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ON CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. A function f, analytic in the unit disk E and given by $f(z) = z + \sum_{k=2}^{\infty} a_n z^k$ is said to be in the family K_n if and only if $D^n f$ is close-to-convex, where $D^n f = \frac{z}{(1-z)^{n+1}} * f, n \in N_0 = \{0,1,2,...\}$ and $*$ denotes the Hadamard product or convolution. The classes K_n are investigated and some properties are given. It is shown that $K_{n+1} \subseteq K_n$ and K_n consists entirely of univalent functions. Some closure properties of integral operators defined on K_n are given.

KEY WORDS AND PHRASES. Univalent, close-to-convex, starlike, convolution, integral operators.
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1. INTRODUCTION.

Let A denote the class of functions $f: f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ analytic in the unit disk $E = \{z: |z| < 1\}$. The Hadamard product or convolution of two functions $f,\ g \in A$ is denoted by $f*g$. For $n \in N_0 = \{0,1,2,3,...\}$, let $D^n f = (\frac{z}{(1-z)^{n+1}} * f$, so that

$$D^n f = z(z^{n-1} f)^{(n)}/n!.$$

Let $S \subset A$ be the class of univalent functions and for $0 \leq \beta < 1$, let $C(\beta)$ and $S^*(\beta)$ denote the subclasses of S consisting of convex functions of order β and starlike functions of order β respectively. The classes C and S^* of convex and starlike functions, respectively, are identified by $C(0) \equiv C$ and $S^*(0) = S^*$.

A function $f \in S$ belongs to the class $K(\alpha, \beta)$ of close-to-convex of order α and type β if and only if for some $g \in S^*(\beta)$ and $0 \leq \alpha < 1$,

$$Re \frac{zf'(z)}{g(z)} > \alpha, \ z \in E.$$

It is clear that $K(0,0) \equiv K$, the class of close-to-convex univalent functions [1].

DEFINITION 1.1. For $n \in N_0$, a function $f \in A$ is said to belong to the classes R_n, if and only if for $z \in E$,

$$Re \frac{z(D^n f(z))'}{D^n f(z)} > 0. \tag{1.1}$$

Thus $R_0 \equiv S^*$ and $R_1 \equiv C$. In [2], Ahuja discussed these classes and showed that $R_{n+1} \subset R_n$ for each $n \in N_0$. This implies that functions in R_n are starlike and hence univalent.

We now extend the classes R_n, as follows:

DEFINITION 1.2. Let $f \in A$. Then $f \in K_n$ if and only if there exists $g \in R_n$ such that for $z \in E$,

$$Re\,\frac{z(D^n f(z))'}{D^n g(z)} > 0. \tag{1.2}$$

We note that $K_0 \equiv K$ and $K_1 \equiv C^*$, the class of quasi-convex functions introduced in [3].

In order to develop some results for K_n, we shall need the following:

LEMMA 1.1 [4]. Let w be analytic in E. If $|\omega|$ assumes its maximum value on the circle $|z| = r$ at a point z_0, then

$$z_0\, \omega'(z_0) = k\, \omega(z_0),$$

where $k \geq 1$.

LEMMA 1.2 [5]. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\psi(u,v)$ be a complex-valued function satisfying the conditions:

(i) $\psi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$,

(ii) $(1,0) \in D$ and $\psi(1,0) > 0$,

(iii) $Re(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{k=2}^{\infty} c_k z^k$ is a function analytic in E, such that $(h(z),\ zh'(z)) \in D$ and $Re\ \psi(h(z),\ zh'(z)) > 0$ for $z \in E$, then $Re\ h(z) > 0$ in E.

LEMMA 1.3. [6]. Let ϕ be convex and g be starlike in E. Then, for F analytic in E with $F(0) = 1$, $\frac{\phi^* Fg}{\phi^* g}$ is contained in the convex hull of $F(E)$.

2. PROPERTIES OF THE FAMILY K_n.

We first prove that all functions in K_n are close-to-convex and hence univalent.

THEOREM 2.1. $K_{n+1} \subset K_n$, for each $n \in N_0$.

PROOF. Let $f \in K_{n+1}$. Then for $z \in E$

$$Re\,\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} > 0, \text{ for some } g \in R_{n+1}.$$

Define $\omega(z)$ in E such that

$$\frac{z(D^n f(z))'}{D^n g(z))} = \frac{1-\omega(z)}{1+\omega(z)}, \tag{2.1}$$

where $\omega(0) = 0$ and $\omega(z) \neq . -1$. We show that $|\omega(z)| < 1$.

From (2.1) we have

$$z(D^n f(z))' = D^n g(z).\ \frac{1-\omega(z)}{1+\omega(z)}. \tag{2.2}$$

So, from (2.2) and the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z), \tag{2.3}$$

we obtain

$$z(D^{n+1} f(z))' = \frac{1}{n+1}\left[z(D^n g(z))'\,\frac{1-\omega(z)}{1+\omega(z)} + D^n g(z)\left\{\frac{-2z\omega'(z)}{(1+\omega(z))^2} + n\frac{1-\omega(z)}{1+\omega(z)}\right\}\right]. \tag{2.4}$$

Now apply (2.3) for the function g, and use (2.4) to obtain

$$\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} = \frac{1-\omega(z)}{1+\omega(z)} + \frac{1}{n+1}\,\frac{D^n g(z)}{D^{n+1} g(z)}\cdot\left[\frac{-2z\omega'(z)}{(1+\omega(z))^2}\right]. \tag{2.5}$$

Since $R_{n+1} \subset R_n$, this implies that $g \in R_n$ and hence there exists an analytic function $\omega_1(z)$ with $\omega_1(0) = 0$ and $|\omega_1(z)| < 1$ such that

$$\frac{D^{n+1}g(z)}{D^n g(z)} = \frac{1-\omega(z)}{1+\omega(z)}. \tag{2.6}$$

Thus using (2.6) in (2.5) we have

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} = \frac{1-\omega(z)}{1+\omega(z)} + \frac{1}{n+1}\left(\frac{1+\omega_1(z)}{1-\omega_1(z)}\right)\left(\frac{-2z\omega'(z)}{(1+\omega(z))^2}\right). \tag{2.7}$$

Suppose now that for $z \in E$

$$\max_{|z| \le |z_0|} |\omega(z)| = |\omega(z_0)| = 1, (\omega(z_0) \ne -1).$$

Then it follows, from Lemma 1.1, that

$$z_0\omega'(z_0) = k\omega(z_0),$$

where $k \ge 1$.
Setting $\omega(z_0) = e^{i\Theta}$ and $\omega_1(z_0) = re^{i\phi}$ in (2.7) gives

$$Re\left\{\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)}\right\} = Re\left[\left(\frac{1}{n+1}\right)\frac{-2k(e^{i\Theta}+e^{-i\Theta}+2)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|(1+e^{i\Theta})^2|^2}\right],$$

$$= \frac{-4k}{n+1}\left[\frac{(\cos\Theta+1)(1+r^2+2r\cos\phi)}{|1+re^{i\phi}|^2|(1+e^{i\Theta})^2|^2}\right].$$

Hence, if $\phi = \frac{\pi}{2}$,

$$Re\,\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} < 0,$$

where $g \in R_{n+1}$ and $k \ge 1$. This contradicts our hypothesis that $f \in K_{n+1}$. Thus $|\omega(z)| < 1$ and so $f \in K_n$.

From Theorem 2.1, we note that $f \in K_n$ implies that $f \in K$ and so f is univalent in E. Also, since $K_n \subset K_1 \equiv C^*$ it follows that f is quasi-convex.

REMARK 2.1. Let $f \in K_n$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z),$$

$$= \left[z + \sum_{k=2}^{\infty}\frac{(k+n-1)!}{n!(k-1)!}z^k\right] * \left[z + \sum_{k=2}^{\infty} a_k z^k\right], \tag{2.8}$$

$$= z + \sum_{k=2}^{\infty}\frac{(k+n-1)!}{n!(k-1)!}a_k z^k.$$

Thus from (2.8) and Definition 1.2 it follows that

$$f \in K_n \text{ if and only if } D^n f \in K. \tag{2.9}$$

THEOREM 2.2. Let $f \in K_n$ and be given by $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then for $k \ge 2$ and $n \ge 0$,

$$|a_k| \le \frac{(k!)(n!)}{(k+n-1)!}.$$

This result is sharp with equality for the function f_0, where

$$D^n f_0(z) = \frac{z}{(1-z)^2}. \tag{2.10}$$

The proof follows immediately from (2.8), (2.9), and the well-known coefficient result for the class K of close-to-convex functions.

THEOREM 2.3. (Covering theorem). Let $f \in K_n$. If B is the boundary of the image of E under f, then every point of B is distance at least $\frac{n+1}{2(n+2)}$ from the origin.

PROOF. Let $f(z) \neq c, c \neq 0$. Then f_1 given by

$$f_1(z) = \frac{cf(z)}{c - f(z)}$$

is univalent in E. Write $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\frac{cf(z)}{c-f(z)} = z + (a_2 + \tfrac{1}{c})z^2 + \cdots\cdots\cdots$$

and, since $f_1 \in S$, it follows that

$$|a_2 + \tfrac{1}{c}| \leq 2.$$

Hence,

$$|\tfrac{1}{c}| \leq 2 + |a_2|,$$

and using Theorem 2.2, we obtain

$$|c| \geq \frac{n+1}{2(n+2)}.$$

This completes the proof.
We note that when $n = 0$, $|c| \geq \frac{1}{4}$ and when $n = 1$, $|c| \geq \frac{1}{2}$ (see [3] and [7]).

THEOREM 2.4. $\bigcap_{n=0}^{\infty} K_n = \{id\}$,
where id is the identity function z.

PROOF. Let $f(z) = g(z) = z$ in (1.2), then it follows trivially that $z \in K_n$ for $n \geq 0$.

On the contrary, assume that $f \in \bigcap_{n=0}^{\infty} K_n$ with $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.
Then it follows from Theorem 2.2. that $f(z) = z$.

3. INTEGRAL OPERATORS.

Let the operator $I_\lambda : A \to A$ be defined by $f = I_\lambda(F)$, as

$$f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \xi^{\frac{1}{\lambda}-2} F(\xi) d\xi, \tag{3.1}$$

where $0 < \lambda \leq 1$.
For $\lambda = \frac{1}{2}$, Libera [8] established that the operator

$$I(F) = \frac{2}{z} \int_0^z F(\xi) d\xi$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [9] greatly generalized Libera's results. Many authors have studied the operators of the form (3.1), see e.g. [7]. Ahuja [2] has discussed the $\lambda = \frac{1}{\gamma+1}$, γ complex and $Re\ \gamma \neq -1$ for the classes R_n. Here we shall consider (3.1) for K_n.

We shall need the following [2]:

Let $I_\lambda: A \to A$ be defined by (3.1) with $0 < \lambda \leq 1$. If $F \in R_n$, then $I_\lambda(F) \in R_n(\alpha)$, i.e., for $z \in E$

$$Re \frac{z(D^n f(z))'}{D^n f(z)} > \alpha,$$

where $0 < \alpha < 1$ and

$$\alpha = \frac{1}{4\lambda}\left[-(2-\lambda) + \sqrt{9\lambda^2 + 4\lambda + 1}\right] \tag{3.2}$$

We now prove:

THEOREM 3.1. Let $F \in K_n$ and let $f = I_\lambda(F)$ be defined as in (3.1) for $0 < \lambda \leq 1$. Then $f \in K_n(\beta, \alpha)$, where α is given by (3.2) and $\beta(0 \leq \beta < 1)$ is defined by (3.9).

PROOF. Let $G \in R_n$ and $I_\lambda(G) = g$, where I_λ is defined by (3.1). So that

$$D^n G(z) = (1-\lambda)D^n g(z) + \lambda z(D^n g(z))'$$

and

$$D^n F(z) = (1-\lambda)D^n f(z) + \lambda z(D^n f(z))', \tag{3.3}$$

Set

$$\frac{z(D^n f(z))'}{D^n g(z)} = (1-\beta)p(z) + \beta, \tag{3.4}$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

We need to show that $Re\ p(z) > 0$ for $z \in E$.

From (3.3), we have

$$\frac{z(D^n F(z))'}{D^n G(z)} = \frac{(1-\lambda)\frac{z(D^n f(z))'}{D^n g(z)} + \frac{\lambda z[z(D^n f(z))']'}{D^n g(z)}}{(1-\lambda) + \lambda z\frac{(D^n g(z))'}{D^n g(z)}} \tag{3.5}$$

Since $g \in R_n(\alpha)$, where α is given by (3.2), we can write

$$\frac{z(D^n g(z))'}{D^n g(z)} = (1-\alpha)p_0(z) + \alpha, \tag{3.6}$$

where $Re\ p_0(z) > 0, z \in E$.

Also, from (3.4) and (3.6), we obtain

$$\frac{z[z(D^n f(z))']'}{D^n g(z)} = \left\{\frac{z(D^n g(z))'}{D^n g(z)}((1-\beta)p(z)) + \beta\frac{z(D^n g(z))'}{D^n g(z)} + (1-\beta)zp'(z)\right\},$$

$$= [(1-\alpha)p_0(z) + \alpha][(1-\beta)p(z) + \beta] + (1-\beta)zp'(z). \tag{3.7}$$

Using (3.4), (3.6), and (3.7) in (3.5), it follows that

$$\frac{z(D^n F(z))'}{D^n G(z)} = \beta + (1-\beta)p(z) + \frac{\lambda(1-\beta)zp'(z)}{(1-\lambda) + \lambda[(1-\alpha)p_0(z) + \alpha]}. \tag{3.8}$$

Next define $\psi(u,v)$ by taking $u = p(z)$ and $v = zp'(z)$ in (3.8) by

$$\psi(u,v) = \beta + (1-\beta)u + \frac{\lambda(1-\beta)v}{\lambda(1-\alpha)p_0 - \lambda(1-\alpha) + 1}.$$

It is clear that $\psi(u,v)$ satisfies conditions (i) and (ii) of Lemma 1.2. To verify condition (iii), we note that

$$Re\ \psi(iu_2,v_1)=\beta+\frac{\lambda(1-\beta)v_1\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}}{\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}^2+\lambda^2(1-\alpha)^2p_2^2},$$

where $p_0(z)=p_1+ip_2$, p_1, p_2 being functions of x and y and $Re\ p_0=p_1>0$.
By putting $v_1\le -\frac{1}{2}(1+u_2^2)$, we obtain

$$Re\ \psi(iu_2,v_1)\le\beta-\frac{\lambda(1-\beta)(1+u_2^2)[\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1]}{2[\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}^2+\lambda^2(1-\alpha)^2p_2^2]}=\frac{A+Bu_2^2}{2C},$$

where

$$C=\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}^2+\lambda^2(1-\alpha)^2p_2^2>0,$$

$$A=2\beta\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}^2+\lambda^2(1-\alpha)^2p_2^2-\lambda(1-\beta)[\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1],$$

and

$$B=-\lambda(1-\beta)[\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1].$$

We note that $Re\ \psi(iu_2,v_1)\le 0$ if and only if $A\le 0$ and $B\le 0$. From $A\le 0$, we obtain $\beta\le\beta_A$ where

$$\beta_A=\frac{\lambda^2(1-\alpha)^2p_2^2+\lambda[\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1]}{2[\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}^2+\lambda^2(1-\alpha)^2p_2^2]+\{\lambda(1-\alpha)p_1-\lambda(1-\alpha)+1\}}\ge 0. \tag{3.9}$$

Also, from $B\le 0$, we have $\beta_A<1$ and the condition (iii) is satisfied to give $Re\ p(z)>0$ for $z\in E$ which implies that $f\in K_n(\beta,\alpha)$.

If we put $n=\left(\frac{1}{\lambda}-1\right)$ in (3.1) we have the following:

THEOREM 3.2. Let $F\in K_n$ and let

$$f(z)=(n+1)z^{-n}\int_0^z\xi^{n-1}F(\xi)d\xi. \tag{3.10}$$

Then $f\in K_{n+1}$.

PROOF. Let $g(z)=(n+1)z^{-n}\int_0^z\xi^{n-1}G(\xi)d\xi,$ (3.11)
where $G\in R_n$. Then from [2] $g\in R_{n+1}$. From (3.10) and (3.11) we have

$$D^nF(z)=\frac{n}{n+1}D^nf(z)+\frac{1}{n+1}z(D^nf(z))' \tag{3.12}$$

and

$$D^nG(z)=\frac{n}{n+1}D^ng(z)+\frac{1}{n+1}z(D^ng(z))'. \tag{3.13}$$

From (3.12) and the identity

$$z(D^nf(z))'=(n+1)D^{n+1}f(z)-nD^nf(z),$$

we have (3.14)

$$D^nF(z)=D^{n+1}f(z).$$

Since $F\in K_n$, it follows that, for $z\in E$,

$$Re\left(\frac{z(D^nF(z))'}{D^nG(z)}\right)>0,\quad G\in R_n$$

and thus, using (3.14), we conclude that, for $z \in E$

$$Re\left[\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}\right] > 0 \text{ for } g \in R_{n+1}.$$

THEOREM 3.3. Let $f \in K_n, n \geq 0$ and $\phi \in C$. Then $\phi^* f \in K_n$.

PROOF. First we prove that, if $g \in R_n$, then $(\phi^* g) \in R_n$. It is sufficient to show that $D^n(\phi^* g) \in S^*$.

Now

$$D^n(\phi^* g)(z) = \frac{z}{(1-z)^{n+1}} * (g^*\phi)(z) = \phi(z) * \frac{z}{(1-z)^{n+1}} * g(z) = (\phi^* D^n g)(z).$$

Since $g \in R_n$ and $\phi \in C$, it follows that $\phi * g \in R_n$, see [2].

Next, we prove that $(\phi^* f) \in K_n$.

$$\frac{z[D^n(\phi^* f)(z)]'}{D^n(\phi^* g)(z)} = \frac{z[\phi(z)^* D^n f(z)]'}{\phi(z)^* D^n g(z)} = \frac{\phi(z)^* z\frac{(D^n f(z))'}{D^n g(z)} \cdot D^n g(z)}{\phi(z)^* D^n g(z)}$$

Applying Lemma 1.3 with $F(z) = \frac{z(D^n f(z))'}{D^n g(z)}$, $D^n g(z) \in S^*$ and $Re\ R(z) > 0$, we obtain

$$Re\frac{z[D^n(\phi^* f)(z)]'}{D^n(\phi^* f)(z)} > 0 \text{ for } z \in E.$$

This proves Theorem 3.3.

REMARK 3.1. Theorem 3.3 is an analogue of the Polya-Schoenberg conjecture [6] for the family K_n. Many results on K_n can be deduced as applications.

We give the following:

THEOREM 3.4. Let $f \in K_n$, $n \geq 0$ and be defined by (3.1). Then $F \in K_n$, $n \geq 0$ for $|z| < r_0$, where r_0 is given by

$$r_0 = \frac{1}{(2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1})}. \tag{3.15}$$

The function f_0, defined by (2.10), shows that this result is sharp.

PROOF. Let $\phi_\lambda(z) = \sum_{k=1}^{\infty} [\lambda(k-1)+1]z^k, 0 < \lambda < 1$.

Then $\lambda_\phi \in C$ for $|z| < r_0$ where r_0 is given by (3.15). Also $F(z) = (\phi_\lambda{}^* f(z))$ and so using Theorem 3.3, we see that $F \in K_n, n \geq 0$ for $|z| < r_0$.

REMARK 3.2. We note that Theorem 3.3 shows that the family K_n is invariant under the following integral operators

$$I_1(f) = \int_0^z \frac{f(\xi)}{\xi} d\xi = (f^*\phi_1)(z),$$

$$I_2(f) = \frac{2}{z}\int_0^z \frac{f(\xi)}{\xi} d\xi = (f^*\phi_2)(z), \qquad \text{(Libera's operator)}$$

$$I_3(f) = \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi, \ |x| < 1, x \neq 1,$$

$$= (f^*\phi_3)(z)$$

and

$$I_4(f) = \frac{1+c}{c}\int_0^z \xi^{c-1} f(\xi)d\xi, \qquad Re(c) > 0$$

$$= (f^*\phi_4)(z),$$

where $\phi_i \in C, i = 1,2,3,4$
and

$$\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -log(1-z),$$

$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + log(1-z)]}{z},$$

$$\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x} log\, \frac{1-xz}{1-z}, \; |x| = 1, x \neq 1,$$

and

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \qquad Re(c) > 0.$$

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