'$, and $A_n' \cap B_n' = \phi$. Therefore $L \subset A_n' \subset B_n \subset L_n$ from this which gives $\nu(A_n') = 1$ and $\nu(B_n) = 1$ by monotonicity of ν. Therefore $u(B_n) = 1$ as $u \le \nu$ on $\underline{L}$. Also $L \subset A_n' \subset B_n \subset L_n' => L \subset \bigcap_{n=1}^{\infty} A_n' \subset \bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} L_n' = L$ which implies that $L = \bigcap_{n=1}^{\infty} A_n' = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} L_n'$, so $u(L) = 0 => u\left(\bigcap_{n=1}^{\infty} B_n\right) = 0$ which $=> u(B_n) = 0$ by $u \in J(\underline{L})$. This is a contradiction as $u(B_n) = 1$. Therefore $u = v \in I_R(\underline{L}) => u \in I_R(\underline{L})$. Now $u \in J(\underline{L}) => u \in I^{\sigma}(\underline{L})$, therefore $u \in I_R^{\sigma}(\underline{L})$.

THEOREM 2.4. Let $\underline{L}$ be a normal lattice, $u \in I_R(\underline{L})$, $u \le \rho(\underline{L}')$ where $\rho \in I_R(\underline{L}')$. Then for $L \in \underline{L}$, $u(L') = sup\{\rho(\tilde{L}) \mid \tilde{L} \subset L' \mid \tilde{L} \in \underline{L}\}$.

PROOF. Suppose $u(L') = 1$, where $L \in \underline{L}$ then since $u \in I_R(\underline{L})$ there exists $\tilde{L} \subset L', \tilde{L} \in \underline{L}, u(\tilde{L}) = 1$. Since $\tilde{L} \subset L' => L \cap \tilde{L} = \phi$, therefore by normality there exists $A, B \in \underline{L}$ such that $L \subset A', \tilde{L} \subset B', A' \cap B' = \phi$. Therefore $\tilde{L} \subset B' \subset A \subset L'$, also $u(\tilde{L}) = 1 => u(B') = 1$ by monotonicity of u. Therefore $\rho(B') = 1$ as $u \le \rho(\underline{L}')$. $\rho(A) = 1$ follows by monotonicity of ρ, proving the theorem.

REMARK. This theorem is equivalent to the following: Let $\underline{L}$ be normal and let $\nu \le u(\underline{L})$ where $\nu \in I(\underline{L})$ and $u \in I_R(\underline{L})$. Then $u(L') = sup\{\nu(\tilde{L}) \mid \tilde{L} \subset L', \tilde{L} \in \underline{L}\}$. Next we show that actually the property in Theorem 2.4 or equivalently the one in the remark characterizes normal lattices, i.e.,

THEOREM 2.5. Suppose $u \in I_R(\underline{L})$ and $\rho \le u(\underline{L})$ where $\rho \in I_R(\underline{L}')$ and $u(L') = 1$, $L \in \underline{L}$ implies $L' \supset A \in \underline{L}$ such that $\rho(A) = 1$. Then $\underline{L}$ is normal.

PROOF. Let $\rho \le u(\underline{L})$, $\rho \le \nu(\underline{L})$ where $u, \nu \in I_R(\underline{L})$ and $\rho \in I_R(\underline{L}')$. Assume $u \ne \nu$, this implies $u(L_1) = 0$, $v(L_1) = 1$, $u(L_2) = 1$, $v(L_2) = 0$ where $L_1, L_2 \in \underline{L}$ and $L_1 \cap L_2 = \phi$. Now $u(L_1) = 0$ implies $u(L_1') = 1$ which implies there exists $L_1' \supset A \in \underline{L}$ such that $\rho(A) = 1$ and $v(L_2) = 0$ implies $v(L_2') = 1$ which implies there exists $L_2' \supset B \in \underline{L}$ such that $\rho(B) = 1$. Since $A \subset L_1'$, $B \subset L_2'$ then $L_1 \subset A'$ and $L_2 \subset B'$. So $\rho(B) = 1$ implies $\rho(B') = 0$ which implies $u(B') = 0$ as $u \le \rho(\underline{L}')$. However, by monotonicity $u(L_2) \le u(B')$ and $u(L_2) = 1$ which implies $u(B') = 1$ which contradicts $u(B') = 0$. Therefore $u = v$ which means $\underline{L}$ is normal.

THEOREM 2.6. Let $\underline{L}$ be a normal lattice, $u \in I_{\sigma}(\underline{L})$, $u \le v(\underline{L})$ where $\nu \in I_R(\underline{L})$. Then $\nu \in I_{\sigma}(\underline{L}')$.

PROOF. Suppose $u \in I_{\sigma}(\underline{L})$ we know $\rho \le u \le \nu(\underline{L})$ where $v \le u \le \rho(\underline{L}')$ and $\nu \in I_R(\underline{L})$, $\rho \in I_R(\underline{L})$. Suppose $L_n' \downarrow \phi$, $v(L_n') = 1$ all n, $L_n' \in \underline{L}'$. Then there exists $\tilde{L}_n \subset L_n'$ such that $\rho(\tilde{L}_n) = 1$ all n by Theorem 2.5. Therefore $u(\tilde{L}_n) = 1$ since $\rho \le u(\underline{L})$. Now $\tilde{L}_n \downarrow \phi$ since $\bigcap_{n=1}^{\infty} \tilde{L}_n \subset \bigcap_{n=1}^{\infty} L_n'$. This contradicts the fact that $u \in I_{\sigma}(\underline{L})$, therefore $v \in I_{\sigma}(\underline{L}')$.

COROLLARY 2.7. If $\underline{L}$ is normal and countably paracompact then the ν (from Theorem 2.6) belongs to $I_R^{\sigma}(\underline{L})$.

PROOF. Since $\underline{L}$ is countably paracompact then $I_{\sigma}(\underline{L}') \subset I_{\sigma}(\underline{L})$ by Theorem 2.2. Then $\nu \in I_{\sigma}(\underline{L})$ and since $\nu \in I_R(\underline{L})$ it follows that $\nu \in I_R^{\sigma}(\underline{L})$.

Next we consider a pair of lattices $\underline{L}_1$, $\underline{L}_2$ of X such that $\underline{L}_1 \subset \underline{L}_2$, then we have:

THEOREM 2.8. If $\underline{L}_1$ separates $\underline{L}_2$ then $\underline{L}_1$ is normal if and only if $\underline{L}_2$ is normal.

PROOF. The proof is not difficult. We just show $\underline{L}_2$ normal implies $\underline{L}_1$ normal. Now let $\underline{L}_2$ be normal and $u \in I(\underline{L}_1)$, $u \le v_1(\underline{L}_1)$, $u \le v_2(\underline{L}_1)$ where v_1, $v_2 \in I_R(\underline{L}_1)$. Now we can extend $u \in I(\underline{L}_1)$ to $\lambda \in I(\underline{L}_2)$ and extend v_1 to $\tau_1 \in I_R(\underline{L}_2)$, v_2 to $\tau_2 \in I_R(\underline{L}_2)$. Now we have $\lambda \le \tau_1(\underline{L}_2)$, $\lambda \le \tau_2(\underline{L}_2)$ which is not difficult to see since $\underline{L}_1$ separates $\underline{L}_2$. Now $\underline{L}_2$ is normal, therefore $\tau_1 = \tau_2$ and $\nu_1 = \tau_1 \mid A(\underline{L}_1) = \tau_2 \mid A(\underline{L}_1) = \nu_2$. Therefore $\underline{L}_1$ is normal.

3. THE WALLMAN SPACE $I_R(\underline{L})$.

We give here a brief discussion of the general Wallman space (see also [11]). Consider the set $I_R(\underline{L})$ and the lattice of subsets $w(\underline{L})$. It is well-known that $w(\underline{L})$ is compact and it is not difficult to show:

THEOREM 3.1. The following are equivalent:

(a) $w(\underline{L})$ is normal;

(b) $w(\underline{L})$ is regular;

(c) $w(\underline{L})$ is T_2.

Now since $w(\underline{L})$ is compact, $\tau w(\underline{L})$ the topology of closed sets, is compact and $w(\underline{L})$ separates $\tau w(\underline{L})$, and by Theorem 2.8 $w(\underline{L})$ is normal if and only if $\tau w(\underline{L})$ is normal. $< I_R(\underline{L}), \tau w(\underline{L}) >$ is a compact topological space and it is always T_1. Assuming $\underline{L}$ is disjunctive, it is T_2 if and only if $\underline{L}$ is normal. Next, let $\underline{L}$ be a δ-normal lattice of subsets of X, then the Alexandroff representation theorem (see [1]) yields for the conjugate space of $C_b(\underline{L})$, namely $C_b(\underline{L})' = M_R(\underline{L})$ where to any $\Phi \in C_b(\underline{L})'$ there corresponds a unique $u \in M_R(\underline{L})$ such that $\Phi(f) = \int_X f du$, for all $f \in C_b(\underline{L})$.

A net $\{u_\alpha\}$ in $M_R(\underline{L})$ converges to u in $M_R(\underline{L})$ in the weak $*$ topology if and only if $\int_X f du_\alpha \to \int_X f du$ for all $f \in C_b(\underline{L})$. We shall denote weak $*$ convergence by $\underset{\rightarrow}{w^*}$.

THEOREM 3.2. Now let $\underline{L}$ be δ-normal and consider convergence in $M_R{}^+(\underline{L})$. The following are equivalent:

(1) $u_\alpha \overset{w^*}{\rightarrow} u$

(2) $u_\alpha(X) \rightarrow u(X)$ and $\overline{\lim_\alpha}\, u_\alpha(A) \leq u(A)$ for all $A \in \underline{L}$

(3) $u_\alpha(X) \rightarrow u(X)$ and $\underline{\lim_\alpha}\, u_\alpha(A') \geq u(A')$ for all $A' \in \underline{L}'$. For the proof in this particular setting see ([7]).

THEOREM 3.3. Let $u_\alpha \in I_R(\underline{L}) \overset{w^*}{\rightarrow} u \in M_R(\underline{L})$ then $u \in I_R(\underline{L})$. Thus $I_R(\underline{L})$ is w^*-closed in $M_R(\underline{L})$.

PROOF. Suppose $u_\alpha \in I_R(\underline{L}) \overset{w^*}{\rightarrow} u \in M_R(\underline{L})$. Therefore $u_\alpha(X) \rightarrow u(X)$ by Theorem 3.2. Now $u_\alpha(X) = 1$ since $u_\alpha \in I_R(\underline{L})$, therefore $u(X) = 1$, which means for $A \in A(\underline{L}): 0 \leq u(A) \leq 1$. Suppose $A \in A(\underline{L})$ and $0 < u(A) < 1$. Since $u \in M_R(\underline{L})$ there exists $L \in \underline{L} \subset A$ such that $0 < u(L) \leq u(A)$ and there exists $A \subset \tilde{L}' \in \underline{L}'$ such that $u(A) \leq u(\tilde{L}') < 1$. Therefore $0 < u(L) \leq u(\tilde{L}') < 1$. Now $L \subset \tilde{L}'$ therefore $L \cap \tilde{L} = \phi$ which implies there exists $A, B \in \underline{L}$ such that $L \subset A', \tilde{L} \subset B', A' \cap B' = \phi$ by $\underline{L}$ normal. Therefore $L \subset A' \subset B \subset \tilde{L}'$ which implies $0 < u(L) \leq u(A') \leq u(B) \leq u(\tilde{L}') < 1$ so that $u(A') \leq u(B) < 1$. Now since $u_\alpha \overset{w^*}{\rightarrow} u$ then $\overline{\lim_\alpha}\, u_\alpha(B) \leq u(B)$ for $B \in \underline{L}$. Now $u(B) < 1$, therefore $\overline{\lim_\alpha}\, u_\alpha(B) < 1$ which means $u_\alpha(B) = 0$ since $u_\alpha \in I_R(\underline{L})$. Also $\underline{\lim_\alpha}\, u_\alpha(A') \geq u(A')$ for $A' \in \underline{L}'$ but $u(A') < 1$, therefore $\underline{\lim_\alpha}\, u_\alpha(A') > 0$ as $0 < u(A') < 1$. Therefore $u_\alpha(A') = 1$ since $u_\alpha \in I_R(\underline{L})$. However for $A' \subset B$ we have $u_\alpha(A') = 1$, $u_\alpha(B) = 0$ which is impossible. Therefore $u(A) = 0$ or 1, which implies $u \in I_R(\underline{L})$.

THEOREM 3.4. $\overline{[\{u_x\}]} = M_R(\underline{L})$

PROOF. The proof of this is not difficult and can be modelled after the well-known special case of $\underline{L}$ being the lattice of zero sets in a Tychonoff space.

THEOREM 3.5. The w^*-topology of $M_R(\underline{L})$ when restricted to $I_R(\underline{L})$ gives the Wallman topology $\tau w(\underline{L})$ for closed sets.

PROOF. Let $u_\alpha \overset{w^*}{\rightarrow} u$ we will show $u_\alpha \overset{w}{\rightarrow} u$ where w is convergence in Wallman. Consider $u_0 \in w(L)'$, therefore $u_0(L') = 1$. Using Theorem 3.2 we have $\underline{\lim_\alpha}\, u_\alpha(L') \geq u_0(L')$, therefore $\underline{\lim_\alpha}\, u_\alpha(L') = 1$. But $1 = \underline{\lim_\alpha}\, u_\alpha(L') \leq \overline{\lim_\alpha}\, u_\alpha(L') \leq 1$, therefore $\lim_\alpha u_\alpha(L') = 1$. So there exists α_0 such that for all $\alpha \geq \alpha_0$ $u_\alpha(L') = 1$, therefore $u_\alpha \in w(L')$ for all $\alpha \geq \alpha_0$. This gives $u_\alpha \overset{w}{\rightarrow} u$ which proves the theorem.

We assume now that $\underline{L}$ is δ-normal, separating and disjunctive. Let $f \in C_b(\underline{L})$ we define $\hat{f}$ on $I_R(\underline{L})$ by $\hat{f}(u) = \int_X f du$ where $u \in I_R(\underline{L})$.

THEOREM 3.6. $\hat{f} \in C(I_R(\underline{L}))$.

PROOF. Let $u_\alpha \xrightarrow{w} u_0$. We must show that $\hat{f}(u_\alpha) \to \hat{f}(u_0)$ which means $u_\alpha \xrightarrow{w^*} u_0$. For $u_0 \in w(L')$ we have $u_\alpha \in w(L')$ for all $\alpha \geq \alpha_0$ as $u_\alpha \xrightarrow{w} u_0$. Therefore, $u_\alpha(L') = 1$, $\alpha \geq \alpha_0$, which implies $\underline{\lim}_\alpha\, u_\alpha(L') = 1$. Therefore $\underline{\lim}_\alpha\, u_\alpha(L') = \overline{\lim}_\alpha\, u_\alpha(L') = \lim u_\alpha(L') = 1$ and $u_0(L') = 1$ as $u_0 \in w(L')$. So $\underline{\lim}_\alpha\, u_\alpha(L') \geq u_0(L')$ and therefore by Theorem 3.2 we have $u_\alpha \xrightarrow{w^*} u_0$ which proves $\hat{f} \in C(I_R(\underline{L}))$.

THEOREM 3.7. The correspondence $f \to \hat{f}$ is a bijection between $C_b(\underline{L})$ and $C(I_R(\underline{L}))$; the continuous functions on the Wallman space $I_R(\underline{L})$.

PROOF. Let $A = \{\hat{f} \mid f \in C_b(\underline{L})\}$. Then $A \subset C(\tau w(\underline{L})) = C(I_R(\underline{L}))$. Since $u_\alpha \xrightarrow{w^*} u \Rightarrow \hat{f}(u) \in C(I_R(\underline{L}))$. Now it is easy to show the following:

(1) $\widehat{f+g} = \hat{f} + \hat{g}$

(2) $\widehat{af} = a\hat{f}$ for $a \in R$

(3) $\widehat{fg} = \hat{f}\hat{g}$

(4) $\|\hat{f}\| = \|f\|$, therefore A is a closed subalgebra of $C(\tau w(\underline{L}))$

(5) A separates points. We can prove this by showing for $u, v \in I_R(\underline{L})$, $u \neq v$ there exists $\hat{f} \in A$ such that $\hat{f}(u) = 1$ and $\hat{f}(v) = 0$. This is done by using the normality of $\underline{L}$.

(6) $1 \in A$. Therefore given $u \in I_R(\underline{L})$ there exists $\hat{f} \in A$ such that $\hat{f}(u) \neq 0$.

So by the Stone-Weierstrass theorem $A = \bar{A} = C(\tau w(\underline{L}))$ which proves the theorem.

4. THE SPACE $Q(\underline{L})$.

In this section, we consider the important measures of $I_R(\underline{L})$ which integrate all $f \in C(\underline{L})$ and consider their relationship to $I_R^\sigma(\underline{L})$. Let $\underline{L}$ be δ-normal lattice. We define $Q(\underline{L}) = \{u \in I_R(\underline{L}) \mid \int_X |f|\, du < \infty \text{ for all } f \in C(\underline{L})\}$.

THEOREM 4.1. $I_R^\sigma(\underline{L}) \subset Q(\underline{L})$.

PROOF. Let $v \in I_R^\sigma(\underline{L})$ and $L_n = (|f| \geq n)$. One can see $L_n \downarrow \phi$ which implies $v(L_n) \to 0$ since $v \in I_R^\sigma(\underline{L})$. Therefore $v(L_N) = 0$ for N big. Now

$$\begin{aligned} \int_X |f|\, dv &= \int_{L_N} |f|\, dv + \int_{L_{N'}} |f|\, dv \\ &\leq NV\, L_{N'} \\ &\leq N \end{aligned}$$

Therefore $\int_X |f|\, dv \leq N$ which proves $v \in Q(\underline{L})$.

THEOREM 4.2. $I_R(\underline{L}) \cap I_\sigma(\underline{L}') \subset Q(\underline{L})$.

PROOF. Let $(|f| > n) = A_n'$. Clearly $A_n' \downarrow \phi$ and $A_n \in \underline{L}$ for all n. Now let $v \in I_R(\underline{L}) \cap I_\sigma(\underline{L}')$, therefore $v(A_n') = 0$, $n \geq N$. Now $\int_X |f|\, dv = \int_{A_N} |f|\, dv + \int_{A_{N'}} |f|\, dv$. Therefore $\int_X |f|\, dv \leq NV(A_N)$, so $\int_X |f|\, dv \leq NV(A_N) < \infty$. Therefore, $v \in Q(\underline{L})$ which provides the theorem.

THEOREM 4.3. $I_R^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}') \subset Q(\underline{L})$.

PROOF. By Theorems 4.1 and 4.2 and the trivial observation that $I_R^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}')$, the result is proved.

Following Varadarajan who considered the lattice of zero sets in a Tychonoff space, we introduce

DEFINITION. The Sequence $\{B_n\}$ in $\underline{L}$ is called regular if $B_n' \downarrow \phi$ and there exists A_n in $\underline{L}$ such that $B_n \subset A_n' \subset B_{n+1}$ for all n.

THEOREM 4.4. Let $\{B_n\}$ be a regular sequence. Then there exists $\{f_n\}$, $f_n \in C_b(\underline{L})$, $0 \leq f_n \leq 1$ such that $f_n \downarrow \phi$, $f_n(B_n) = 0, f_n(B'_{n+1}) = 1$ for $n = 1, 2, \ldots$

PROOF. Omitted.

THEOREM 4.5. Let X be an abstract set and $\underline{L}$ a δ-normal lattice of subsets which is also countably paracompact. Let $\{A_n\}$ in $\underline{L}$, $A_n \downarrow \phi$. Then there exists a regular sequence $\{C_n\}$ such that $C_n \subset A_n{}'$ for all n.

PROOF. Since $A_n \downarrow \phi$ and since $\underline{L}$ is countably paracompact then there exists $\{B_n\}$ in $\underline{L}$ with $A_n \subset B_n{}' \downarrow \phi$. Now we show by induction that for any n we have $\{C_K\},\{D_K\}$ in $\underline{L}$ with $A_K \subset C_K{}' \subset D_K \subset (B_K{}' \cap C'_{K-1})$ where $K = 1,\ldots n$: (1) For $n = 1$, take $C_0 = \phi$, and $A_1 \subset C_1{}' \subset D_1 \subset B_1{}'$ follows by normality. (2) Assume expression is true for n. Now $A_{n+1} \subset B'_{n+1}$ and $A_{n+1} \subset A_n \subset C_n{}'$, therefore $A_{n+1} \subset B'_{n+1} \cap C_n{}'$. Using normality, there exists $C_{n+1}, D_{n+1} \in \underline{L}$ such that $A_{n+1} \subset C'_{n+1} D_{n+1} \subset (B'_{n+1} \cap C_n{}')$ which finishes the induction argument. Since $C_n \subset A_n{}'$ we must show $\{C_n\}$ is regular. Now $C_n{}' \subset B_n{}'$ implies $C_n{}' \downarrow \phi$ and $C_n \subset D'_{n+1} \subset C_{n+1}$. Therefore $\{C_n\}$ is regular as $D_{n+1} \in \underline{L}$. Finally using the previous two results it is not difficult to show using an argument similar to Varadarajan that the following holds:

THEOREM 4.6. Let $\underline{L}$ be δ-normal and countably paracompact, then $Q(\underline{L}) \subset I_R{}^{\sigma}(\underline{L})$.

So using Theorems 4.1 and 4.6 we have:

THEOREM 4.7. Let $\underline{L}$ be δ-normal and countably paracompact, then $Q(\underline{L}) = I_R{}^{\sigma}(\underline{L})$.

We also have:

THEOREM 4.8. If $Q(\underline{L}) = I_R(\underline{L}) \cap I_{\sigma}(\underline{L}')$ and if $I_{\sigma}(\underline{L}') \subset I_{\sigma}(\underline{L})$ then $Q(\underline{L}) = I_R{}^{\sigma}(\underline{L})$.

PROOF. $Q(\underline{L}) = I_R(\underline{L}) \cap I_{\sigma}(\underline{L}') \subset I_R(\underline{L}) \cap I_{\sigma}(\underline{L})$, but we know if $v \in M_R(\underline{L})$ and $v \in M_{\sigma}(\underline{L})$ then $v \in M_R{}^{\sigma}(\underline{L})$. Therefore $Q(\underline{L}) \subset I_R(\underline{L}) \cap I_{\sigma}(\underline{L}) = I_R{}^{\sigma}(\underline{L})$, so $Q(\underline{L}) \subset I_R{}^{\sigma}(\underline{L})$. However, from Theorem 4.1 we have $I_R{}^{\sigma}(\underline{L}) \subset Q(\underline{L})$, therefore $Q(\underline{L}) = I_R{}^{\sigma}(\underline{L})$.

Note: $I_{\sigma}(\underline{L}') \subset I_{\sigma}(\underline{L})$ if $\underline{L}$ is countably paracompact, also if $\underline{L}$ is regular and Lindelöf.

Now we consider two lattices $\underline{L}_1$ and $\underline{L}_2$ such that $\underline{L}_1 \subset \underline{L}_2$. Then $C(\underline{L}_1) \subset C(\underline{L}_2)$.

THEOREM 4.9. Let $\underline{L}_1$, $\underline{L}_2$ be lattices of subsets such that $\underline{L}_1$ semi-separates $\underline{L}_2$. If $v \in Q(\underline{L}_2)$ and if $u = \nu \mid A(\underline{L}_1)$, then $u \in Q(\underline{L}_1)$.

PROOF. Since $\underline{L}_1$ semi-separates $\underline{L}_2, u \in I_R(\underline{L}_1)$. Also, since $C(\underline{L}_1) \subset C(\underline{L}_2)$ and since v integrates all $f \in C(\underline{L}_2)$, u integrates all $g \in C(\underline{L}_1)$. Hence $u \in Q(\underline{L}_1)$.

THEOREM 4.10. Let $\underline{L}_1$, $\underline{L}_2$ be lattice of subsets such that $\underline{L}_1$ separates $\underline{L}_2$. Let $v \in Q(\underline{L}_2)$ and $u = \nu \mid A(\underline{L}_1)$. If $Q(\underline{L}_1) = I_R{}^{\sigma}(\underline{L}_1)$ then $v \in I_{\sigma}(\underline{L}_2{}')$.

PROOF. By the previous theorem $u \in Q(\underline{L}_1) = I_R{}^{\sigma}(\underline{L}_1)$ by hypothesis, and since $\underline{L}_1$ separates $\underline{L}_2$ it is easy to see ν, the extension of u, is in $I_{\sigma}(\underline{L}_2{}')$.

THEOREM 4.11. Let $\underline{L}_1, \underline{L}_2$ be lattice of subsets such that $\underline{L}_1$ separates $\underline{L}_2$. If $Q(\underline{L}_1) = I_R{}^{\sigma}(\underline{L}_1)$ then $Q(\underline{L}_2) = I_R(\underline{L}_2) \cap I_{\sigma}(\underline{L}_2{}')$.

PROOF. $v \in Q(\underline{L}_2)$ implies $v \in I_R(\underline{L}_2)$, but $v \in I_{\sigma}(\underline{L}_2{}')$ from Theorem 4.10, therefore $Q(\underline{L}_2) \subset I_R(\underline{L}_2) \cap I_{\sigma}(\underline{L}_2{}')$. However we know if $v \in I_R(\underline{L}_2) \cap I_{\sigma}(\underline{L}_2{}')$ then $v \in Q(\underline{L}_2)$ from Theorem 4.2 which proves the result.

We have the following application: For $\underline{L}$ δ-normal, $z(\underline{L}) \subset \underline{L}$ where $z(\underline{L})$ consists of all sets of $\underline{L}$ of the form $L = \bigcap_{n=1}^{\infty} L_n{}'$, $L_n \in \underline{L}$ for all n, (see [1]). Now $z(\underline{L})$ separates $\underline{L}$ and $z(\underline{L})$ is normal and countably paracompact. Therefore by Theorem 4.7 we have $I_R{}^{\sigma}(z(\underline{L})) = Q(z(\underline{L}))$. Now using Theorem 4.11 we have $Q(\underline{L}) = I_R(\underline{L}) \cap I_{\sigma}(\underline{L}')$. Also if $I_{\sigma}(\underline{L}') \subset I_{\sigma}(\underline{L})$ then $Q(\underline{L}) = I_R{}^{\sigma}(\underline{L})$ by Theorem 4.8.

REMARK. We recall that if X is Tychonoff space and if $\underline{L} = z$, the lattice of zero sets then $(I_R{}^{\sigma}(z), \tau w_{\sigma}(z))$ is the realcompactification $\nu(X)$ of X.

Now we consider other criterion for $Q(\underline{L}) = I_R(\underline{L}) \cap I_{\sigma}(\underline{L}')$. If X is a topological space and if $A \subset X$ we denote by $\overline{A}^{\delta}$ the G_{δ}-closure of A. Now if X is an abstract set and $\underline{L}$ as usual is a separating disjunctive δ-normal lattice of subsets then we can view X embedded in $Q(\underline{L})$; we have $X \subset Q(\underline{L}) \subset I_R(\underline{L})$. In fact, using Theorem 4.3 we have $X \subset I_R{}^\sigma(\underline{L}) \subset I_R(\underline{L}) \cap I_\sigma(\underline{L}') \subset Q(\underline{L}) \subset I_R(\underline{L})$.

THEOREM 4.12. $\bar{X}^\delta \subset Q(\underline{L})$ where $\bar{X}^\delta$ is the G_δ-closure of X in the Wallman space $I_R(\underline{L})$.

PROOF. Suppose $u \in \bar{X}^\delta$. If $u \notin Q(\underline{L})$ then there exists $f \in C(\underline{L})$, $f \geq 0$ such that $\int_\infty f du = \infty$. Let $A_n{}' = (f > n) \in \underline{L}'$. Then $A_n{}' \downarrow \phi$ and $u(A_n{}') = 1$. Therefore $u \in \bigcap_{n=1}^\infty w(A_n{}') \subset I_R(\underline{L}) - X$ which contradicts the fact $u \in \bar{X}^\delta$. Therefore $u \in Q(\underline{L})$, so $\bar{X}^\delta \subset Q(\underline{L})$.

THEOREM 4.13. If $Q(\underline{L}) \subset \bar{X}^\delta$, then G_δ-closure of X in $I_R(\underline{L})$, then $u \in I_\sigma(\underline{L}')$ where $u \in Q(\underline{L})$.

PROOF. Suppose $u \in Q(\underline{L})$ which implies $u \in I_R(\underline{L})$. If $u \notin I_\sigma(\underline{L}')$ then there exists $L_n{}' \downarrow \phi$, $L_n \in \underline{L}$, $u(L_n{}') = 1$. Therefore $u \in \bigcap_{n=1}^\infty w(L_n{}') \subset I_R(\underline{L}) - X$. Therefore $u \notin \bar{X}^\delta$, so $Q(\underline{L}) \subset \bar{X}^\delta$ implies $u \in I_\sigma(\underline{L}')$.

THEOREM 4.14. $Q(\underline{L}) = \bar{X}^\delta$ if and only if $u \in I_\sigma(\underline{L}')$ for all $u \in Q(\underline{L})$.

PROOF. If $Q(\underline{L}) = \bar{X}^\delta$ and if $u \in Q(\underline{L})$ then $u \in I_\sigma(\underline{L}')$ by the previous theorem. While if $Q(\underline{L}) \subset I_\sigma(\underline{L}')$ then we must have $Q(\underline{L}) \subset \bar{X}^\delta$ for if not then there exists $G \in G_\delta$ such that $u \in G \subset I_R(\underline{L}) - X$ where $u \in I_R(\underline{L})$. Therefore $u \in \bigcap_{n=1}^\infty O_n \subset I_R(\underline{L})$ where O_n is an open set, which implies $u \in O_n$ for all n. Now $w(L_n{}')$ is an open set for $L_n \in \underline{L}$, therefore $u \in w(L_n{}') \subset O_n$ which yields $u \in \bigcap_{n=1}^\infty w(L_n{}') \subset \bigcap_{n=1}^\infty O_n$. Therefore there exists $u \in Q(\underline{L})$ such that $u \in \bigcap_{n=1}^\infty w(L_n{}')$ where the $w(L_n{}') \downarrow \phi$ and where $\bigcap_{n=1}^\infty L_n \in \underline{L}$ and $\bigcap_{n=1}^\infty w(L_n{}') \subset I_R(\underline{L}) - X$, but then $u(L_n{}') = 1$ for all n and $L_n{}' \downarrow \phi$ which is a contradiction. Thus $Q(\underline{L}) \subset \bar{X}^\delta$ and then by Theorem 4.12, $Q(\underline{L}) = \bar{X}^\delta$.

Using the previous theorem and Theorem 4.2 we have:

COROLLARY 4.15. If $\underline{L}$ is δ-normal separating and disjunctive then $Q(\underline{L}) = \bar{X}^\delta$ if and only if $Q(\underline{L}) = I_R(\underline{L}) \cap I_\sigma(\underline{L}')$.

REMARK. We note that $Q(\underline{L}) = I_R(\underline{L})$ if and only if $C_b(\underline{L}) = C(\underline{L})$; this situation arises in particular if $C(\underline{L})$ consists only of constant functions. (see below)

5. THE WALLMAN SPACE $I_R{}^\sigma(\underline{L})$.

First we note $I_R{}^\sigma(\underline{L})$ may be empty. Let $X = \{0,1,2,\ldots\}$ where $\underline{L}$ consists of ϕ and all sets of the form $\{n, n+1, \ldots\}$ for all n, and $v_1, v_2 \in I_R(\underline{L})$. If $v_1 \neq v_2$ then there exists $L_1, L_2 \in \underline{L}$ such that $v_1(L_1) = 1$, $V_2(L_1) = 0$, $v_1(L_2) = 0$, $v_2(L_2) = 1$ and $L_1 \cap L_2 = \phi$. However, this is impossible here as $L_1 \cap L_2 \neq \phi$ unless L_1 or $L_2 = \phi$. Therefore $I_R(\underline{L}) = \{u\}$. Now clearly if $L_n = \{n, n+1, \ldots\}, L_n \in \underline{L}$ and $L_n \downarrow \phi$. However, $u(L_n) = 1$ for all n, therefore $I_R{}^\sigma(\underline{L}) = \phi$. We also have in this example: $C(\underline{L}) = C_b(\underline{L}) =$ constant functions; $\underline{L}$ is not disjunctive, $\underline{L}$ is not countably paracompact; $\underline{L}$ is not regular; $\underline{L}$ is a δ-lattice.

Now we state a familiar result:

THEOREM 5.1. Let $\underline{L}$ be disjunctive then $< I_R{}^\sigma(\underline{L}), W_\sigma(\underline{L}) >$ is replete.

Next we give facts about $C(\underline{L})$: we denote by $M_R{}^I(\underline{L})$ the set $\{u \in M_R{}^\sigma(\underline{L}) \mid \int_X |f| d|u| < \infty$ for all $f \in C(\underline{L})\}$. Note $I_R{}^\sigma(\underline{L}) \subset M_R{}^I(\underline{L})$ and we denote by, similar to Varadarajan, W_I the topology on $M_R{}^I(\underline{L})$. A net $\{u_\alpha\}$ in $M_R{}^I(\underline{L})$ converges to u in $M_R{}^I(\underline{L})$ with respect to W_I if and only if $\int_X f du_\alpha \to \int_X f du$ for all $f \in C(\underline{L})$. The topology W_I restricted to $I_R{}^\sigma(\underline{L})$ is the Wallman topology. Now using this it is easy to show that $\hat{f}(u) = \int_X f du$, $u \in I_R{}^\sigma(\underline{L})$ is continuous with respect to the Wallman topology $\tau w_\sigma(\underline{L})$ on $I_R{}^\sigma(\underline{L})$, i.e., $\hat{f}(u) \in C(I_R{}^\sigma(\underline{L})) = C(\tau w_\sigma(\underline{L}))$. Let $\underline{L}$ be separating, disjunctive and δ-normal throughout and $f \in C(\underline{L})$.

THEOREM 5.2. Let $f \in C(\underline{L})$ then $\hat{f}^{-1}[a, \infty) = Z(\hat{g})$ where $g = (f - a) \wedge 0 \in C(\underline{L})$ and similarly $\hat{f}^{-1}(-\infty, a] = Z(\hat{h})$ where $h \in C(\underline{L})$.

PROOF. Omitted.

THEOREM 5.3. Let $z(\underline{L})$ be the zero lattice of $C(\underline{L})$ then $w_\sigma(z(\underline{L})) = z(w_\sigma(\underline{L}))$.

PROOF. Let $Z \in z(\underline{L}) \subset \underline{L}$. Therefore by a theorem of Alexandroff $Z = \bigcap_{n=1}^{\infty} L_n', L_n \in \underline{L}$ all n. Thus $w_\sigma(Z) = \bigcap_{n=1}^{\infty} W_\sigma(L_n)'$. But $w_\sigma(\underline{L})$ is δ-normal therefore by Alexandroff theorem again we get $w_\sigma(Z) \in z(w_\sigma(\underline{L}))$. Converse if $w_\sigma(L) \in z(w_\sigma(\underline{L}))$, where $L \in \underline{L}$ then $w_\sigma(L) = \bigcap_{n=1}^{\infty} w_\sigma(L_n') = w_\sigma(\bigcap_{n=1}^{\infty} L_n')$ and since $\underline{L}$ is disjunctive, $L = \bigcap_{n=1}^{\infty} L_n' \in z(\underline{L})$ again by Alexandroff's result and the proof is completed.

We have seen that if $f \in C(\underline{L})$ then $\hat{f} \in C(\tau w_\sigma(\underline{L}))$, i.e., it is continuous with respect to Wallman topology on $I_R^\sigma(\underline{L})$. However we can do better.

THEOREM 5.4. If $f \in C(\underline{L})$ then $\hat{f} \in C(w_\sigma(\underline{L}))$ (where $\hat{f}(u) = \int_X f du$ for all $u \in I_R^\sigma(\underline{L})$).

PROOF. We must show $\hat{f}^{-1}(E) \in w_\sigma(\underline{L})$ for any closed set $E \subset R$. It will suffice to show this for $E = [a,b] \subset R$. Now $[a,b] = (-\infty, b] \cap [a, \infty)$ so $\hat{f}^{-1}[a,b] = \hat{f}^{-1}[(-\infty,b] \cap [a,\infty)] = \hat{f}^{-1}(-\infty,b] \cap \hat{f}^{-1}[a,\infty) = Z(\hat{h}) \cap Z(\hat{g})$ using Theorem 5.2. Next we note if $g \in C(\underline{L})$ then $Z(\hat{g}) = \overline{Z(g)}$ where the closure is taken in the Wallman space $I_R^\sigma(\underline{L})$ with topology of closed sets $\tau w_\sigma(\underline{L})$. Therefore $\hat{f}^{-1}[a,b] = \overline{Z(g)} \cap \overline{Z(h)}$, and $Z(g), Z(h) \in z(\underline{L}) \subset \underline{L}$ so $\hat{f}^{-1}[a,b] = \overline{Z(g) \cap Z(h)}$ (using $\overline{A} \cap \overline{B} = \overline{A \cap B}$ for $A, B \in \underline{L}$)). In addition $\hat{f}^{-1}[a,b] = \overline{Z(g^2+h^2)}$ and $Z(g^2+h^2) = Z \in z(\underline{L})$ and $\overline{Z(g^2+h^2)} = \overline{Z} = w_\sigma(Z)$. Therefore $\hat{f}^{-1}[a,b] = \overline{Z} = w_\sigma(Z)$ which implies $\hat{f}^{-1}[a,b] \in w_\sigma(z(\underline{L}))$. However using Theorem 5.3 we get $\hat{f}^{-1}[a,b] \in z(w_\sigma(\underline{L}))$. However $z(w_\sigma(\underline{L})) \subset w_\sigma(\underline{L})$ therefore $\hat{f}^{-1}[a,b] \in w_\sigma(\underline{L})$ which implies $\hat{f} \in C(w_\sigma(\underline{L}))$.

Now we intend to prove the converse. Suppose that $h \in (w_\sigma(\underline{L}))$ then clearly $h|_X \in C(\underline{L})$ and let $h|_X = f \in C(\underline{L})$ then $h = \hat{f}$ since both are continuous with respect to the Wallman topology and they agree on X which is dense in $I_R^\sigma(\underline{L})$.

Using the above results we have the following:

THEOREM 5.5. The correspondence $f \to \hat{f}$ is a bijection between $C(\underline{L})$ and $C(w_\sigma(\underline{L}))$; the $w_\sigma(\underline{L})$-continuous functions on the Wallman space $I_R^\sigma(\underline{L})$.

Next let $u \in I_R(\underline{L})$, then we define $M^u = \{f \in C(\underline{L}) \mid u \in \overline{Z(f)}^{I_R(\underline{L})}\}$. The following facts we list for completeness (proofs can be found for this setting in [8]):

1) If $u_1, u_2 \in I_R(\underline{L})$ and if $u_1 \neq u_2$ then $M^{u_1} \neq M^{u_2}$.

2) M^u is a maximal ideal in $C(\underline{L})$.

3) (Generalized Gelfand-Kolmogoroff) If M is a maximal ideal in $C(\underline{L})$ then there exists $u \in I_R(\underline{L})$ such that $M = M^u$.

Thus there exists a one to one correspondence between elements of $I_R(\underline{L})$ and maximal ideals of $C(\underline{L})$.

Now we return to the Wallman space $< I_R^\sigma(\underline{L}), \tau w_\sigma(\underline{L}) >$ and give conditions when this topological space is realcompact. We know that for $\underline{L}$ disjunctive $w_\sigma(\underline{L})$ is replete; the question we are now concerned with is: when is the lattice $z(\tau w_\sigma(\underline{L}))$ replete? or i.e., when is the Wallman space realcompact?

THEOREM 5.6. Let $\underline{L}$ be δ-normal, separating, disjunctive, and countably paracompact then $Q(\underline{L}) = I_R^\sigma(\underline{L})$ and if $I_R^\sigma(\underline{L})$ with the Wallman topology is a c.b. space then it is realcompact.

PROOF. $Q(\underline{L}) = I_R^\sigma(\underline{L})$ by Theorem 4.7. Now $< I_R^\sigma(\underline{L}), w_\sigma(\underline{L}) >$ is replete from Theorem 5.1. Now $w_\sigma(\underline{L}) \subset \tau w_\sigma(\underline{L})$ (of course) and consequently $z(\tau w_\sigma(\underline{L})) \subset \tau w_\sigma(\underline{L})$. Now $\underline{L}$ δ-normal implies $w_\sigma(\underline{L})$ δ-normal and $\underline{L}$ countably paracompact implies $w_\sigma(\underline{L})$ is countably paracompact. Then by Theorem 5.3 of [2] we have $\tau w_\sigma(\underline{L})$ is replete. Now by hypothesis $\tau w_\sigma(\underline{L})$ is $z(\tau w_\sigma(\underline{L}))$ countably bounded (c.b.). Thus $z(\tau w_\sigma(\underline{L}))$ is replete by Theorem 3.4 of [2]. Hence $< I_R^\sigma(\underline{L}), \tau w_\sigma(\underline{L}) >$ is realcompact.

Note. If $\tau w_\sigma(\underline{L})$ is $z(\tau w_\sigma(\underline{L}))$ countably paracompact the same conclusion can be drawn.

We continue to assume that $\underline{L}$ is separating, disjunctive and δ-normal. Let $h \in C(I_R{}^{\sigma}(\underline{L}))$ or, i.e., $h \in C(\tau w_{\sigma}(\underline{L}))$ in lattice notation, then $f = h \mid_X \in C(\tau \underline{L})$, clearly. If $f \in C(\underline{L})$ then by our earlier work in this section we would have $h = \hat{f} \in C(w_{\sigma}(\underline{L}))$. This situation arises if X is a Tychonoff topological space and $\underline{L} = z$ lattice of zero sets of continuous functions on X for in this case if $h \in C(I_R{}^{\sigma}(z))$ then $h \mid_X \in C(\tau z) = C(F) = C(z)$ where F is the lattice of closed sets of X. Thus, in this case, $w_{\sigma}(z) = z(\tau w_{\sigma}(z))$ and since $w_{\sigma}(z)$ is replete, we have that $I_R{}^{\sigma}(z)$ is realcompact with respect to the Wallman space.

THEOREM 5.7. Let $\underline{L}$ be separating, disjunctive and δ-normal. If $C(\tau w_{\sigma}(\underline{L})) = C(w_{\sigma}(\underline{L}))$ then $z(w_{\sigma}(\underline{L})) = z(\tau w_{\sigma}(\underline{L}))$ and if $w_{\sigma}(\underline{L})$ is $z(w_{\sigma}(\underline{L}))$ c.b. or countably paracompact then $I_R{}^{\sigma}(\underline{L})$ with the Wallman topology is realcompact.

PROOF. Since $w_{\sigma}(\underline{L}) \subset \tau w_{\sigma}(\underline{L})$ then $z(w_{\sigma}(\underline{L})) \subset z(\tau w_{\sigma}(\underline{L}))$. Now let $Z(f) \in z(\tau w_{\sigma}(\underline{L}))$ where $f \in C(\tau w_{\sigma}(\underline{L}))$, but $C(\tau w_{\sigma}(\underline{L})) = C(w_{\sigma}(\underline{L}))$. This implies $Z(f) \in z(w_{\sigma}(\underline{L}))$. Therefore $z(\tau w_{\sigma}(\underline{L})) \subset z(w_{\sigma}(\underline{L}))$. Now if $w_{\sigma}(\underline{L})$ is $z(w_{\sigma}(\underline{L}))$ countably bounded or countably paracompact then since $w_{\sigma}(\underline{L})$ is replete we have using the same argument as in the proof of Theorem 5.6 that $z(w_{\sigma}(\underline{L}))$ is replete, therefore $z(\tau w_{\sigma}(\underline{L}))$ is replete.

Finally we extend Theorem 5.7 but first note $z(w_{\sigma}(\underline{L})) \subset w_{\sigma}(\underline{L}) \subset \tau w_{\sigma}(\underline{L})$ and $z(w_{\sigma}(\underline{L})) \subset z(\tau w_{\sigma}(\underline{L})) \subset \tau w_{\sigma}(\underline{L})$.

THEOREM 5.8. Let $\underline{L}$ be separating, disjunctive and δ-normal. If $\underline{L}$ is $z(\underline{L})$ countably bounded (c.b.) or $\underline{L}$ is $z(\underline{L})$-countably paracompact and assume $z(\tau w_{\sigma}(\underline{L})) \subset \tau z(w_{\sigma}(\underline{L}))$, then $z(\tau w_{\sigma}(\underline{L}))$ is replete, i.e., $I_R{}^{\sigma}(\underline{L})$ with the Wallman topology is realcompact.

PROOF. $z(w_{\sigma}(\underline{L}))$ is complement generated since $z(\underline{L})$ is complement generated. (Use Theorem 5.3) and $z(w_{\sigma}(\underline{L})) \subset z(\tau w_{\sigma}(\underline{L})) \subset \tau z(w_{\sigma}(\underline{L}))$, therefore by Theorem 3.1 part (1) of [2] we have $z(\tau w_{\sigma}(\underline{L}))$ is replete, as $z(w_{\sigma}(\underline{L}))$ is replete from the fact $\underline{L}$ is $z(\underline{L})$ countably bounded or $\underline{L}$ is $z(\underline{L})$ countably paracompact.

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