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ON CERTAIN MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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ABSTRACT. In this paper, we introduce a new class $T_p(\alpha)$ of meromorphic functions with positive coefficients in $D = \{z: 0 < |z| < 1\}$. The aim of the present paper is to prove some properties for the class $T_p(\alpha)$.

KEY WORDS AND PHRASES. Meromorphic function, meromorphically starlike and convex.
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1. INTRODUCTION.

Let A_p denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \qquad (p = 1, 3, 5, \cdots) \tag{1.1}$$

which are analytic in $D = \{z: 0 < |z| < 1\}$ with a simple pole at the origin with residue one there.

A function $f(z) \in A_p$ is said to be **meromorphically starlike** of order α if it satisfies

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \le \alpha < 1)$ and for all $z \in D$.

Further, a function $f(z) \in A_p$ is said to be meromorphically convex of order α if it satisfies

$$Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \tag{1.3}$$

for some α $(0 \le \alpha < 1)$ and for all $z \in D$.

Some subclasses of A_1 when $p = 1$ were recently introduced and studied by Pommerenke [1], Miller [2], Mogra, et al [3], and Cho, et al [4].

Let T_p be the subclass of A_p consisting of functions

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \qquad (a_n \ge 0). \tag{1.4}$$

A function $f(z) \in T_p$ is said to be a member of the class $T_p(\alpha)$ if it satisfies

$$\left|\frac{z^{p+1}f^{(p)}(z) + p!}{z^{p+1}f^{(p)}(z) - p!}\right| < \alpha. \tag{1.5}$$

for some $\alpha(0 \le \alpha < 1)$ and for all $z \in D$.

In this paper we present a systematic study of the various properties of the class $T_p(\alpha)$ including distortion theorems and starlikeness and convexity properties.

2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following coefficient inequality.

THEOREM 2.1. A function $f(z) \in T_p$ is in the class $T_p(\alpha)$ if and only if

$$\sum_{n=p}^{\infty} \binom{n}{p} a_n \leq \frac{2\alpha}{1+\alpha}, \tag{2.1}$$

where

$$\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!}.$$

PROOF. Assuming that (2.1) holds for all admissible α, we have

$$|z^{p+1}f^{(p)}(z)+p!| - \alpha|z^{p+1}f^{(p)}(z)-p!| \tag{2.2}$$

$$= \left|\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}\right| - \alpha\left|2\cdot p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}\right|$$

$$\leq \sum_{n=p}^{\infty} \frac{n!}{(n-p)!}(1+\alpha)a_n|z|^{n+1} - 2\alpha\cdot p!.$$

Therefore, letting $z \to 1^-$, we obtain

$$\sum_{n=p}^{\infty} \frac{n!}{(n-p)!}(1+\alpha)\, a_n - 2\alpha\cdot p! \leq 0 \tag{2.3}$$

which shows that $f(z) \in T_p(\alpha)$.

Conversely, if $f(z) \in T_p(\alpha)$, then

$$\left|\frac{z^{p+1}f^{(p)}(z)+p!}{z^{p+1}f^{(p)}(z)-p!}\right| = \left|\frac{\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}}{2\cdot p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}}\right| < \alpha \qquad (z \in D). \tag{2.4}$$

Since $Re(z) \leq |z|$ for all z, (2.4) gives

$$Re\left\{\frac{\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}}{2\cdot p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n z^{n+1}}\right\} < \alpha \qquad (z \in D). \tag{2.5}$$

Choose values of z on the real axis so that $z^{p+1}f^{(p)}(z)$ is real. Upon clearing the denominator in (2.5) and letting $z \to 1^-$, we have

$$\sum_{n=p}^{\infty} \frac{n!}{(n-p)!}(1+\alpha)a_n \leq 2a\cdot p! \tag{2.6}$$

which is equivalent to (2.1). Thus we complete the proof of Theorem 2.1.

Taking $p = 1$ in Theorem 1, we have

COROLLARY 2.1. $f(z) \in T_1(\alpha)$ if and only if

$$\sum_{n=1}^{\infty} na_n \leq \frac{2\alpha}{1+\alpha}. \tag{2.7}$$

THEOREM 2.2. If $f(z) \in T_p(\alpha)$, then

$$|f^{(j)}(z)| \geq \frac{j!}{|z|^{j+1}} - \frac{p!2\alpha}{(p-j)!(1+\alpha)}|z|^{p-j} \tag{2.8}$$

and

$$|f^{(j)}(z)| \leq \frac{j!}{|z|^{j+1}} + \frac{p!2\alpha}{(p-j)!(1+\alpha)}|z|^{p-j} \tag{2.9}$$

for $z \in D$, where $0 \leq j \leq p$ and $0 < \alpha \leq \frac{j!(p-j)}{p!2 - j!(p-j)!}$.
Equalities in (2.8) and (2.9) are attained for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{1+\alpha} z^p. \tag{2.10}$$

PROOF. It follows from Theorem 2.1 that

$$\frac{(p-j)!(1+\alpha)}{p!} \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n \leq \sum_{n=p}^{\infty} \binom{n}{p}(1+\alpha)a_n \leq 2\alpha. \tag{2.11}$$

Therefore, we have

$$|f^{(j)}(z)| \geq \frac{j!}{|z|^{j+1}} - \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n |z|^{n-j} \geq \frac{j!}{|z|^{j+1}} - \frac{p!2\alpha}{(p-j)!(1+\alpha)} |z|^{p-j} \tag{2.12}$$

and

$$|f^{(j)}(z)| \leq \frac{j!}{|z|^{j+1}} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n |z|^{n-j} \leq \frac{j!}{|z|^{j+1}} + \frac{p!2\alpha}{(p-j)!(1+\alpha)} |z|^{p-j}. \tag{2.13}$$

Taking $j = 0$ in Theorem 2.2, we have

COROLLARY 2.2 If $f(z) \in T_p(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha}{1+\alpha} |z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\alpha}{1+\alpha} |z|^p \tag{2.14}$$

for $z \in D$. Equalities in (2.14) are attained for the function $f(z)$ given by (2.10).

Making $j = 1$ in Theorem 2, we have

COROLLARY 2.3. If $f(z) \in T_p(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha p}{1+\alpha} |z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{2\alpha p}{1+\alpha} |z|^{p-1} \tag{2.15}$$

for $z \in D$, where $0 < \alpha \leq \frac{1}{2p-1}$. Equalities in (2.15) are attained for the function (z) given by (2.10).

Letting $p = 1$ in Theorem 2.2, we have

COROLLARY 2.4. If $f(z) \in T_1(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha}{1+\alpha} |z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\alpha}{1+\alpha} |z| \tag{2.16}$$

and

$$\frac{1}{|z|^2} - \frac{2\alpha}{1+\alpha} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{2\alpha}{1+\alpha} \tag{2.17}$$

for $z \in D$. Equalities in (2.16) and (2.17) are attained for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{1+\alpha} z. \tag{2.18}$$

3. STARLIKE AND CONVEXITY.

THEOREM 3.1. If $f(z) \in T_p(\alpha)$, then $f(z)$ is meromorphically starlike of order δ $(0 \leq \delta < 1)$ in $|z| < r_1$, where

$$r_1 = \inf_{n \geq p} \left\{ \frac{\binom{n}{p}(1+\alpha)(1-\delta)}{2\alpha(n+2-\delta)} \right\}^{\frac{1}{n+1}}. \tag{3.1}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{\binom{n}{p}(1+\alpha)} z^n \quad (n \geq p). \tag{3.2}$$

PROOF. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \delta \tag{3.3}$$

for $|z| < r_1$. We note that

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{\sum_{n=p}^{\infty}(n+1)a_n z^n}{\frac{1}{z}+\sum_{n=p}^{\infty}a_n z^n}\right|\le\frac{\sum_{n=p}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=p}^{\infty}a_n|z|^{n+1}}. \tag{3.4}$$

Therefore, if

$$\sum_{n=p}^{\infty}\frac{n+2-\delta}{1-\delta}a_n|z|^{n+1}\le 1, \tag{3.5}$$

then (3.3) holds true. Further, using Theorem 2.1, it follows from (3.5) that (3.3) holds true if

$$\frac{n+2-\delta}{1-\delta}|z|^{n+1}\le\frac{\binom{n}{p}(1+\alpha)}{2\alpha}\qquad(n\ge p), \tag{3.6}$$

or

$$|z|\le\left\{\frac{\binom{n}{p}(1+\alpha)(1-\delta)}{2\alpha(n+2-\delta)}\right\}^{\frac{1}{n+1}}\qquad(n\ge p). \tag{3.7}$$

This completes the proof of Theorem 3.1

THEOREM 3.2. If $f(z)\in T_p(\alpha)$, then $f(z)$ is meromorphically convex of order δ $(0\le\delta<1)$ in $|z|<r_2$, where

$$r_2=\inf_{n\ge p}\left\{\frac{\binom{n}{p}(1+\alpha)(1-\delta)}{2\alpha n(n+2-\delta)}\right\}^{\frac{1}{n+1}}. \tag{3.8}$$

The result is sharp for the function $f(z)$ given by (3.2).

PROOF. Note that we have to prove that

$$\left|\frac{zf''(z)}{f'(z)}+2\right|\le 1-\delta \tag{3.9}$$

for $|z|<r_2$. Since

$$\left|\frac{zf''(z)}{f'(z)}+2\right|=\left|\frac{\sum_{n=p}^{\infty}n(n+1)a_n z^{n-1}}{-\frac{1}{z^2}+\sum_{n=p}^{\infty}na_n z^{n-1}}\right|\le\frac{\sum_{n=p}^{\infty}n(n+1)a_n|z|^{n+1}}{1-\sum_{n=p}^{\infty}na_n|z|^{n+1}}. \tag{3.10}$$

we see that if

$$\sum_{n=p}^{\infty}\frac{n(n+2-\delta)}{1-\delta}a_n|z|^{n+1}\le 1, \tag{3.11}$$

Or

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1}\le\frac{\binom{n}{p}(1+\alpha)}{2\alpha}\qquad(n\ge p), \tag{3.12}$$

then (3.9) holds true. Therefore, $f(z)$ is meromorphically convex of order δ in $|z|<r_2$.

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