Internat. J. Math. & Math. Sci.
VOL. 16 NO. 1 (1993) 33-40

ON REGULAR AND SIGMA-SMOOTH TWO VALUED MEASURES AND LATTICE GENERATED TOPOLOGIES

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(Received November 21, 1991 and in revised form April 7, 1992)

ABSTRACT . Let X be an abstract set and **L** a lattice of subsets of X. I(**L**) denotes the non-trivial zero one valued finitely additive measures on A(**L**) , the algebra generated by **L** , and IR(**L**) those elements of I(**L**) that are **L**-regular . It is known that I(**L**)=IR(**L**) if and only if **L** is an algebra. We first give several new proofs of this fact and a number of characterizations of this in topologicial terms.

Next we consider , I(σ*,**L**) the elements of I(**L**) that are σ–smooth on **L** , and IR(σ,**L**) those elements of I(σ*,**L**) that are **L**-regular. We then obtain necessary and sufficent conditions for I(σ*,**L**)=IR(σ,**L**) , and in particuliar ,we obtain conditions in terms of topologicial demands on associated Wallman spaces of the lattice.

KEY WORDS AND PHRASES. Regular and sigma smooth two valued measures,normal lattices,regular lattices, T_2 lattices, countably paracompact and countably bounded,separation and semi-separation of lattices,pre-measures,I-lattice , etc.
1992 AMS SUBJECT CLASSIFICATION CODES. 28A60, 28A32.

1. INTRODUCTION

In this paper we wish to determine when certain classes of measures are equal, and to obtain necessary and sufficent conditions for such equality to hold,emphazing topologicial characterizations.

To be specific let X be an abstract set,**L** a lattice of subsets of X.Let A(**L**) denote the algebra generated by the lattice **L**,and I(**L**) the collection of non-trivial zero-one valued fintely additive measures on A(**L**).IR(**L**) will denote measures in I(**L**) that are **L**-regular on A(**L**),i.e. if $\mu \varepsilon IR(\mathbf{L})$ and $B \varepsilon A(\mathbf{L})$ then there exists a $L \varepsilon \mathbf{L}$ st $B \supseteq L$ and $\mu(B)=\mu(L)$.I(σ*,**L**) will denote those elements of I(**L**) that are sigma-smooth on **L**,i.e. if $L_n \varepsilon \mathbf{L}$ $n=1,2,...\infty$ and $L_n \downarrow \emptyset$ then for $\mu \varepsilon I(\sigma*,\mathbf{L})$, $\lim \mu(L_n)=0$.IR(σ,**L**) will denote those measures in I(σ*,**L**) that are **L**-regular.

The first area of concern is when I(**L**)=IR(**L**).It is well known that this is true iff **L** is an algebra .We give several proofs of this,highlighting topologicial considerations ,to be more precise I(**L**)=IR(**L**) is equivalent to the following:

a) The lattice V(**L**) (see below for definitions) in the space I(**L**) is regular.

b) The topology of closed sets τV(**L**) in I(**L**) is T_1.

c) The lattice of sets V(**L**) in I(**L**) is disjunctive.

The second main area of concern is determining conditions for I(σ*,**L**)=IR(σ,**L**),and conversely what this implies for the lattice.We show (see below for definitions) that I(σ*,**L**)=IR(σ,**L**) is equivalent to: The lattice V(σ,**L**) in the space I(σ*,**L**) is regular.We also show that if **L** is disjunctive and W(σ,**L**) is prime complete or **L** is normal and countably compact and if I(σ*,**L**),V(σ,**L**) is T_1 then I(σ*,**L**)=IR(σ,**L**).Also suppose τ**L**⊇E⊇A(**L**)⊇**L** then if τ**L** is **L** cb or more generally if E (and thus A(**L**)) is **L** cb and a) Either S(**L**)⊇σ(**L**) (Where S(**L**) are the lattice Souslin derived sets.In particuliar if ρ(**L**)=σ(**L**) .) and **L** is delta or b) If **L** is complement generated (and not necessarily delta) then IR(σ,**L**)=I(σ*,**L**).

2. BACKROUND AND NOTATION

We begin by reviewing some notation and terminology which is fairly standard (see Alexandroff [1],Frolik [4],andSzeto [7]).We supply backround material for the readers convenience.

Let X be abstract set,and **L** a lattice of subsets of X st X,Øε**L**.A delta lattice is one that is closed under countable intersections,and the delta lattice generated by **L** is denoted δ(**L**).In addition **L** is complement generated iff for every element Lε**L** there exists a sequence of subsets A_iε**L** i=1,2... and L=∩A_i' i=1,2...∞ (where ' denotes complement) .**L** is countably paracompact if for every sequence L_nε**L** and L_n↓Ø there exists B_n'ε**L**' st B_n'↓Ø and B_n'⊇L_n for every n. A tau lattice is one that is closed under arbitrary intersections,and the tau lattice generated by **L** is denoted by τ**L**.A(**L**) will denote the algebra generated by the lattice **L**.

Let $\mathbf{L}_1$,$\mathbf{L}_2$ be two lattices st $\mathbf{L}_2$⊇$\mathbf{L}_1$,then $\mathbf{L}_1$ semi-separates (ss) $\mathbf{L}_2$ if for L_1ε$\mathbf{L}_1$,L_2ε$\mathbf{L}_2$ and L_1∩L_2=Ø, then there exists an A_1ε$\mathbf{L}_1$, A_1⊇L_2 such that A_1∩L_1=Ø.

Let I(**L**) denote the set of non-trivial two valued {0,1} finitely additive measures on the algebra generated by **L**,and let I(σ*,**L**) denote those elements of I(**L**) that are sigma-smooth on **L**,i.e. if {L_n}ε**L**, L_n↓Ø and μεI(σ*,**L**),limμ(L_n)=0 n→∞.I(σ,**L**) denotes those elements of I(**L**) that are sigma-smooth on A(**L**),i.e. if {A_n}εA(**L**), A_n↓Ø,and μεI(σ,**L**) limμ(A_n)=0 as n→∞.This is equivalent to countably additivity on A(**L**).IR(**L**) will stand for the measures on A(**L**) that are **L**-regular on A(**L**),i.e. μεIR(**L**) μ(A)=supμ(L) Lε**L** A⊇L and AεA(**L**).This is equivalent to being **L**-regular on **L**'.IR(σ,**L**) denotes the set of μεIR(**L**) that are σ-smooth on **L**.The obvious relations hold,I(**L**)⊇I(σ*,**L**)⊇I(σ,**L**)⊇IR(σ,**L**) and I(**L**)⊇IR(**L**).The support of a measure S(μ),μεI(**L**) is defined as S(μ)=∩{Lε**L** | μ(L)=1}.

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two lattices of sets of X st $\mathbf{L}_2$⊇$\mathbf{L}_1$ then $\mathbf{L}_2$ is $\mathbf{L}_1$ countably bounded (cb) if for $L2_n$ε$\mathbf{L}_2$ and $L2_n$↓Ø, then there exists $L1_n$ε$\mathbf{L}_1$,$L1_n$↓Ø and $L1_n$⊇$L2_n$.

A lattice is said to be disjunctive if for any xεX and Lε**L** such that x∉L there exists L~ε**L** such that xεL~ and L∩L~=Ø.**L** is said to be regular if for xεX x∉L Lε**L** then there exists L_1,L_2ε**L** xεL_1' L_2'⊇L and L_1'∩L_2'=Ø.**L** is said to be normal if for L_1,L_2ε**L** and L_1∩L_2=Ø,there exist L_3,L_4ε**L** st L_3'⊇L_1 L_4'⊇L_2 and L_3'∩L_4'=Ø.**L** is said to be coutably compact if for any {L_n}ε**L** and ∩L_n=Ø n=1,2...∞ ,then there exists a finite subindexing st ∩L_{n_i}=Ø n_i=1 to N.A lattice is said to be T_1 if for x,yεX there exist L_1,L_2ε**L** st xεL_1' y∉L_1'and yεL_2' x∉L_2'.

ρ(**L**) denotes the smallest collection of sets that is closed under countable unions and intersections and contains **L**.σ(**L**) will stand for the smallest σ-algebra containing **L**.S(**L**) will stand for the collection of Souslin sets generated by **L**.

Note: For $\mu,\mu_1 \varepsilon I(\mathbf{L})$,we write $\mu \le \mu_1$ ($\mathbf{L}$) if $\mu(L) \le \mu_1(L)$ for all $L \varepsilon \mathbf{L}$.

We now note some measure equivalences of topologicial properties:1) $\mathbf{L}$ is disjunctive iff for all $x \varepsilon X$ $\mu_x \varepsilon IR(\sigma,\mathbf{L})$ where μ_x is the point mass measure,i.e. $\mu_x(A)=1$ if $x\varepsilon A$,$\mu_x(A)=0$ if $x \notin A$,$A \varepsilon A(\mathbf{L})$.2) $\mathbf{L}$ is regular iff $\mu \le \mu_1$ ($\mathbf{L}$) where $\mu,\mu_1 \varepsilon I(\mathbf{L})$ implies $S(\mu)=S(\mu_1)$.3) $\mathbf{L}$ is normal iff $\mu \varepsilon I(\mathbf{L})$ and $\mu_1,\mu_2 \varepsilon IR(\mathbf{L})$ implies that if $\mu \le \mu_1$ ($\mathbf{L}$) and $\mu \le \mu_2$ ($\mathbf{L}$) then $\mu_1=\mu_2$.4) $\mathbf{L}$ is countably compact iff $\mu \varepsilon I(\mathbf{L})$ implies that $\mu \varepsilon I(\sigma^*,\mathbf{L})$.The proofs are not difficult and thus we will only prove the third result for the sake of completeness. Further facts about regular and normal lattices appear in Eid [3] and Grassi [5].

Theorem 2.1: $\mathbf{L}$ is normal iff $\mu \varepsilon I(\mathbf{L})$ $\mu_1,\mu_2 \varepsilon IR(\mathbf{L})$ and $\mu \le \mu_1$ ($\mathbf{L}$),$\mu \le \mu_2$ ($\mathbf{L}$) implies $\mu_1=\mu_2$.

Proof: Let $\mathbf{L}$ be normal and let $\mu \varepsilon I(\mathbf{L})$ $\mu \le \mu_1$ ($\mathbf{L}$) ,$\mu \le \mu_2$ ($\mathbf{L}$) $\mu_1,\mu_2 \varepsilon IR(\mathbf{L})$,assume $\mu_1 \ne \mu_2$.Then there exists a $L \varepsilon \mathbf{L}$ st $\mu_1(L)=1$ $\mu_2(L)=0$ say,and since $\mu_2 \varepsilon IR(\mathbf{L})$ and $\mu_2(L')=1$ there exists $L^{\sim} \varepsilon \mathbf{L}$ st $\mu_2(L^{\sim})=1$ and $L' \supseteq L^{\sim}$,thus $L \cap L^{\sim}=\emptyset$.But $\mathbf{L}$ is normal thus there exists $L_1,L_2 \varepsilon \mathbf{L}$ st $L_1' \supseteq L$ $L_2' \supseteq L^{\sim}$ and $L_1' \cap L_2'=\emptyset$,which implies that $\mu_1(L_1')=1$ $\mu_2(L_2')=1$ or $\mu_1(L_1)=0$ $\mu_2(L_2)=0$ and thus $\mu(L_1)=\mu(L_2)=0$.Also $L_1 \cup L_2=X$,and therefore $\mu(X)=\mu(L_1)+\mu(L_2)-\mu(L_1 \cap L_2)=0$,a contradiction.Therefore $\mu_1=\mu_2$ and the condition holds.

Note: A fact we will use in the second part of the proof and in the proceeding parts of the paper is that there exists a one-one correspondence between prime $\mathbf{L}$-filters and elements of $I(\mathbf{L})$,and a one-one correspondence between $\mathbf{L}$-ultrafilters and elements of $IR(\mathbf{L})$.This correspondence is set up by letting $\mu \varepsilon I(\mathbf{L})$ and $H=\{ L \varepsilon \mathbf{L} \mid \mu(L)=1\}$.Then H is a prime $\mathbf{L}$-filter and conversely if H is a prime $\mathbf{L}$-filter there exists a measure $\mu \varepsilon I(\mathbf{L})$ associated with H st if $L \varepsilon H$, $\mu(L)=1$.A similiar correspondence holds for H and $\mu \varepsilon IR(\mathbf{L})$ in which case H is an $\mathbf{L}$-ultrafilter.

Now we return to the proof of the theorem,conversely,let $\mu \varepsilon I(\mathbf{L})$ and $\mu \le \mu_1$ ($\mathbf{L}$),$\mu_1 \le \mu_2$ ($\mathbf{L}$) for $\mu_1,\mu_2 \varepsilon IR(\mathbf{L})$ imply $\mu_1=\mu_2$,and assume $\mathbf{L}$ is not normal.Then there exists $L_1,L_1^{\sim} \varepsilon \mathbf{L}$ st $L_1 \cap L_1^{\sim}=\emptyset$ and $H=\{ L \varepsilon \mathbf{L} \mid L' \supseteq L_1$ or $L' \supseteq L_1^{\sim} \}$ has the finite intersection property and thus there exists an associated measure $\mu \varepsilon I(\mathbf{L}')$ associated with the filter base H st $\mu(L')=1$ $L \varepsilon H$.Now let $L_2 \varepsilon \mathbf{L}$ and suppose that $\mu(L_2')=0$, then L_2' does not contain L_1 thus $L_1 \cap L_2 \ne \emptyset$.Since the collection $\{L_1 \cap L_2 \mid \mu(L_2)=1$ $L_2 \varepsilon \mathbf{L}\}$ has the fip thus there exists a $\mu_1 \varepsilon IR(\mathbf{L})$ st $\mu_1(L_1)=1$ and $\mu \le \mu_1$ ($\mathbf{L}$).By similiar reasoning there exists a $\mu_2 \varepsilon IR(\mathbf{L})$ st $\mu \le \mu_2$ ($\mathbf{L}$) $\mu_2(L_1^{\sim})=1$.By hypothesis $\mu_1=\mu_2$.Hence $\mu_1(L_1)=\mu_1(L_1^{\sim})=1$.Therefore $\mu_1(L_1 \cap L_1^{\sim})=1$.But $L_1 \cap L_1^{\sim}=\emptyset$ thus $\mu_1(L_1 \cap L_1^{\sim})=0$,a contradiction.Thus $\mathbf{L}$ must be normal.

We now prove a result that will be useful in the sequel.

Theorem 2.2: Let $\mathbf{L}$ be normal and countably paracompact ,then if $\mu \varepsilon I(\sigma^*,\mathbf{L})$ there exists a unique $\mu_1 \varepsilon IR(\sigma,\mathbf{L})$ st $\mu \le \mu_1$ ($\mathbf{L}$).

Proof: Let $\mu \varepsilon I(\sigma^*,\mathbf{L})$ and $\mu_1 \varepsilon IR(\mathbf{L})$ st $\mu \le \mu_1$ ($\mathbf{L}$).Then we must prove $\mu_1 \varepsilon I(\sigma,\mathbf{L})$.Let $A_n \varepsilon \mathbf{L}$ $A_n \downarrow \emptyset$.Since $\mathbf{L}$ is countably paracompact there exists $\{B_n'\} \downarrow \emptyset$,$B_n \varepsilon \mathbf{L}$ and $B_n' \supseteq A_n$ for every n.Since $B_n' \supseteq A_n$ and $\mathbf{L}$ is normal and $A_n \cap B_n=\emptyset$,there exists $C_n,D_n \varepsilon \mathbf{L}$ st $C_n' \supseteq A_n$ and $D_n' \supseteq B_n$ st $D_n' \cap C_n'=\emptyset$.Then $B_n' \supseteq D_n \supseteq C_n' \supseteq A_n$ and we can assume without loss of generality that these inclusions hold with $D_n \downarrow \emptyset$.Then $\mu_1(A_n) \le \mu_1(C_n') \le \mu(C_n') \le \mu(D_n)$ and since $B_n' \downarrow \emptyset$ $D_n \downarrow \emptyset$ plus the fact $\mu \varepsilon I(\sigma^*,\mathbf{L})$ imply that $\lim \mu(D_n)=0$ as $n \to \infty$.Then $\lim \mu_1(A_n)=0$ as $n \to \infty$,and $\mu_1 \varepsilon IR(\sigma,\mathbf{L})$).Uniqueness follows from normality.

Next we consider various sets of measures defined on the algebra generated by a lattice $\mathbf{L}$.For example consider $I(\mathbf{L})$,$I(\sigma^*,\mathbf{L})$,$IR(\mathbf{L})$,or $IR(\sigma,\mathbf{L})$.Denote such sets by I.Also consider the collection of sets $\mathcal{H}(\mathbf{L})$ where $\mathcal{H}(\mathbf{L})=\{\mathcal{H}(L) : L \varepsilon \mathbf{L}\}$ and $\mathcal{H}(L)=\{\mu \varepsilon I : \mu(L)=1\}$.Then the following hold: a) $\mathcal{H}(A \cup B)=\mathcal{H}(A) \cup \mathcal{H}(B)$ for $A,B \varepsilon \mathbf{L}$.b) $\mathcal{H}(A) \cap \mathcal{H}(B)=\mathcal{H}(A \cap B)$ $A,B \varepsilon \mathbf{L}$.c) $\mathcal{H}(A)'=\mathcal{H}(A')$ for $A \varepsilon \mathbf{L}$.d) If $A \supseteq B$ then $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ $A,B \varepsilon \mathbf{L}$.e) If $\mathbf{L}$ is disjunctive (if necessary) and $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ then $A \supseteq B$ $A,B \varepsilon \mathbf{L}$.f) The collection $\mathcal{H}(\mathbf{L})$ is a lattice and $\mathcal{H}(A(\mathbf{L}))=A(\mathcal{H}(\mathbf{L}))$.

We will assume in discussing $\mathcal{H}(\mathbf{L})$ for convenience,that $\mathbf{L}$ is disjunctive,although it will be clear that this assumption is not always needed.

If $\mu\varepsilon I(\mathbf{L})$ then define a measure on $A(\mathcal{H}(\mathbf{L}))$ $\mu^\wedge\varepsilon I(\mathcal{H}(\mathbf{L}))$ by $\mu^\wedge(\mathcal{H}(A))=\mu(A)$ for $A\varepsilon A(\mathbf{L})$.Conversely if $\mu^\wedge\varepsilon I(\mathcal{H}(\mathbf{L}))$ define a measure on $A(\mathbf{L})$ $\mu\varepsilon I(\mathbf{L})$ by $\mu(A)=\mu^\wedge(\mathcal{H}(A))$ $\mathcal{H}(A)\varepsilon A(\mathcal{H}(\mathbf{L}))$.Then the following hold:

Theorem 2.3: If $\mathbf{L}$ is disjunctive (if necessary) then there is a 1-1 correspondence between the sets $I(\mathbf{L})$ and $I(\mathcal{H}(\mathbf{L}))$ given by $\mu\leftrightarrow\mu^\wedge$.Further $\mu\varepsilon I(\mathbf{L})$ is σ-smooth or regular iff $\mu^\wedge\varepsilon I(\mathcal{H}(\mathbf{L}))$ is σ–smooth or $\mathcal{H}(\mathbf{L})$ regular.

If $I=I(\mathbf{L})$ we let $\mathcal{H}(\mathbf{L})=V(\mathbf{L})$.
If $I=I(\sigma*,\mathbf{L})$ we let $\mathcal{H}(\mathbf{L})=V(\sigma,\mathbf{L})$.
If $I=IR(\mathbf{L})$ we let $\mathcal{H}(\mathbf{L})=W(\mathbf{L})$.
If $I=IR(\sigma,\mathbf{L})$ we let $\mathcal{H}(\mathbf{L})=W(\sigma,\mathbf{L})$.

These sets are topologized by taking $\mathcal{H}(L)$,$\mathcal{H}(L)\varepsilon\mathcal{H}(\mathbf{L})$ as a basis for the closed sets, and will be referred to as generalized Wallman spaces

3 THE SPACES IR(L) AND I(L)

In this section we investigate a variety of conditions which are equivalent to $IR(\mathbf{L})=I(\mathbf{L})$ both abstractly and from a topologicial point of view with respect to the space $I(\mathbf{L}),\tau V(\mathbf{L})$.This will useful for out subsequent analysis of $I(\sigma*,\mathbf{L})$,as well as being interesting in its own right.

Theorem 3.1: Let $\mathbf{L}$ be a lattice of subsets of X,then the following are equivalent: a)$IR(\mathbf{L})=I(\mathbf{L})$
b)$IR(\mathbf{L}')=IR(\mathbf{L})$
c) $V(\mathbf{L})$ in the space $I(\mathbf{L})$ is regular
d) The topology of closed sets $\tau V(\mathbf{L})$ in $I(\mathbf{L})$ is T_1
e) The lattice of sets $V(\mathbf{L})$ in $I(\mathbf{L})$ is disjunctive
f) $\mathbf{L}$ is an algebra.

Proof: We show first that $IR(\mathbf{L})=I(\mathbf{L})$ iff $IR(\mathbf{L}')=IR(\mathbf{L})$.

If $IR(\mathbf{L})=I(\mathbf{L})$ and if $\mu\varepsilon IR(\mathbf{L}')$,then $\mu\varepsilon I(\mathbf{L})$ and thus $\mu\varepsilon IR(\mathbf{L})$.Also if $\mu\varepsilon IR(\mathbf{L})$ then $\mu\varepsilon I(\mathbf{L}')$ and there exists $\mu_1\varepsilon IR(\mathbf{L}')$ st $\mu\leq\mu_1$ $(\mathbf{L}')$.But $\mu_1\varepsilon I(\mathbf{L})=IR(\mathbf{L})$ and therefore $\mu=\mu_1\varepsilon IR(\mathbf{L}')$. Conversely if $IR(\mathbf{L}')=IR(\mathbf{L})$ let $\mu\varepsilon I(\mathbf{L})$,then there exists $\mu_1\varepsilon IR(\mathbf{L})$ st $\mu\leq\mu_1$ $(\mathbf{L})$ or $\mu_1\leq\mu$ $(\mathbf{L}')$.But $\mu_1\varepsilon IR(\mathbf{L}')$,thus $\mu_1=\mu\varepsilon IR(\mathbf{L}')=IR(\mathbf{L})$ and $I(\mathbf{L})=IR(\mathbf{L})$.

Next we wish to show $IR(\mathbf{L})=I(\mathbf{L})$ iff the lattice $V(\mathbf{L})$ in the space $I(\mathbf{L})$ is regular.

Let $IR(\mathbf{L})=I(\mathbf{L})$ assume that $V(\mathbf{L})$ is not regular then there exists then there exists $V(L)\varepsilon V(\mathbf{L})$ $\mu\varepsilon I(\mathbf{L})$ st $\mu\notin V(L)$ $H=\{\ V(L^{\sim\prime})\mid V(L^{\sim\prime})\supseteq V(L)$ or $\mu\varepsilon V(L^{\sim\prime})\}$ has the finite intersection property and thus there exists a $\mu_1{}^\wedge\varepsilon I(V(\mathbf{L}'))$ st $\mu_1{}^\wedge(V(L^{\sim\prime}))=1$ $V(L^{\sim\prime})\varepsilon H$ and $\mu^\wedge\leq\mu_1{}^\wedge$ $(V(\mathbf{L}'))$.Projecting down $\mu\leq\mu_1$ $(\mathbf{L}')$, and since $IR(\mathbf{L})=I(\mathbf{L})$ $\mu=\mu_1$. Then projecting upward $\mu_1{}^\wedge=\mu^\wedge$, $\mu^\wedge(V(L'))=\mu_1{}^\wedge(V(L'))=1$ and $\mu^\wedge(V(L))=\mu_1{}^\wedge(V(L))=1$,a contradiction .

Conversely let $V(\mathbf{L})$ be regular in $I(\mathbf{L})$ let $\mu\varepsilon I(\mathbf{L})$,$L\varepsilon\mathbf{L}$, and $\mu(L')=1$. Therefore $\mu\notin V(\mathbf{L})$.Since $V(\mathbf{L})$ is regular there exists $V(L_1'),V(L_2')\varepsilon V(\mathbf{L})$ st $V(L_1')\cap V(L_2')=\emptyset$ and $\mu\varepsilon V(L_1')$ and $V(L_2')\supseteq V(L)$.But this implies that $\mu\varepsilon V(L_2)$ and $L'\supseteq L_2$ $\mu(L_2)=1$ and $\mu\varepsilon IR(\mathbf{L})$,therefore $IR(\mathbf{L})\supseteq I(\mathbf{L})$ and $I(\mathbf{L})=IR(\mathbf{L})$.

Next we show $IR(\mathbf{L})=I(\mathbf{L})$ iff the topology of closed sets $\tau V(\mathbf{L})$ in $I(\mathbf{L})$ is T_1.

$\tau V(\mathbf{L})$ is T_1 iff $V(\mathbf{L})$ is T_1.Assume that $IR(\mathbf{L})=I(\mathbf{L})$ let $\mu_1,\mu_2\varepsilon I(\mathbf{L})=IR(\mathbf{L})$ and $\mu_1\neq\mu_2$.Then there exists $L_1,L_2\varepsilon\mathbf{L}$ st $\mu_1(L_1)=1,\mu_2(L_1)=0,\mu_1(L_2)=0$,and $\mu_2(L_2)=1$,which implies that $\mu_1\varepsilon V(L_2'),\mu_2\notin V(L_2'),\mu_1\notin V(L_1'),\mu_2\varepsilon V(L_1')$,i.e. $V(\mathbf{L})$ is T_1.

Conversely let $V(\mathbf{L})$ be T_1 and let $\mu\varepsilon I(\mathbf{L})$. If $\mu\notin IR(\mathbf{L})$ then there exists $\nu\varepsilon IR(\mathbf{L})$ and $L_1\varepsilon\mathbf{L}$ such that $\mu\varepsilon V(L_1')$, $\nu\notin V(L_1')$ and $\mu\leq\nu$ $(\mathbf{L})$. Since $V(\mathbf{L})$ is T_1 ,there exists $L_2\varepsilon\mathbf{L}$ such that $\mu\notin V(L_2')$ and $\nu\varepsilon V(L_2')$, i.e. $\mu(L_2)=1$ and $\nu(L_2)=0$, a contradiction.

Next we show $IR(\mathbf{L})=I(\mathbf{L})$ iff the lattice of sets $V(\mathbf{L})$ in $I(\mathbf{L})$ is disjunctive.
Assume that $IR(\mathbf{L})=I(\mathbf{L})$. Let $\mu\varepsilon I(\mathbf{L})$ and suppose $L\varepsilon\mathbf{L}$ $\mu\notin V(L)$ .Since $IR(\mathbf{L})=I(\mathbf{L})$ there exists $L_1\varepsilon L$ st $L'\supseteq L_1$ $\mu(L_1)=1$ and $\mu\varepsilon V(L_1)$ and $V(L)\cap V(L_1)=\emptyset$,thus $V(\mathbf{L})$ is disjunctive.

Conversely let $V(\mathbf{L})$ be disjunctive, let $\mu\varepsilon I(\mathbf{L}$, let $L\varepsilon\mathbf{L}$,and let $\mu(L')=1$ (and hence $\mu\notin V(L')$) .Since $V(\mathbf{L})$ is disjunctive there exists a $V(L_1)\varepsilon V(\mathbf{L})$ st $\mu\varepsilon V(L_1)$ and $V(L_1)\cap V(L)=\emptyset$.But this implies that $L'\supseteq L_1$ $\mu(L_1)=1$,and $\mu\varepsilon IR(\mathbf{L})$.Therfore $IR(\mathbf{L})=I(\mathbf{L})$.

Finally,now we claim $I(\mathbf{L})=IR(\mathbf{L})$ iff L is an algebra,i.e. $\mathbf{L}=\mathbf{L}'$.

Let $\mathbf{L}$ be an algebra and $\mu\varepsilon I(\mathbf{L})$ then since $\mathbf{L}=\mathbf{L}'$ μ is trivially regular and $IR(\mathbf{L})=I(\mathbf{L})$.Conversely let $I(\mathbf{L})=IR(\mathbf{L})$ and assume that $\mathbf{L}\neq\mathbf{L}'$,i.e. that $\mathbf{L}$ is not an algebra.Thus there exists a $L^{\sim}\varepsilon\mathbf{L}$ st $L^{\sim\prime}\notin\mathbf{L}$ and look at $H=\{ L \mid L\supseteq L^{\sim}$ or $L\supseteq L^{\sim\prime}\}$.Then H has the fip and thus there exists a $\mu\varepsilon I(\mathbf{L})$ st $\mu(L)=1,L\varepsilon H$.For $\mu(L_1')=1$ $L_1\varepsilon\mathbf{L}$ implies that L_1 does not contain $L^{\sim}$ or $L^{\sim\prime}$.Thus there exists $\mu_1\varepsilon IR(\mathbf{L})$ st $\mu_1(L^{\sim})=1$ $\mu\leq\mu_1$ $(\mathbf{L})$ and also a $\mu_2\varepsilon IR(\mathbf{L}')$ st $\mu_2(L^{\sim\prime})=1$ and $\mu\leq\mu_2$ $(\mathbf{L}')$.But since $I(\mathbf{L})=IR(\mathbf{L})$ and this implies from above that $IR(\mathbf{L}')=IR(\mathbf{L})$,$\mu_1(L^{\sim})=\mu_1(L^{\sim\prime})=1$ or $\mu_1(L_{\sim}\cap L^{\sim\prime})=1=\mu_1(\emptyset)=0$,a contradiction.$\mathbf{L}=\mathbf{L}'$ and $\mathbf{L}$ is an algebra.

Note: It is well known even for abstract distributive lattices that $I(\mathbf{L})=IR(\mathbf{L})$ iff $\mathbf{L}=\mathbf{L}'$.(See Bourbaki [2], Huerta [6]).

Because of the importance of the last result we present an alternative approach which is of importance because of its relevance to lattice separation properties.

Theorem 3.2: Suppose $\mathbf{L}_1,\mathbf{L}_2$ are lattices of subsets of X st $\mathbf{L}_2\supseteq\mathbf{L}_1$.If $\mathbf{L}_2$ is disjunctive and $\mathbf{L}_1$ is normal and if $\psi:IR(\mathbf{L}_2)\rightarrow IR(\mathbf{L}_1)$ where ψ is the restriction map,i.e. $\psi(\nu)=\mu$ the restriction of ν to $A(\mathbf{L}_1)$,then $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$.

Proof: Suppose $L_1\varepsilon\mathbf{L}_1$ and $L_2\varepsilon\mathbf{L}_2$ and $L_1\cap L_2=\emptyset$. Then $W_2(L_1)\cap W_2(L_2)=\emptyset$ and $\psi(W_2(L_2))\cap W_1(L_1)=\emptyset$ for if $\mu=\psi(\nu)$ where $\nu\varepsilon W_2(L_2)$, then $\nu(L_2)=1$. Therefore $\mu(L_1)=\nu(L_1)=0$ and thus $\mu\notin W_1(L_1)$.

$\psi(W_2(L_2))=\cap W_1(L1_i)$ $i\varepsilon I$ an arbitrary index set I,and $L1_i\supseteq L_2$ .This holds since $W_2(L_2)$ is closed and thus compact $(W_2(X)\supseteq W_2(L_2)$ and $W_2(X)$ is compact).Also ψ is continuous since $\psi^{-1}(W_1(L_1))=W_2(L_1)$.$\mathbf{L}_1$ is normal which is equivalent to $W_1(\mathbf{L}_1)$ being T_2 by a known result (see Bourbaki [2]). Therefore since $W_2(L_2)$ is compact and ψ is continuous then $\psi(W_2(L_2))$ is compact and since $W_1(\mathbf{L}_1)$ is T_2 ,$\psi(W_2(L_2))$ is closed and thus $\psi(W_2(L_2))=\cap W_1(L1_i)$ $i\varepsilon I$ an arbitary index set.Also since $\mathbf{L}_2$ is disjunctive then since ψ: $IR(\mathbf{L}_2)\rightarrow IR(\mathbf{L}_1)$ is well defined $\mathbf{L}_1$ is also disjunctive.But this implies that $L1_i\supseteq L_2$.Thus $\psi(W_2(L_2))=\cap W_1(L1_i)$ $i\varepsilon I$ $L1_i\supseteq L_2$.

Now look at $\psi(W_2(L_2))\cap W_1(L_1)=(\cap W_1(L1_i))\cap W_1(L_1)=\emptyset$,$i\varepsilon I$.Then by compactness $(\cap W_1(L1_\alpha))\cap W_1(L_1)=\emptyset$ $\alpha=1,2...N$.Since $\mathbf{L}_1$ is disjunctive,this implies that $\cap L1_\alpha\supseteq L_2$ $\alpha=1,...N$,$L_1^{\sim}=\cap L1_\alpha$ $\alpha=1,...N$,$L_1^{\sim}\varepsilon\mathbf{L}_1$ and $L_1\cap L_1^{\sim}=\emptyset$.Thus $\mathbf{L}_1$ semi-separates $\mathbf{L}_2$.

Corollary 3.1: If $\mathbf{L}$ is a lattice of subsets of X st $I(\mathbf{L})=IR(\mathbf{L})$ then $\mathbf{L}$ is an algebra.

Proof:Set $\mathbf{L}_1=\mathbf{L}$ and $\mathbf{L}_2=A(\mathbf{L})$ .Since $I(\mathbf{L})=IR(\mathbf{L})$, $\mathbf{L}$ is normal. Then the hypotheses of the theorem hold,thus $\mathbf{L}$ ss $A(\mathbf{L})$. Let $L'\cap L=\emptyset$ $L\varepsilon\mathbf{L}$,then since $L'\varepsilon A(\mathbf{L})$ and $\mathbf{L}$ ss $A(\mathbf{L})$,this implies that $L'\varepsilon\mathbf{L}$ $\mathbf{L}=\mathbf{L}'$,i.e. $\mathbf{L}$ is an algebra.

Note:Suppose $\mathbf{L}_2$ is disjunctive and cc then $IR(\sigma,\mathbf{L}_2)=IR(\mathbf{L}_2)$ and if $\nu\varepsilon IR(\mathbf{L}_2)$ then $\mu=\psi(\nu)\varepsilon I(\sigma,\mathbf{L}_1)$,and if $\mathbf{L}_1$ is a delta lattice and $S(\mathbf{L}_1)\supseteq\sigma(\mathbf{L}_1)$ then $\mu\varepsilon IR(\sigma,\mathbf{L}_1)$,in which case if $\mathbf{L}_1$ is also normal then $\mathbf{L}_1$ ss $\mathbf{L}_2$ by theorem 3.2.

Another application arises if $\mathbf{L}_2$ is $\mathbf{L}_1$ cb and $\mathbf{L}_1$ is cc then $IR(\sigma,\mathbf{L}_2)=IR(\mathbf{L}_2)$.If $\mathbf{L}_2$ is disjunctive and $\mathbf{L}_1$ is a delta lattice,$\sigma(\mathbf{L}_1)\supseteq S(\mathbf{L}_1)$ and $\mathbf{L}_1$ is normal,then theorem 3.2 can be applied and $\mathbf{L}_1$ ss $\mathbf{L}_2$.

4. THE SPACES $I(\sigma^*,\mathbf{L})$ AND $IR(\sigma,\mathbf{L})$

In this section we wish to consider,analogous matters concerning the spaces $I(\sigma^*,\mathbf{L})$ and $IR(\sigma,\mathbf{L})$ to those considered earlier for $I(\mathbf{L})$ and $IR(\mathbf{L})$.First we obtain conditions when $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$ implies $\mathbf{L}$ is an algebra.In this connection we introduce a definition.

Definition 4.1 The lattice of subsets of X is almost countably compact (acc) if $\mu\varepsilon IR(\mathbf{L}')$ implies $\mu\varepsilon I(\sigma^*,\mathbf{L})$.

Remark: Clearly $\mathbf{L}$ cc implies $\mathbf{L}$ acc.It is easy to show that if $\mathbf{L}$ is normal and countably paracompact then $\mathbf{L}$ acc implies $\mathbf{L}$ cc: Namely let $\mu\varepsilon I(\mathbf{L})$ then $\mu\varepsilon I(\mathbf{L}')$ and also there exists a $\mu_1\varepsilon IR(\mathbf{L}')$ st $\mu\leq\mu_1$ $(\mathbf{L}')$ or $\mu_1\leq\mu$ $(\mathbf{L})$.But $\mathbf{L}$ acc implies that $\mu_1\varepsilon I(\sigma^*,\mathbf{L})$ and $\mathbf{L}$ normal and cp implies there exists $\mu_2\varepsilon IR(\sigma,\mathbf{L})$ (see introduction) st $\mu_1\leq\mu_2$ $(\mathbf{L})$ and thus $\mu\leq\mu_2$ $(\mathbf{L})$.Thus $\mu\varepsilon I(\sigma^*,\mathbf{L})$ and $\mathbf{L}$ is cc.

Theorem 4.1: If $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$ and if $\mathbf{L}$ is acc then $\mathbf{L}$ is an algebra.

Proof: Let $\mu\varepsilon I(\mathbf{L})$ then there exists a $\mu_1\varepsilon IR(\mathbf{L}')$ st $\mu\leq\mu_1$ $(\mathbf{L}')$.But $\mathbf{L}$ is acc,therefore $\mu_1\varepsilon I(\sigma^*,\mathbf{L})$.But $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$,thus $\mu_1\varepsilon IR(\sigma,\mathbf{L})$ and since $\mu_1\leq\mu$ $(\mathbf{L})$ $\mu\varepsilon IR(\sigma,\mathbf{L})$.Thus $I(\mathbf{L})=IR(\mathbf{L})$,which implies that $\mathbf{L}$ is complemented or that is ,$\mathbf{L}$ is an algebra.

Theorem 4.2: Consider the set $I(\sigma^*,\mathbf{L})$,then the lattice $V(\sigma,\mathbf{L})$ in $I(\sigma^*,\mathbf{L})$ is regular iff $IR(\sigma,\mathbf{L})=I(\sigma^*,\mathbf{L})$.

Proof: Assume that $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$ and that $V(\sigma,\mathbf{L})$ is not regular .Then there exists $\mu\varepsilon I(\sigma^*,\mathbf{L})$ and $V(\sigma,L)\varepsilon V(\sigma,\mathbf{L})$ such that $\mu\notin V(\sigma,L)$ and $H=\{V(\sigma,L^{\sim\prime})\mid V(\sigma,L^{\sim\prime})\supseteq V(\sigma,L)$ or $\mu\varepsilon V(\sigma,L^{\sim\prime})\}$ has the fip and thus there exists a $\mu_1{}^\wedge\varepsilon I(V(\sigma,\mathbf{L}'))$ st $\mu_1{}^\wedge(V(\sigma,L^{\sim\prime}))=1$ $V(\sigma,L^{\sim\prime})\varepsilon H$.In addition $\mu^\wedge\varepsilon I(\sigma^*,V(\sigma,\mathbf{L}))$ and $\mu^\wedge\leq\mu_1{}^\wedge$ on $V(\sigma,L')$ or $\mu_1{}^\wedge\leq\mu^\wedge$ on $V(\sigma,\mathbf{L})$.Therefore $\mu_1{}^\wedge\varepsilon I(\sigma^*,V(\sigma^*,\mathbf{L}))$ and projecting downward $\mu_1\leq\mu$ $(\mathbf{L})$.But $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$,therefore $\mu=\mu_1$ and $\mu^\wedge=\mu_1{}^\wedge$ on $V(\sigma,\mathbf{L})$ projecting upward,also $\mu^\wedge=\mu_1{}^\wedge\varepsilon IR(\sigma,V(\sigma,\mathbf{L}))$.Since $\mu^\wedge\varepsilon IR(\sigma,V(\sigma,\mathbf{L}))$ and $\mu^\wedge(V(\sigma,L'))=1$ there exists a $V(\sigma,L_1{}^\sim)$ st $\mu^\wedge(V(\sigma,L_1{}^\sim))=1$ and $V(\sigma,L')\supseteq V(\sigma,L_1{}^\sim)$ or $V(\sigma,L_1{}^{\sim\prime})\supseteq V(\sigma,L)$ and by definition of $\mu_1{}^\wedge$,$\mu^\wedge(V(\sigma,L_1{}^{\sim\prime}))=\mu_1{}^\wedge(V(\sigma,L_1{}^{\sim\prime}))=1$ a contradiction.$I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ is regular.

Conversely let $I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ be regular and let $\mu\varepsilon I(\sigma^*,\mathbf{L})$ $\mu\notin V(\sigma,L)$,then $\mu\varepsilon V(\sigma,L')$ and $\mu(L')=1$.Consider the projection upward,then $\mu^\wedge\varepsilon I(\sigma^*,V(\sigma^*,\mathbf{L}))$ $\mu\notin V(\sigma^*,L)$.Then since $I(\sigma^*,\mathbf{L}),V(\sigma^*,\mathbf{L})$ is regular there exists $V(\sigma,L_1),V(\sigma,L_2)\varepsilon V(\sigma,\mathbf{L})$ st $V(\sigma,L_1{}')\supseteq V(\sigma,L)$ $\mu\varepsilon V(\sigma,L_2{}')$ and $V(\sigma,L_1{}')\cap V(\sigma,L_2{}')=\emptyset$ or $V(\sigma,L_1)\cup V(\sigma,L_2)=I(\sigma^*,\mathbf{L})$.Then $\mu^\wedge(V(\sigma,L_1))=1$ or $\mu^\wedge(V(\sigma,L_2))=1$.Since $\mu^\wedge(V(\sigma,L_2{}'))=1$ $\mu^\wedge(V(\sigma,L_2))=0$ and $\mu^\wedge(V(\sigma,L_1))=1$ $V(\sigma,L_1{}')\supseteq V(\sigma,L)$ or $V(\sigma,L')\supseteq V(\sigma,L_1)$.This implies that $L'\supseteq L_1$ and $\mu(L_1)=1$.Therefore $\mu\varepsilon IR(\sigma,\mathbf{L})$ and $I(\sigma^*,\mathbf{L})=IR(\sigma^*,\mathbf{L})$.

Theorem 4.3: Suppose $\tau\mathbf{L}\supseteq E\supseteq A(\mathbf{L})\supseteq\mathbf{L}$ then if $\tau\mathbf{L}$ is $\mathbf{L}$ cb or more generally if E(and thus $A(\mathbf{L})$) is $\mathbf{L}$ cb and a) $S(\mathbf{L})\supseteq\sigma(\mathbf{L})$ (in particuliar if $\rho(\mathbf{L})=\sigma(\mathbf{L})$) and $\mathbf{L}$ is delta or b) If $\mathbf{L}$ is complement generated (and not neccessarily delta) then $IR(\sigma,\mathbf{L})=I(\sigma^*,\mathbf{L})$.

Proof: Let $\mu\varepsilon I(\sigma^*,\mathbf{L})$.Since $A(\mathbf{L})$ is $\mathbf{L}$ countably bounded $I(\sigma,\mathbf{L})=I(\sigma^*,\mathbf{L})$.Now let $\mathbf{L}$ be cg and $\mu(L')=1$,$\mu\varepsilon I(\sigma,L)$ $L\varepsilon\mathbf{L}$,then $L'=\cup L_i$ $i=1,2,\ldots$ or $L=\cap L_i{}'$ $i=1,2,\ldots$ .Then since $\mu\varepsilon I(\sigma,\mathbf{L})$, $0=\lim\mu(L)=\lim\mu(\cap L_i{}')$ $i=1,2..N$ and therefore $\mu(\cap L_i{}')=0$ where $i=1,2,\ldots N$ for some N.Since $L'\supseteq\cup L_i$ $i=1,2\ldots N$ $\mu\varepsilon IR(\sigma,\mathbf{L})$ and $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$.

Suppose instead that $\mathbf{L}$ is delta and $S(\mathbf{L})\supseteq\sigma(\mathbf{L})$ in particuliar $\rho(\mathbf{L})=\sigma(\mathbf{L})$.Since $A(\mathbf{L})$ is countably bounded by $\mathbf{L}$ $I(\sigma^*,\mathbf{L})=I(\sigma,\mathbf{L})$.Consider μ^* the outer measure induced by μ and its restriction to the μ^*-measurable sets.Then the μ^*-measurable sets include $\sigma(\mathbf{L})$and thus $A(\mathbf{L})$,and μ^* is delta regular on such sets by the hypotheses $S(\mathbf{L})\supseteq\sigma(\mathbf{L})$ (or more particuliarly $\sigma(\mathbf{L})=\rho(\mathbf{L})$).Since $\mathbf{L}$ is delta this implies that $\mu\varepsilon IR(\sigma,\mathbf{L})$ and $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$.

Note: If $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$ then $IR(\sigma,\mathbf{L}')=IR(\sigma,\mathbf{L})$ and $\mu_x \varepsilon IR(\sigma,\mathbf{L})$ and $\mu_x \varepsilon IR(\sigma,\mathbf{L}')$.Thus $\mathbf{L}$ and $\mathbf{L}'$ is disjunctive,thus $\tau\mathbf{L}\supseteq\mathbf{L}'$ and $\tau\mathbf{L}$ contains $\mathbf{L}$ and $\mathbf{L}'$,which implies that $\mathbf{L}$ is contained in the algebra of closed-open sets determined by the topology of $\mathbf{L}$ on X.

Defintion 4.2: A lattice $\mathbf{L}$ is said to be <u>prime complete</u> if for any $\mu\varepsilon I(\sigma^*,\mathbf{L})$ $S(\mu)\neq\emptyset$.

Theorem 4.4: If $\mathbf{L}$ is disjunctive,and $W(\sigma,\mathbf{L})$ in $IR(\sigma,\mathbf{L})$ is prime complete,then for any $\mu\varepsilon I(\sigma^*,\mathbf{L})$,there exists a $\mu_1\varepsilon IR(\sigma,\mathbf{L})$ st $\mu\leq\mu_1$ $(\mathbf{L})$.

Proof: Since $\mathbf{L}$ is disjunctive there exists a one to one correspondence between measures on X,$\mathbf{L}$ and $IR(\sigma,\mathbf{L})$,$W(\sigma,\mathbf{L})$.Thus let $\mu\varepsilon I(\sigma^*,\mathbf{L})$ then $\mu^\wedge\varepsilon I(\sigma^*,W(\sigma,\mathbf{L}))$ and since $W(\sigma,\mathbf{L})$ is prime complete $S(\mu^\wedge)\neq\emptyset$,then there exists a $\{\mu_1\}\varepsilon S(\mu^\wedge)$ $\mu_1\varepsilon IR(\sigma,\mathbf{L})$.Further if $\mu(L)=1$ $L\varepsilon\mathbf{L}$ then $\mu^\wedge(W(\sigma,L))=\mu(L)=1$,and since $\mu_1\varepsilon W(\sigma,L)$ $\mu_1(L)=1$ therefore $\mu\leq\mu_1$ $(\mathbf{L})$ with $\mu_1\varepsilon IR(\sigma,\mathbf{L})$.

Theorem 4.5: If a) $\mathbf{L}$ is disjunctive and $W(\sigma,\mathbf{L})$ is prime complete or alternately b) $\mathbf{L}$ is normal and countably paracompact then if $I(\sigma^*,\mathbf{L}),V(\sigma,,\mathbf{L})$ is T_1 then $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$.

Proof: Let $\mu\varepsilon I(\sigma^*,\mathbf{L})$ then by hypothesis a) or b) there exists a $\nu\varepsilon IR(\sigma,\mathbf{L})$ st $\mu\leq\nu$ $(\mathbf{L})$.Then since $I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ is T_1 there exists a L', $L\varepsilon\mathbf{L}$ st $\nu\varepsilon V(\sigma,L')$ $\mu\not\varepsilon V(\sigma,L')$ which implies that $\mu(L)=1$ $\nu(L)=0$ a contradiction unless $\mu=\nu$ and $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$.

Theorem 4.6: If $I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ is T_1 then $\mu,\mu_1\varepsilon I(\sigma^*,\mathbf{L})$ implies that if $\mu\neq\mu_1$ then neither $\mu\leq\mu_1$ $(\mathbf{L})$ or $\mu_1\leq\mu$ $(\mathbf{L})$ holds.Conversely if neither $\mu\leq\mu_1$ $(\mathbf{L})$ or $\mu_1\leq\mu$ $(\mathbf{L})$ $\mu\neq\mu_1$ holds then $I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ is T_1.

Proof: Let $\mu,\mu_1\varepsilon I(\sigma^*,\mathbf{L})$ $\mu\neq\mu_1$.Since $V(\sigma,\mathbf{L})$ is T_1,this implies that there exists $V(\sigma,L_1),V(\sigma,L_2)\varepsilon V(\sigma,\mathbf{L})$ st $\mu\varepsilon V(\sigma,L_1')$ $\mu_1\not\varepsilon V(\sigma,L_1')$ $\mu\not\varepsilon V(\sigma,L_2')$ $\mu_1\varepsilon V(\sigma,L_2')$ or $\mu(L_1)=0$ $\mu_1(L_1)=1$ $\mu(L_2)=1$ $\mu_1(L_2)=0$.Thus neither $\mu\leq\mu_1$ $(\mathbf{L})$ or $\mu_1\leq\mu$ $(\mathbf{L})$ can hold.

Conversely suppose $\mu,\mu_1\varepsilon I(\sigma^*,\mathbf{L})$ $\mu\neq\mu_1$ and neither $\mu\leq\mu_1$ $(\mathbf{L})$ or $\mu_1\leq\mu$ $(\mathbf{L})$ holds.This implies that there exists $L_1,L_2\varepsilon\mathbf{L}$ st $\mu(L_1)=1$ $\mu_1(L_1)=0$ $\mu(L_2)=0$ $\mu_1(L_2)=1$ or $\mu(L_1')=0$ $\mu_1(L_1')=1$ $\mu(L_2')=1$ $\mu_1(L_2')=0$ or $\mu\varepsilon V(\sigma,L_2')$ $\mu_1\not\varepsilon V(\sigma,L_2')$ $\mu_1\varepsilon V(\sigma,L_1')$ $\mu\not\varepsilon V(\sigma,L_1')$ and $I(\sigma^*,\mathbf{L}),V(\sigma,\mathbf{L})$ is T_1.

Definition 4.3: Denote by $\Pi(\sigma,\mathbf{L})$ the collection of premeasures that are sigma-smooth.A premeasure $p\varepsilon\Pi(\mathbf{L})$ is defined on $\mathbf{L}$ and satisfies 1) $p(\emptyset)=0$;2) If $p(A)=1$ $p(B)=1$ $A,B\varepsilon\mathbf{L}$ then $p(A\cap B)=1$;3) If $p(B)=1$ and $A\supseteq B$ where $A,B\varepsilon\mathbf{L}$ then $p(A)=1$.It is sigma-smooth if $\{A_n\}\downarrow\emptyset$ $A_n\varepsilon\mathbf{L}$ then $\lim p(A_n)=0$ as $n\to\infty$.

Definition 4.4: $\mathbf{L}$ is an <u>I-lattice</u> iff for every $p\varepsilon\Pi(\sigma,\mathbf{L})$ there exists a $\mu\varepsilon IR(\sigma,\mathbf{L})$ st $p\leq\mu$ $(\mathbf{L})$.

Theorem 4.7: Let $\mathbf{L}$ be an I-lattice which is also a delta lattice and suppose $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$,then $\mathbf{L}$ is complemented.

Proof: Assume $\mathbf{L}$ is not complemented,then for some $L\varepsilon\mathbf{L}$ $L'\not\varepsilon\mathbf{L}$.Consider $H=\{L^\sim \mid L^\sim\supseteq L',L^\sim\varepsilon\mathbf{L}\}$. Then since $\mathbf{L}$ is delta ,H has the countable intersection property since $L'\neq\emptyset$. Thus there exists a $p\varepsilon\Pi(\sigma,\mathbf{L})$ associated with H.Since $\mathbf{L}$ is I-lattice there exists $\mu\varepsilon IR(\sigma,\mathbf{L})$ st $p\leq\mu$ $(\mathbf{L})$.Also $I(\sigma^*,\mathbf{L})=IR(\sigma,\mathbf{L})$ implies that $IR(\sigma,\mathbf{L})=IR(\sigma,\mathbf{L}')$.

Now if $L^\sim\supseteq L'$then $L^\sim\cap L\neq\emptyset$,since $L'\not\varepsilon\mathbf{L}$.Thus $\mu(L)=1$ since associated with μ is an ultrafilter. But since $\mu\varepsilon IR(\sigma\mathbf{L}')$ $\mu(L)=0$, a contradiction and hence $\mathbf{L}$ is complemented.

ACKNOWLEDGEMENTS. I wish to thank the referees for their helpful comments and corrections that greatly enhanced the readability of this paper.

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