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SOME RESULTS ON CONVERGENCE RATES FOR PROBABILITIES OF MODERATE DEVIATIONS FOR SUMS OF RANDOM VARIABLES

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ABSTRACT. Let X, X_n, $n\geq 1$ be a sequence of *iid* real random variables, and $S_n = \sum_{k=1}^{n} X_k$, $n\geq 1$. Convergence rates of moderate deviations are derived, i.e., the rate of convergence to zero of certain tail probabilities of the partial sums are determined. For example, we obtain equivalent conditions for the convergence of series $\sum_{n\geq 1}(\psi^2(n)/n)P(|S_n| \geq \sqrt{n}\varphi(n))$ only under the assumptions that $EX = 0$ and $EX^2 = 1$, where φ and ψ are taken from a broad class of functions. These results generalize and improve some recent results of Li (1991) and Gafurov (1982) and some previous work of Davis (1968). For $b\epsilon[0,1]$ and $\epsilon > 0$, let

$$\lambda_{\epsilon,b} = \sum_{n\geq 3}((\log\log n)^b/n)\ I(|S_n| \geq \sqrt{(2+\epsilon)n\log\log n}).$$

The behaviour of $E\lambda_{\epsilon,b}$ as $\epsilon\downarrow 0$ is also studied.

KEY WORDS AND PHRASES. Moderate deviations, Rates of convergence, Tail probabilities.

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1. INTRODUCTION

Let $X, X_n, n\geq 1$ be *iid* random variables with $EX = 0$ and $EX^2 = 1$. Let $S_n = \sum_{k=1}^{n} X_k, k\geq 1$. It is well known that by the law of the iterated logarithm

$$\lim\sup_{n\to\infty} S_n/\sqrt{2n \log\log n} = 1 \quad \text{a.s.}$$

The study of the estimate of the rate of convergence in the above relation has engaged the attention of some probabilists over the last few decades. This paper is concerned about the rate of convergence in the law of the iterated logarithm. Recently, Li (1991) obtained some convergence rates for particular cases which are nearly the best possible. See Corollary 2.5. The papers by Darling and Robbins(1967), Davis(1968), Gafurov(1982), Li(1991), and Strassen (1967) are close to the present one.

Davis (1968), Theorem 3, p.1483 proved the following result. Let φ be a positive nondecreasing function on $[1,\infty)$. Suppose

$$E(X^2(\log^+|X|)(\log^+\log^+|X|) < \infty.$$

Then the following are equivalent.

$$\sum_{n\geq 1}(\varphi^2(n)/n)P(|S_n| \geq \sqrt{n}\,\varphi(n)) < \infty. \tag{1.1}$$

$$\int_1^\infty (\varphi(t)/t) \exp\{-\varphi^2(t)/2\}\, dt < \infty. \tag{1.2}$$

Gafurov (1982) showed that (1.1) and (1.2) are equivalent under the weaker condition that $E(X^2\log^+|X|) < \infty$. Gafurov(1982) also established the following result. If $E(X^2\log^+|X|) < \infty$, then

$$\lim_{\epsilon\downarrow 0} \epsilon^{3/2} \sum_{n\geq 3}((\log\log n)/n)\, P(|S_n| \geq \sqrt{(2+\epsilon)n \log\log n}) = \sqrt{2}. \tag{1.3}$$

In this paper, we obtain some general results in the spirit of (1.1), (1.2), and (1.3) which seem to be the best possible in a certain sense. These results generalize and improve the above results and some more. Some errors from Davis (1967) and Gafurov (1982) are pointed out.

We now proceed to describe the contents of each section. In Section 2, we state the main results of this paper proofs of which are given in Section 4. One of the objectives of Section 2 is to find equivalent conditions for

$$\sum_{n\geq 1}(\psi^2(n)/n)P(|S_n| \geq \sqrt{n}\,\varphi(n)) < \infty$$

for a broad class of functions φ and ψ. Theorem 2.1 is a very general result from which quite a number of results in the literature follow as special cases. For example, Corollary 2.2 improves Theorem 3 of Davis (1967) and the related result of Gafurov (1982), p.141. Corollary 2.5 gives a recent result of Li (1991) as a corollary to Theorem 2.1. The second objective is to study the limit behaviour of

$$\lambda_{\epsilon,b} = \sum_{n\geq 3}((\log\log n)^{b}/n)\ I(|S_n| \geq \sqrt{(2+\epsilon)n\log\log n})$$

as $\epsilon\downarrow 0$ for $b \in [0,1)$. Theorem 2.8 improves and generalizes Theorem 2 of Gafurov (1982). In Section 3, we collect some auxiliary results needed in the proofs of the main results of Section 2. Lemma 3.3 plays a crucial role in the derivation of direct and powerful estimates of the convergence rates involved. This lemma is inspired by the results of the same genre established by Heyde (1967) and (1969). Lemma 3.3 seems to be new, although the proof is along the lines of Heyde (1967). In Section 5, the main results of this paper are analyzed vis-a-vis with some well known results.

2. MAIN RESULTS

Let $X, X_n, n\geq 1$ be a sequence of real valued random variables and $S_n = X_1 + X_2 + \cdots + X_n$, $n\geq 1$. Let $\varphi(\cdot)$ and $\psi(\cdot)$ be two positive real valued functions on $[1,\infty)$ such that $\varphi(\cdot)$ is nondecreasing, $\lim_{t\to\infty}\varphi(t) = \infty$ and $\psi(t) = O(\varphi(t))$ as $t\to\infty$. For $t \geq 0$, let $\sigma^2(t) = E(X^2I(|X|<\sqrt{t}))-(EXI(|X|<\sqrt{t}))^2$. For ease in writing, we use the symbol σ_n^2 for $\sigma^2(n\varphi^2(n))$ for $n \geq 1$, unless, otherwise specified. Let $L(x) = L_1(x) = \log\max\{e,x\}$ and $L_k(x) = L(L_{k-1}(x))$ for $k \geq 2$. We use $L(x)$ and $\log x$ interchangeably. We do the same for $L_2(x)$ and $\log\log x$. $\log^+x$ stands for $\max\{1,\log x\}$. Consider the following statements.

$$\sum_{n\geq 1} (\psi^2(n)/n)\ P(|S_n| \geq n^{1/2}\varphi(n)) < \infty. \tag{2.1}$$

$$\sum_{n\geq 1} (\psi^2(n)/n\varphi(n))\ \exp\{-\varphi^2(n)/2\sigma_n^2\} < \infty. \tag{2.2}$$

$$\sum_{n\geq 1} (\psi^2(n)/n\varphi(n))\ \exp\{-\varphi^2(n)/2\} < \infty. \tag{2.3}$$

$$\sum_{n\geq 1} (\varphi^2(n)/n)\ P(|S_n| \geq n^{1/2}\varphi(n)) < \infty. \tag{2.4}$$

$$\sum_{n\geq 1} (\varphi(n)/n)\ \exp\{-\varphi^2(n)/2\sigma_n^2\} < \infty. \tag{2.5}$$

$$\sum_{n\geq 1} (1/n)\ P(|S_n| \geq n^{1/2}\varphi(n)) < \infty. \tag{2.6}$$

$$\sum_{n\geq 1} (1/n\varphi(n)) \exp\{-\varphi^2(n)/2\sigma_n^2\} < \infty. \tag{2.7}$$

$$\int_1^\infty (\varphi(x)/x) \exp\{-\varphi^2(x)/2\} \, dx < \infty. \tag{2.8}$$

$$\int_1^\infty (1/x\varphi(x)) \exp\{-\varphi^2(x)/2\} \, dx < \infty. \tag{2.9}$$

The following is a very general result which generalizes quite a number of results in the literature.

THEOREM 2.1. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$. Then (2.1) and (2.2) are equivalent. If, in addition, $E(X^2 I(|X| \geq t)) = O(1/\log\log t)$ as $t\to\infty$, then (2.1) and (2.3) are equivalent.

Some remarks are in order how delicate Theorem 2.1 is. Some classical results follow as special cases of Theorem 2.1. For example, see Corollary 2.4 below. Further, Theorem 2.1 generalizes Theorem 3 of Davis (1968) and Theorem 1 of Gafurov (1982). See Corollary 2.2. In addition, Theorem 4 of Davis (1968) is not true. See Remark 2 below. We now set out amplifying these statements.

Now consider the important special case : $\varphi(\cdot) = \psi(\cdot)$. The following corollary is concerned with this special case.

COROLLARY 2.2. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$. Then (2.4) and (2.5) are equivalent. If, in addition, $E(X^2 I(|X| \geq t)) = O(1/\log\log t)$ as $t\to\infty$, then (2.4) and (2.8) are equivalent.

We now look at another important special case: $\psi(\cdot) \equiv 1$ which is covered by the following corollary.

COROLLARY 2.3. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$. Then (2.6) and (2.7) are equivalent. If, in addition, $E(X^2 I(|X| \geq t)) = O(1/\log\log t)$ as $t\to\infty$, then (2.6) and (2.9) are equivalent.

REMARKS. 1. Davis(1968,Theorem 3,p.1483) proved the equivalence of (2.4) and (2.8) under the assumption that $E(X^2 \log^+|X| \log^+\log^+|X|) < \infty$. Theorem 1 of Gafurov (1982) p.139 implies that (2.4) and (2.8) are equivalent under the weaker condition $EX^2\log^+|X| < \infty$. Corollary 2.2 generalizes this result in view of the fact that $E(X^2\log^+\log^+|X|) < \infty$ implies that $E(X^2 I(|X| \geq t)) = O(1/\log\log t)$ as $t\to\infty$.

2. Let $\varphi(x) = (2\log^+ \log^+ x)^{1/2}$, $x\geq 1$. With this choice of φ, there are *iid* random variables X, X_n, $n\geq 1$ such that $EX = 0$, $EX^2 = 1$, $E(X^2 I(|X| \geq t)) = O((\log\log\log t)/(\log\log t))$ as $t\to\infty$,

$$\sum_{n\geq 3}(\log\log n)/n)\, P(|S_n| \geq \sqrt{2n\log\log n}) < \infty \tag{2.10}$$

and

$$\sum_{n\geq 3} (1/n)\, P(|S_n| \geq \sqrt{2n\log\log n}) < \infty. \tag{2.11}$$

It is easy to check that

$$\int_3^\infty (\sqrt{2\log\log x}\,/x) \exp\{-\log\log x\}\, dx = \infty, \tag{2.12}$$

and

$$\int_3^\infty (1/x\sqrt{2\log\log x}) \exp\{-\log\log x\}\, dx = \infty. \tag{2.13}$$

This example is useful to bring into focus some finer points of some of the results established in this paper which will be pointed out at appropriate junctures. For example, Theorem 4 of Davis (1968), p.1484 is not true. The above serves as a counter-example in view of (2.11) and (2.13).

More generally, to demonstrate that the results are really the "best possible", we can, for any $f(t) \uparrow \infty$, exhibit a random variable X satisfying

$$EX^2 I(|X| \geq t) = O(f(t)/\log\log t) \quad \text{as} \quad t\to\infty$$

for which (2.4) holds but (2.8) fails.

Taking $\varphi(t) = \psi(t) = \epsilon\sqrt{t}$, for $\epsilon > 0$ and $t \geq 1$, we obtain the following classical result on complete convergence due to Hsu and Robbins (1947) as a a consequence of Theorem 2.1 above.

COROLLARY 2.4. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 < \infty$. Then

$$\sum_{n\geq 1} P(|S_n| \geq \epsilon n) < \infty \tag{2.14}$$

for every $\epsilon > 0$.

Another consequence of Theorem 2.1 is the following result of Li (1991).

COROLLARY 2.5. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables. Then the following are equivalent.

(i) $EX = 0$ and $EX^2 = 1$. (2.15)

(ii) $\sum_{n\geq 3} (1/n)\, P(|S_n| \geq \sqrt{(2+\epsilon)n \log\log n})$

$< \infty$, for every $\epsilon > 0$,
$= \infty$, for $-2 < \epsilon < 0$. (2.16)

(iii) $\sum_{n\geq 3} ((\log\log n)/n)\, P(|S_n| \geq \sqrt{(2+\epsilon)n \log\log n})$

$< \infty$, for every $\epsilon > 0$,
$= \infty$, for $-2 < \epsilon < 0$. (2.17)

(iv) $\sum_{n\geq 3} (1/(n \log n))\, P(\sup_{k\geq n} |S_k|/\sqrt{(2+\epsilon)\, k \log\log k} \geq 1)$

$< \infty$, for every $\epsilon > 0$,
$= \infty$, for $-2 < \epsilon < 0$. (2.18)

Under the additional assumption that $E(X^2 I(|X| \geq t)) = 0(1/\log\log t)$ as $t\to\infty$, using Corollary 2.2, we can obtain a more precise result than the one provided by Corollary 2.5.

COROLLARY 2.6. Let $X, X_n, n\geq 1$ be a sequence of *iid* random variables with $EX = 0$, $EX^2 = 1$, and $E(X^2 I(|X| \geq t)) = 0(1/\log\log t)$ as $t\to\infty$. Then for any $k \geq 4$,

$$\sum_{n\geq 3} ((\log\log n)/n)\, P(|S_n| \geq \sqrt{2n(L_2(n) + (3/2)L_3(n) + \cdots + (1+\epsilon)L_k(n))})$$

$< \infty$, for every $\epsilon > 0$,
$= \infty$, for $-1 < \epsilon < 0$. (2.19)

From the example alluded to in Remark 2 above, the condition on the tail behaviour of the distribution in Corollary 2.6 cannot be improved in the sense exemplified in the following corollary.The formulation of Corollary 2.5 cannot be improved in the same sense.

COROLLARY 2.7. Let $X, X_n, n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$, and $\varphi(n), n\geq 1$ a positive nondecreasing sequence of numbers satisfying $\lim_{n\to\infty} \varphi(n) = \infty$ and $\log\log n = o(\varphi^2(n))$ as $n\to\infty$. Then

$$\sum_{n\geq 1} (\varphi^2(n)/n)\, P(|S_n| \geq \epsilon\sqrt{n}\varphi(n)) < \infty \qquad (2.20)$$

for every $\epsilon > 0$. In particular, we have

$$\sum_{n\geq 1} ((\log n)/n)\, P(|S_n| \geq \epsilon\sqrt{n \log n}) < \infty \qquad (2.21)$$

for every $\epsilon > 0$.

For $\epsilon > 0$ and $b \geq 0$, consider the random variable

$$\lambda_{\epsilon,b} = \sum_{n\geq 3} ((\log\log n)^b/n)\, I(|S_n| \geq \sqrt{(2+\epsilon)n \log\log n}),$$

which represents the number of jumps the random walk S_n, $n\geq 3$ makes with the "weights" $(\log\log n)^b/n$ over the boundary $\pm\sqrt{(2+\epsilon)n\log\log n}$. If $\epsilon \downarrow 0$, then observe that $\lambda_{\epsilon,b}\uparrow$. Therefore, it is interesting to study the behavior of $E\lambda_{\epsilon,b}$ when $\epsilon \downarrow 0$. The following theorem generalizes Theorem 2 of Gafurov (1982), p.141 under much weaker conditions.

THEOREM 2.8. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$. If $E(X^2 I(|X| \geq t)) = o(1/\log\log t)$ as $t\to\infty$, then for for every $b \in [0,1]$, we have

$$\lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^b/n)\, P(|S_n| \geq \sqrt{(2+\epsilon)n\log\log n})$$
$$= 2^b\sqrt{2/\pi}\,\Gamma(b+(1/2)), \qquad (2.22)$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$, $s > 0$.

From Remark 2, we can see that the conditions of Theorem 2.8 are the best possible for the validity of its conclusion. Moreover, Theorem 2.8 is not true if "o" is replaced by "O".

THEOREM 2.9. Let X, X_n, $n\geq 1$ be *iid* random variables and $b \geq 2$. Then (2.22) is equivalent to $EX = 0$, $EX^2 = 1$, and $E(X^2(\log^+ \log^+|X|)^{b-1}) < \infty$.

THEOREM 2.10. Let X, X_n, $n\geq 1$ be a sequence of *iid* random variables and $b > 0$. Then $EX = 0$, $EX^2 = 1$, $E(X^2(\log^+|X|)^b/L_2|X|) < \infty$, and

$$\lim_{\epsilon\downarrow 0} \sqrt{\epsilon} \sum_{n\geq 3} ((\log n)^b/n)\, P(|S_n| \geq \sqrt{(b+1)(2+\epsilon)n\log\log n})$$
$$= \sqrt{2}/(b+1) \qquad (2.23)$$

are equivalent.

3. AUXILIARY RESULTS

In this section, we collect some auxiliary results needed in the subsequent sections. We need some additional notation. Let $F_n(x) = P(n^{-1/2}S_n \leq x)$ for $-\infty < x < \infty$ and $n\geq 1$, and $\Phi(\bullet)$ the distribution function of the standard normal distribution.

LEMMA 3.1. Let X be random variable with $EX = 0$ and $EX^2 < \infty$. If $\varphi(\cdot)$ is a positive nondecreasing function on $[1,\infty)$, then

$$\sum_{n\geq 1} \varphi^2(n)\ P(|X| \geq \sqrt{n}\ \varphi(n)) < \infty,$$

$$\sum_{n\geq 1} (\varphi^2(n)/\sqrt{n})\ E(|X|I(|X| \geq \sqrt{n}\ \varphi(n))) < \infty, \tag{3.1}$$

and

$$\sum_{n\geq 1} (1/[n^{3/2}\varphi(n)])\ E(|X|^3 I(|X| < \sqrt{n}\ \varphi(n))) < \infty.$$

PROOF. We will establish the last statement of (3.1). The other two can be established in the same vein. Let $i_k = \min\{i \geq 1 : k \leq i\varphi^2(i) + 2\}$, $k \geq 1$. Let $[x]$ denote the integral part of x. Take $c > 0$ such that $1/[\sqrt{n}\ \varphi(n)] \leq c/(n\varphi^2(n) + 2)^{1/2}$ for every $n \geq 1$. We then have

$$\begin{aligned}
&\sum_{n\geq 1} (1/[n^{3/2}\varphi(n)])\ E(|X|^3 I(|X| < \sqrt{n}\ \varphi(n))) \\
&\leq \sum_{n\geq 1} (1/[n^{3/2}\varphi(n)]) \sum_{k=1}^{[n\varphi^2(n)]+1} k^{3/2} P(k-1 \leq X^2 < k) \\
&\leq \sum_{k\geq 1} \Big(\sum_{n\geq i_k} (1/[n^{3/2}\varphi(n)])\Big)\ k^{3/2}\ P(k-1 \leq X^2 < k) \\
&\leq \sum_{k\geq 1} (4/[i_k^{1/2}\varphi(i_k)])\ k^{3/2}\ P(k-1 \leq X^2 < k) \\
&\leq 4c \sum_{k\geq 1} kP(k-1 \leq X^2 < k) < \infty.
\end{aligned}$$

We need the following lemma which is an important result on the non-uniform estimates of the remainder term in the central limit theorem. See Nagaev (1965), Theorem 3, p.215.

LEMMA 3.2. Let $X, X_n, n\geq 1$ be a sequence of *iid* random variables with $EX = 0$, $EX^2 = 1$ and $E|X|^3 < \infty$. Then for every x,

$$|F_n(x) - \Phi(x)| \leq AE|X|^3/[n^{1/2}(1+|x|)^3], \tag{3.2}$$

where A is an absolute constant.

The following lemma plays an important role in the derivation of main results in this paper.

LEMMA 3.3. Let $X, X_n, n\geq 1$ be a sequence of *iid* random variables with $EX = 0$ and $EX^2 = 1$. If $\varphi(\cdot)$ is a positive nondecreasing function on $[1,\infty)$

then

$$\sum_{n\geq 1} (\varphi^2(n)/n) \sup_{|x|\geq\varphi(n)} |F_n(x) - \Phi(x/\sigma_n)| < \infty. \tag{3.3}$$

PROOF. Let $X_{n,k} = X_k I(|X_k| < \sqrt{n}\,\varphi(n))$, $k = 1,2,\cdots,n$, and $\mu_n = E(XI(|X| < \sqrt{n}\,\varphi(n)))$, $n \geq 1$. Using the information that $EX = 0$ and $EX^2 = 1$, we observe that $1 \geq \sigma_n \to 1$ and $\sqrt{n}\,\mu_n = -\sqrt{n}\,E(XI(|X| \geq \sqrt{n}\,\varphi(n))) \to 0$ as $n \to \infty$. Also note that for every $x \in (-\infty,\infty)$ and $n \geq 1$,

$$|P(n^{-1/2}S_n < x) - P(n^{-1/2}\textstyle\sum_{k=1}^n X_{n,k} < x)| \leq nP(|X| \geq \sqrt{n}\,\varphi(n)).$$

Consequently, by Lemma 3.2,

$$\begin{aligned}|F_n(x) - \Phi(x/\sigma_n)| &\leq |P(n^{-1/2}S_n < x) - P(n^{-1/2}\textstyle\sum_{k=1}^n X_{n,k} < x)|\\
&+ |P(n^{-1/2}\textstyle\sum_{k=1}^n (X_{n,k} - \mu_n)/\sigma_n < (x - \sqrt{n}\,\mu_n)/\sigma_n) - \Phi((x - \sqrt{n}\,\mu_n)/\sigma_n)|\\
&+ |\Phi(x - \sqrt{n}\,\mu_n)/\sigma_n) - \Phi(x/\sigma_n)|\\
&\leq nP(|X| \geq \sqrt{n}\,\varphi(n)) + c(E(|X|^3 I(|X| < \sqrt{n}\,\varphi(n))) + |\mu_n|^3)/[\sqrt{n}(1+|x|^3)])\\
&+ c(\sqrt{n}\,|\mu_n|),\end{aligned}$$

where $c > 0$ is a constant depending only on the distribution of X. By Lemma 3.1, we have

$$\begin{aligned}&\sum_{n\geq 1} (\varphi^2(n)/n) \sup_{|x|\geq\varphi(n)} |F_n(x) - \Phi(x/\sigma_n)|\\
&\leq \sum_{n\geq 1} \varphi^2(n)\, P(|X| \geq \sqrt{n}\,\varphi(n)) + c\sum_{n\geq 1} n^{-3/2}\\
&+ c\sum_{n\geq 1} E(|X|^3 I(|X| < \sqrt{n}\,\varphi(n)))/[n^{3/2}\varphi(n)]\\
&+ c\sum_{n\geq 1} (\varphi^2(n)/\sqrt{n})\, E(|X| I(|X| \geq \sqrt{n}\,\varphi(n))) < \infty.\end{aligned}$$

The form of the following lemma has its origins in Feller (1946), Lemma 1, p.633. Its proof can be obtained using arguments as outlined in Feller (1946) with obvious modifications, and is therefore omitted.

LEMMA 3.4. Let $\varphi(n)$, $n\geq 1$, $\psi(n)$, $n\geq 1$, and a_n, $n\geq 1$ be sequences of positive numbers with $a_n \to 1$ as $n \to \infty$.

(1) Suppose $\psi(n) = O(\varphi(n))$ as $n \to \infty$. Let

$$\varphi_1(n) = 2(\log^+\log^+ n)^{1/2}, \quad \text{if} \quad \varphi(n) \ge 2(\log^+\log^+ n)^{1/2},$$
$$= \varphi(n), \quad \text{otherwise},$$

and

$$\psi_1(n) = 2(\log^+\log^+ n)^{1/2}, \quad \text{if} \quad \psi(n) \ge 2(\log^+\log^+ n)^{1/2},$$
$$= \psi(n), \quad \text{otherwise}.$$

Then the following

$$\sum_{n\ge1} (\psi^2(n)/n\varphi(n)) \exp\{-\varphi^2(n)/2a_n\} < \infty \tag{3.4}$$

and

$$\sum_{n\ge1} (\psi_1^2(n)/n\varphi_1(n)) \exp\{-\varphi_1^2(n)/2a_n\} < \infty \tag{3.5}$$

are equivalent.

(2) Suppose there exists $b \in (0,\infty)$ such that $\psi(n) = 0((\log n)^{b/2})$ as $n \to \infty$. Let

$$\varphi_2(n) = (2(b+2)\log^+\log^+ n)^{1/2}, \quad \text{if} \quad \varphi(n) \ge (2(b+2)\log^+\log^+ n)^{1/2},$$
$$= \varphi(n), \quad \text{otherwise}.$$

Then (3.4) and

$$\sum_{n\ge1}(\psi_1^2(n)/n\varphi_2(n)) \exp\{-\varphi_2^2(n)/2a_n\} < \infty \tag{3.6}$$

are equivalent.

The following lemma is useful in the study of behaviour of $\lambda_{\epsilon,b}$.

LEMMA 3.5. For every $b \ge 0$, we have

$$\lim_{\epsilon\downarrow0} \epsilon^{(2b+1)/2} \sum_{n\ge3} ((\log\log n)^b)/n)\Phi(-\sqrt{(2+\epsilon)\log\log n})$$
$$= 2^{b-1}\sqrt{2/\pi}\,\Gamma(b+(1/2)), \tag{3.7}$$

and

$$\lim_{\epsilon\downarrow0} \sqrt{\epsilon} \sum_{n\ge3} ((\log n)^b/n)\Phi(-\sqrt{(b+1)(2+\epsilon)\log\log n})$$
$$= 2^{-1/2}(b+1)^{-1}. \tag{3.8}$$

PROOF. To prove (3.7), note that

$$\lim_{\epsilon\downarrow0} \epsilon^{(2b+1)/2} \sum_{n\ge3} ((\log\log n)^b/n)\ \Phi(-\sqrt{(2+\epsilon)\log\log n})$$

$$= \lim_{\epsilon\downarrow0} \epsilon^{(2b+1)/2} \int_3^\infty ((\log\log x)^b/x)\ \Phi(-\sqrt{(2+\epsilon)\log\log x})\, dx$$

$$= \lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} u^{2b} e^{u^2} \Phi(-\sqrt{2+\epsilon}\, u)\Big|_3^\infty$$

$$+ \lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \int_3^\infty 2bu^{2b-1} e^{u^2} \Phi(-\sqrt{2+\epsilon}\, u)\, du$$

$$+ \lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \int_3^\infty u^{2b} e^{u^2} \Phi'(-\sqrt{2+\epsilon}\, u) \sqrt{2+\epsilon}\, du$$

The first term above is obviously zero. The convergence of the third term implies that the second term is zero. Thus it remains to be shown that the third term = $2^{b-1}\sqrt{2/\pi}\,\Gamma(b+(1/2))$. But this is clear. To prove (3.8), we first note that

$$\lim_{\epsilon\downarrow 0} \sqrt{\epsilon} \sum_{n\geq 3} ((\log n)^b/n)\, \Phi(-\sqrt{(b+1)(2+\epsilon)\log\log n})$$

$$= \lim_{\epsilon\downarrow 0} \sqrt{\epsilon} \int_3^\infty ((\log x)^b/x)\, \Phi(-\sqrt{(b+1)(2+\epsilon)\log\log x})\, dx.$$

Now we have

$$\int_3^\infty ((\log x)^b/x)\, \Phi(-\sqrt{(b+1)(2+\epsilon)\log\log x})\, dx$$

$$= \int_a^\infty e^{bt^2} \Phi(-\sqrt{(b+1)(2+\epsilon)}\, t)\, 2t\, e^{t^2} dt, \text{ where } a = \sqrt{\log\log 3},$$

$$= -(b+1)^{-1} (\log 3)^{b+1} \Phi(-\sqrt{(b+1)(2+\epsilon)\log\log 3})$$

$$+ \int_a^\infty (2\pi)^{-1/2} (b+1)^{-1} \sqrt{(b+1)(2+\epsilon)}\, e^{-(\epsilon(b+1)/2)t^2} dt$$

$$= 2^{-1/2}(b+1)^{-1} \epsilon^{-1/2} + o(\epsilon^{-1/2}) \text{ as } \epsilon\downarrow 0.$$

This completes the proof.

Theorem 2.2 of Li (1991) is required in the proof of Theorem 2.9 below. We state it in the following lemma adapted to our needs.

LEMMA 3.6. Let $X, X_n, n \geq 1$ be a sequence of *iid* random variables and $b \geq 0$. Then the following are equivalent.

(i) $EX = 0$, $EX^2 = 1$ and $EX^2(\log^+\log^+|X|)^b < \infty$.

(ii) $\sum_{n\geq 3} [(\log\log n)^{b+1}/n]\, P(|S_n| \geq \sqrt{(2+\epsilon)\, n \log\log n})$

$< \infty$ for any $\epsilon > 0$,

$= \infty$ for $-2 < \epsilon < 0$.

We would like to point out that Theorem 2.2 of Li (1991) is concerned with Banach space valued random variables. For the case of real valued random variables, the statement that the infinite series in (ii) above is $< \infty$ for any $\epsilon > 0$ is enough to get (i).

4. PROOFS OF THE MAIN RESULTS

PROOF OF THEOREM 2.1

Using Lemma 3.3, we show that (2.1) and

$$\sum_{n\geq 1} (\psi^2(n)/n)\ \Phi(-\varphi(n)/\sigma_n) < \infty \tag{4.1}$$

are equivalent. Note that $\int_a^\infty e^{-t^2/2}\,dt \sim (1/a)e^{-a^2/2}$ as $a \to \infty$, and $\sigma_n^2 \to 1$ as $n \to \infty$. It now follows that (4.1) and (2.2) are equivalent.

To prove the second part of Theorem 2.1, we note that $0 \leq 1 - \sigma_n^2 \leq 2E(X^2 I(|X| \geq \sqrt{n}\,\varphi(n))) = 0(1/(\log\log n))$ as $n \to \infty$ from $EX = 0$, $EX^2 = 1$, and $E(X^2 I(|X| \geq t)) = 0(1/(\log\log t))$ as $t \to \infty$. In view of Lemma 3.4, we can assume, without loss of generality, that $\varphi(n) = 0((\log\log n)^{1/2})$ as $n \to \infty$. Thus (2.2) and (2.3) are equivalent since $(\varphi^2(n)/2)((1/\sigma_n^2) - 1)$ is nonnegative and bounded.

CONSTRUCTION OF AN EXAMPLE CITED IN REMARK 2

Let

$$g(x) = 3(L_3(x^2) - 1)/(|x|^3 L(x^2)(L_2(x^2) + 3L_3(x^2))^2, \quad \text{if } |x| \geq c_1$$
$$= 0, \quad \text{if } |x| < c_1, \tag{4.2}$$

where $c_1 > 0$ is such that $p_1 = \int_{-\infty}^{\infty} g(x)\,dx \leq 1/2$ and $p_2 = \int_{-\infty}^{\infty} x^2 g(x)\,dx \leq 1/2$. For this choice of c_1, let $c_2 = \sqrt{(1-p_2)/(1-p_1)}$. Let $X, X_n, n \geq 1$ be *iid* random variables such that the distribution function F of X is given by

$$F(x) = (1-p_1)F_1(x) + p_1 F_2(x), \quad -\infty < x < \infty, \tag{4.3}$$

where

$$F_1(x) = 0, \quad \text{if } x < -c_2,$$
$$= 1/2, \quad \text{if } -c_2 \leq x < c_2,$$
$$= 1, \quad \text{if } x \geq c_2,$$

and

$$F_2(x) = (1/p_1)\int_{-\infty}^{x} g(t)\,dt, \quad -\infty < x < \infty. \tag{4.4}$$

From the above choice of F, it now follows that $EX = 0$, $EX^2 = 1$, and for large t, $E(X^2 I(|X| < \sqrt{t})) = 1 - (3L_3(t)/(L_2(t) + 3L_3(t))$. Consequently, $E(X^2 I(|X| \geq t)) = 3L_3(t^2)/(L_2(t^2) + 3L_3(t^2)) = O(L_3(t)/L_2(t))$ as $t \to \infty$. Thus for this particular choice of F,

$$\sum_{n\geq 3} (\sqrt{\log\log n}/n) \exp\{-(\log\log n)/\sigma^2(n\log\log n)\} < \infty \tag{4.5}$$

holds as is easily verified. Thus (2.10) and (2.11) hold.

PROOF OF THEOREM 2.8

By Lemma 3.3, we have, for every $b \in [0,1]$, that

$$\lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^b/n) P(|S_n| \geq \sqrt{(2+\epsilon)n\log\log n})$$

$$= \lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^b/n) 2\Phi(-\sqrt{(2+\epsilon)\log\log n}\,/\sigma_n), \tag{4.7}$$

where $\sigma_n^2 = \sigma^2(2n \log\log n)$, $n\geq 1$. From $EX = 0$, $EX^2 = 1$, and $E(X^2 I(|X| \geq t)) = O(1/(\log\log t))$ as $t \to \infty$, it follows that $0 \leq 1 - \sigma_n = O(1/(\log\log n))$ as $n \to \infty$. Assume, without loss of generality, that $\sigma_1 > 0$. Observe that for $0 < \epsilon < 1$,

$$|\Phi(-\sqrt{(2+\epsilon)\log\log n}\,/\sigma_n) - \Phi(-\sqrt{(2+\epsilon)\log\log n})|$$

$$\leq (1/(\sqrt{2\pi}\,\sigma_1)) \exp\{-((2+\epsilon)\log\log n)/2)\} \sqrt{(2+\epsilon)\log\log n}\,(1-\sigma_n)$$

$$\leq c\,(\log n)^{-(1+(\epsilon/2))} (\log\log n)^{-1/2} \alpha_n, \tag{4.8}$$

where $c = 1/\sigma_1$, and $\alpha_n \to 0$ as $n \to \infty$. Using a similar argument as in the proof of Lemma 3.5, we have

$$\lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} (\log\log n)^{b-1/2}/n(\log n)^{1+\epsilon/2} = 2^b\Gamma(b+(1/2)). \tag{4.9}$$

Hence

$$\lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^{b-1/2}/n(\log n)^{1+\epsilon/2})\alpha_n = 0. \tag{4.10}$$

By (3.7) of Lemma 3.5, we have

$$\lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^b/n) P(|S_n| \geq \sqrt{(2+\epsilon)n\log\log n})$$

$$= \lim_{\epsilon\downarrow 0} \epsilon^{(2b+1)/2} \sum_{n\geq 3} ((\log\log n)^b/n) 2\Phi(-\sqrt{(2+\epsilon)\log\log n})$$

$$= 2^b(2/\pi)^{1/2}\Gamma(b+(1/2)). \quad (4.11)$$

PROOF OF THEOREM 2.9

By Lemma 3.6, it is easy to prove that (2.22) implies $EX = 0$, $EX^2 = 1$, and $EX^2(\log^+\log^+|X|)^{b-1} < \infty$. If $b \geq 2$, it follows that $EX^2I(|X| \geq t) = o(1/\log\log t)$ as $t \to \infty$ since $EX^2(\log^+\log^+|X|)^{b-1} < \infty$. Using the same argument as in the proof of Theorem 2.8,one can write down a proof of Theorem 2.9 using Lemmas 3.3 and 3.5.

PROOF OF THEOREM 2.10

Lemmas 3.3 and 3.5, and ideas in the proof of Theorem 2.8 can be used to write down a proof of Theorem 2.10.

5. MISCELLANY

In this section, we present some remarks derivative of the results presented above. They provide some useful comparisons with some relevant results available in the literature.

(1) Feller (1946) proved the following result. Let X, X_n, $n\geq1$ be a sequence of *iid* random variables with $EX = 0$, $EX^2 = 1$, and $EX^2I(|X| \geq t) = O(1/\log\log t)$ as $t \to \infty$. Let $\varphi(\bullet)$ be a positive nondecreasing function on $[1,\infty)$. Then the following are equivalent.

(i) $P(S_n \geq \sqrt{n}\,\varphi(n)$ infinitely often$) = 0$.

(ii) $\int_1^\infty (\varphi(t)/t)\exp\{-\varphi^2(t)/2\}\,dt < \infty$.

Our results show that (ii) and

(iii) $\sum_{n\geq1} (\varphi^2(n)/n)\,P(|S_n| \geq \sqrt{n}\,\varphi(n)) < \infty$

are equivalent.

(2) We now show that the result presented in the last paragraph of Gafurov (1982) p.143 is not right. We justify our statement as follows. Let

$$\sigma_n^2(\epsilon) = EX^2I(|X| < \sqrt{(2-\epsilon)n\log\log n})$$
$$- (EXI(|X| < \sqrt{(2-\epsilon)n\log\log n}))^2,\ n \geq 3,\ 0 \leq \epsilon < 2.$$

If $EX = 0$, $EX^2 = 1$, and $EX^2I(|X| \geq t) = O(1/\log\log t)$ as $t \to \infty$, then 1 -

$\sigma_n^2(\epsilon) = o(1/\log\log n)$ as $n \to \infty$. Then using an argument similar to the one used in the proof of (4.8), one can show that

$$\Phi(-\sqrt{(2-\epsilon)n\log\log n}/\sigma_n(\epsilon)) \simeq \Phi(-\sqrt{(2-\epsilon)n\log\log n})$$

as $n \to \infty$. By Lemma 3.3,

$$\lim_{n\to\infty}(\textstyle\sum_{k=3}^n((\log\log k)/k)\, P(|S_k| \geq \sqrt{(2-\epsilon)k\log\log k}))/[(\log n)^{\epsilon/2}\sqrt{\log\log n}]$$

$$= \lim_{n\to\infty}\textstyle\sum_{k=3}^n((2\log\log k)/k)\Phi(-\sqrt{(2-\epsilon)k\log\log k}/\sigma_k(\epsilon))/[(\log n)^{\epsilon/2}\sqrt{\log\log n}]$$

$$= \lim_{n\to\infty}(\textstyle\sum_{k=3}^n((2\log\log k)/k)\Phi(-\sqrt{(2-\epsilon)k\log\log k}))/[(\log n)^{\epsilon/2}\sqrt{\log\log n}]$$

$$= \lim_{n\to\infty}(\textstyle\sum_{k=3}^n((2\log\log k)/k)(1/\sqrt{2\pi})(\exp\{-(2-\epsilon)(\log\log k)/2\})) \div$$

$$\sqrt{(2-\epsilon)\log\log k}\,(\log n)^{\epsilon/2}\sqrt{\log\log n}$$

$$\doteq \sqrt{2/(2-\epsilon)\pi}\,\lim_{n\to\infty}\textstyle\sum_{k=3}^n(\sqrt{\log\log k})/[k(\log k)^{1-\epsilon/2}(\log n)^{\epsilon/2}\sqrt{\log\log n}]$$

$$= (1/\epsilon)\sqrt{8/[(2-\epsilon)\pi]}.$$

In the steps above, we have used the fact that $\int_a^\infty \exp\{-x^2/2\}\,dx \sim (1/a)e^{-a^2/2}$ as $a\to\infty$. In a similar fashion, it follows that

$$\lim_{n\to\infty}\textstyle\sum_{k=3}^n((\log\log k)/k)P(|S_k| \geq \sqrt{2k\log\log k})/(\log\log n)^{3/2}$$

$$= \sqrt{1/\pi}\,\lim_{n\to\infty}\textstyle\sum_{k=3}^n\sqrt{\log\log k}/[(k\log k)(\log\log n)^{3/2}]$$

$$= 2/[3\sqrt{\pi}].$$

The gist of the above deliberations can be summarized as follows. Let X, X_n, $n \geq 1$ be a sequence of *iid* random variables with $EX = 0$, $EX^2 = 1$, and $EX^2I(|X| \geq t) = o(1/\log\log t)$ as $t \to \infty$. Then for $0 < \epsilon < 2$

$$\lim_{n\to\infty}\epsilon\sqrt{(2-\epsilon)\pi/8}\,A_n(\epsilon)/[(\log n)^{\epsilon/2}\sqrt{\log\log n}] = 1 \tag{5.1}$$

and

$$\lim_{n\to\infty}(3\sqrt{\pi}/2)\,A_n(0)/(\log\log n)^{3/2} = 1, \tag{5.2}$$

where $A_n(\epsilon) = \sum_{k=3}^n((\log\log k)/k)P(|S_k| \geq \sqrt{(2-\epsilon)k\log\log k})$, $n \geq 3$, and $0 \leq \epsilon < 2$. But (5.1) is not compatible with the limit

$$\lim_{n\to\infty}\sup_{\epsilon>0}|\epsilon^{3/2}A_n(\epsilon)/[(\log n)^{\epsilon/2}(\log\log n)] - 1| = 0$$

given by Gafurov (1982) on the last page.

(3) Let B be a real separable Banach space with norm $||\bullet||$, and B^* its dual space. Let X, X_n, $n\geq 1$ be a sequence of *iid* B-valued random variables. Let K be the unit ball of the Hilbert space determined by the covariance function of X. For a study of the properties of K, see Li (1991) who showed that the following are equivalent.

(i) $EX = 0$, $E||X||^2 < \infty$, and $S_n/\sqrt{2n \log \log n} \to 0$ in probability.

(ii) K is compact in B, and for every $\epsilon > 0$,

$$\sum_{n\geq 3} ((\log \log n)/n) P(\inf_{x\in K} ||S_n/\sqrt{2n \log \log n} - x|| \geq \epsilon) < \infty.$$

(iii) K is compact in B, and for every $\epsilon > 0$,

$$\sum_{n\geq 3} (1/(n \log n)) P(\sup_{k\geq n} \inf_{x\in K} ||S_k/\sqrt{2k \log \log k} - x|| \geq \epsilon) < \infty.$$

It is of considerable interest to compare (i), (ii), and (iii) with (2.15), (2.17), and (2.18), respectively. Ledoux and Talagrand (1986) gave necessary and sufficient conditions that X satisfies the bounded Law of Iterated Logarithm and Compact Law of Iterated Logarithm. Li (1991) pointed out that the following are equivalent.

(iv) $EX = 0$, $E||X||^2/L_2(||X||) < \infty$, $\{f^2(X),\ f \in B^*,\ ||f|| \leq 1\}$ is uniformly integrable, and $S_n/\sqrt{2n \log \log n} \to 0$ in probability.

(v) K is comapct in B, and for every $\epsilon > 0$,

$$\sum_{n\geq 3} (1/n) P(\inf_{x\in K} ||S_n/\sqrt{2n \log \log n} - x|| \geq \epsilon) < \infty.$$

(vi) $P(\{S_n/\sqrt{2n \log \log n},\ n\geq 3\}$ is conditionally compact$) = 1$.

The remarkable result of the equivalence of (iv) and (vi) is due to Ledoux and Talagrand (1986). Note the similarity between (v) and (2.16).

(4) We do not know whether an analogue of Theorem 2.1 holds for Banach space valued random variables.

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