Internat. J. Math. & Math. Sci.
VOL. 15 NO. 3 (1992) 543-552

ON A NONLINEAR DEGENERATE EVOLUTION EQUATION WITH STRONG DAMPING

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(Received June 26, 1990 and revised form October 12, 1990)

ABSTRACT. In this paper we consider the nonlinear degenerate evolution equation with strong damping,

$$(*) \qquad \begin{cases} K(x,t)u_{tt} - \Delta u - \Delta u_t + F(u) = 0 & \text{in} \quad Q = \Omega \times]0,T[\\ u(x,0) = u_0, (Ku')(x,0) = 0 & \text{in} \quad \Omega \\ u(x,t) = 0 & \text{on} \quad \Sigma = \Gamma \times]0,T[\end{cases}$$

where K is a function with $K(x,t) \geq 0$, $K(x,0) = 0$ and F is a continuous real function satisfying

$$(**) \qquad sF(s) \geq 0, \quad \text{for all} \quad s \in \mathbb{R},$$

Ω is a bounded domain of $\mathbb{R}^n$, with smooth boundary Γ. We prove the existence of a global weak solution for (*).

KEY WORDS AND PHRASES. Weak solutions, evolution equation with damping.
1991 AMS SUBJECT CLASSIFICATION CODE. 35 K 22

1. INTRODUCTION.

In this work we study the existence of global weak solutions for the degenerate problem

$$\begin{cases} K(x,t)u'' - \Delta u - \Delta u' + F(u) = 0 \\ u(0) = u_0 \\ (Ku')(0) = 0 \\ u = 0 & \text{in} \quad \Sigma \end{cases} \tag{1.1}$$

in the cylinder $Q = \Omega \times]0,T[$ where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $T > 0$ is an arbitrary real number, Σ is a lateral boundary of Q, F is a continuous real function such that $sF(s) \geq 0$, for all $s \in \mathbb{R}$, $K : Q \to \mathbb{R}$ is a function such that $K(x,t) \geq 0$, $(x,t) \in Q$, $K(x,0) = 0$, Δ is the Laplace operator and $u' = \frac{\partial u}{\partial t}$.

Equation (1.1) is a nonlinear perturbation of the wave equation. For $n = 1$ or $n = 2$, (1.1) governs the motion of a linear Kelvin solid (a bar if $n = 1$ and a plate if $n = 2$) subject to no nonlinear elastic constraints, where $K(x,t)$ is a mass density.

Problem (1.1) with $K(x,t)=1$ without the term $-\Delta u'$ was studied by Strauss [1]. He proves the existence of global weak solutions and the asymptotic behavior as t approaches to infinity. The global weak solutions for the equation

$$K_1(x,t)u'' + K_2(x,t)u' - \Delta u + F(u) = 0 \tag{1.2}$$

with $K_1(x,t)\geq 0, K_1(x,0)\geq \alpha > 0$ and $K_2(x,t)\geq \beta > 0$ was studied by Maciel [2].

Problem (1.2) was also studied by Mello [3] for $F\in C^1(\mathrm{R}), F(0)=0, \int_0^s F(\xi)d\xi \geq 0$, F' dominated by $|s|^p, p>0, K_2$ independent of t non-zero inital data.

In [4] and [5], Larkin studied problem (1.2) with $F(u)=|u|^p u$ and $F(u)=|u'|^p u', p>0$, respectively. In both cases the initial data are zero.

Problem (1.1) with $K(x,t)=1$ was studied by Ang and Dinh [6] with $F\in C^1(\mathrm{R}), F(0)=0$ and $F'\geq -C$ with $C>0$ "small." They proved the existence of global weak solutions and the asymptotic behavior when t approaches to infinity.

We denote by $(,), |\cdot|, ((,)), \|\cdot\|$ the inner and norm of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, and $a(u,v) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx$ represents Dirichlet's form in $H_0^1(\Omega)$.

2. ASSUMPTIONS AND MAIN RESULTS.

We consider the following hypothesis:

(H.1) $F:\mathrm{R}\to\mathrm{R}$ is continuous with $sF(s)\geq 0, \ \forall s\in\mathrm{R}$;

(H.2) $K\in C^1([0,T]:L^\infty(\Omega))$ with $K(x,t)\geq 0, (x,t)\in Q$ and $K(x,0)=0$

(H.3) $\left|\frac{\partial K}{\partial t}\right| \leq \delta + C(\delta)K, \forall\delta>0$ where $C(\delta)$ is a positive constant.

Then we have the following result:

THEOREM 1. Under hypothesis (H.1)-(H.3) if $G(s)=\int_0^s F(\xi)d\xi$ and $u_0\in H_0^1(\Omega), G(u_0)\in L^1(\Omega)$ then there exists a function $u:[0,T]\to L^2(\Omega)$ such that:

$$u\in L^\infty(0,T:H_0^1(\Omega)) \tag{2.1}$$

$$u'\in L^\infty(0,T:H_0^1(\Omega)) \tag{2.2}$$

$$\sqrt{K(x,t)}\,u'\in L^\infty(0,T:L^2(\Omega)) \tag{2.3}$$

$$K'(x,t)u'\in L^2(0,T:H_0^1(\Omega)) \tag{2.4}$$

$$\frac{d}{dt}(Ku',v)-(K'u',v)+a(u,v)+a(u',v)+(F(u),v)=0 \text{ in } \mathcal{D}'(0,T), \forall v\in H_0^1(\Omega) \tag{2.5}$$

$$u(0)=u_0 \tag{2.6}$$

$$(Ku')(0)=0 \tag{2.7}$$

We divide the proof in two parts:

i) We consider F Lipschitzian and derivable except on a finite number of points with $sF(s)\geq 0$, $\forall s\in\mathrm{R}$.

ii) We consider F continuous with $F(s) \geq 0$, $\forall s \in \mathbb{R}$ and approximate F by a sequence $(F_\eta)_{\eta \in \mathbb{N}}$, F_η Lipschitzian and derivable except on a finite number of points with $sF_\eta(s) \geq 0$, $\forall s \in \mathbb{R}$, $\forall \eta \in \mathbb{N}$, with $F_\eta \to F$ uniformly on bounded sets of $\mathbb{R}$.

2.1 LIPSCHITZIAN CASE

We have the following result:

THEOREM 2. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $sF(s) \geq 0$, Lipschitzian and derivable except on a finite number of points. Let be $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ with $G(u_0) \in L^1(\Omega)$, where $G(s) = \int_0^s F(\xi)d\xi$.

Then there exists a unique function $u : Q \to \mathbb{R}$ satisfying:

$$u \in L^\infty(0,T;H_0^1(\Omega)) \tag{2.8}$$

$$u' \in L^\infty(0,T;H_0^1(\Omega)) \tag{2.9}$$

$$u'' \in L^2(0,T;H_0^1(\Omega)) \tag{2.10}$$

$$K(x,t)u'' - \Delta u - \Delta u' + F(u) = 0 \ \text{ in } \ L^2(0,T;H^{-1}(\Omega)) \tag{2.11}$$

$$u(0) = u_0, \qquad u'(0) = 0\,. \tag{2.12}$$

PROOF. Let $(w_\nu)_{\nu \in \mathbb{N}}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ and $V_m = [w_1, \ldots, w_m]$ the subspace generated by the m first vectors of $(w_\nu)_{\nu \in \mathbb{N}}$.

2.1.1 APPROXIMATION PERTURBED PROBLEM

Fix $\varepsilon > 0$ and for each $m \in \mathbb{N}$ consider a function of the form

$$u_{\varepsilon m}(t) = \sum_{j=1}^{m} g_{j\varepsilon m}(t) w_j$$

such that $u_{\varepsilon m}(t)$ is a solution of the problem:

$$((K+\varepsilon)u''_{\varepsilon m}, w) + a(u_{\varepsilon m}, w) + a(u'_{\varepsilon m}, w) + (F(u_{\varepsilon m}), w) = 0, \quad \forall w \in V_m \tag{2.13}$$

$$u_{\varepsilon m}(0) = u_{0m} \to u_0 \quad \text{strongly in} \quad H_0^1(\Omega) \cap H^2(\Omega) \tag{2.14}$$

$$u'_{\varepsilon m}(0) = 0 \tag{2.15}$$

By Caratheodory's theorem, $u_{\varepsilon m}(t)$ exists on $[0, T_{\varepsilon m}[$, $T_{\varepsilon m} < T$. The a priori estimates will allow us to extend $u_{\varepsilon m}(t)$ to whole interval $[0,T]$.

2.1.2 A PRIORI ESTIMATES

I) Consider $w = u'_m(t)$ in (2.13). We obtain

$$\frac{1}{2}\frac{d}{dt}\left[(K, u'^2_{\varepsilon m}) + \varepsilon|u'_{\varepsilon m}|^2 + \|u_{\varepsilon m}\|^2 + 2\int_\Omega G(u_{\varepsilon m})dx\right] + \|u'_{\varepsilon m}\|^2 = \frac{1}{2}\left[\frac{\partial K}{\partial t}, u'^2_{\varepsilon m}\right]$$

Integrating from 0 to $t \leq T_{\varepsilon m}$ and using (H.3) we get:

$$(K, u'^2_{\varepsilon m}) + \varepsilon|u'_{\varepsilon m}|^2 + \|u_{\varepsilon m}\|^2 + 2\int_\Omega G(u_{\varepsilon m})dx + 2\int_0^t \|u'_{\varepsilon m}\|^2 ds$$

$$\leq \|u_{0m}\|^2 + 2\int_\Omega G(u_{0m})dx + \int_0^t \left[\delta|u'_{\varepsilon m}|^2 + C(\delta)(K, u'^2_{\varepsilon m})\right]ds$$

By (2.14) and because $G(u_0) \in L^1(\Omega)$ we have:

$$\int_\Omega G(u_{0m})dx \to \int_\Omega G(u_0)dx \tag{2.16}$$

By (2.14)-(2.16) and Gronwall's inequality, it follows that:

$$(K, u_{\varepsilon m}^{'2}) + \varepsilon |u_{\varepsilon m}^{'}|^2 + \| u_{\varepsilon m} \|^2 + 2 \int_\Omega G(u_{\varepsilon m})dx + (2 - \tilde{C}\delta) \int_0^t \| u_{\varepsilon m}^{'} \|^2 ds \le M$$

where M is a positive constant independent of $\varepsilon, m, t, \tilde{C}$ is a positive constant such that $|v|^2 \le \tilde{C} \|v\|^2$ and $\delta < \min\left\{2, \frac{2}{\tilde{C}}\right\}$. Thus

$$\left(K^{\frac{1}{2}} u_{\varepsilon m}^{'}\right) \text{ is bounded in } L^\infty(0,T; L^2(\Omega)) \tag{2.17}$$

$$(u_{\varepsilon m}) \text{ is bounded in } L^\infty(0,T; H_0^1(\Omega)) \tag{2.18}$$

$$(u_{\varepsilon m}^{'}) \text{ is bounded in } L^2(0,T; H_0^1(\Omega)) \tag{2.19}$$

$$(\sqrt{\varepsilon} u_{\varepsilon m}^{'}) \text{ is bounded in } L^\infty(0,T; L^2(\Omega)) \tag{2.20}$$

II) Since F is Lipschitzian and derivable except on a finite number of points of R, we can differentiate with respect to t to obtain

$$\left[\frac{\partial K}{\partial t} u_{\varepsilon m}^{''}, w\right] + (K u_{\varepsilon m}^{'''}, w) + \varepsilon (u_{\varepsilon m}^{'''}, w) + a(u_{\varepsilon m}^{'}, w) + a(u_{\varepsilon m}^{''}, w) + (F^{'}(u_{\varepsilon m}) u_{\varepsilon m}^{'}, w) = 0 \tag{2.21}$$

Taking $w = u_{\varepsilon m}^{''}(t)$ in (2.21), we get

$$\frac{d}{dt}\left[(K, u_{\varepsilon m}^{''2}) + \varepsilon |u_{\varepsilon m}^{''}|^2 + \| u_{\varepsilon m}^{'} \|^2\right] + 2\| u_{\varepsilon m}^{''} \|^2 + \left[\frac{\partial K}{\partial t}, u_{\varepsilon m}^{''2}\right] + 2(F^{'}(u_{\varepsilon m}, u_{\varepsilon m}^{''}) = 0 \tag{2.22}$$

But

$$2(F^{'}(u_{\varepsilon m}) u_{\varepsilon m}^{'}, u_{\varepsilon m}^{''}) \le 2|F^{'}(u_{\varepsilon m})\| u_{\varepsilon m}^{''}|\, |u_{\varepsilon m}^{'} 1 \le 2\beta |u_{\varepsilon m}^{'}|\, |u_{\varepsilon m}^{''}| \tag{2.23}$$

where β is a positive constant.

Integrating (2.22) from 0 to t and using (2.14)-(2.15), (2.23) and (H.3), it follows that

$$(K, u_{\varepsilon m}^{''2} + \varepsilon |u_{\varepsilon m}^{''}|^2 + \| u_{\varepsilon m}^{'} \|^2 + (2 - \delta) \int_0^t \| u_{\varepsilon m}^{''} \|^2 ds$$

$$\le \varepsilon |u_{\varepsilon m}^{''}(0)|^2 + C_1 \int_0^t \left[\| u_{\varepsilon m}^{'} \|^2 + (K, u_{\varepsilon m}^{''2})\right] ds \tag{2.24}$$

where C_1 is a positive constant.

Now, we are going to estimate the term $\varepsilon\, |u_{\varepsilon m}^{''}(0)|^2$. Consider $t = 0$ in (2.13), and $w = u_{\varepsilon m}^{''}(0)$. Then we get

$$\varepsilon |u_{\varepsilon m}^{''}(0)| \le |\Delta u_{0m}| + |F(u_{0m}) \le C \tag{2.25}$$

where C is a positive constant independent of ε, m and t.

By (2.24), (2.25) and Gronwall's inequality, there exists a positive constant M_1, independent of ε, m and t, such that:

$$(K, u_{\varepsilon m}^{''2}) + \varepsilon \,|\, u_{\varepsilon m}^{'} \,|^2 + \| u_{\varepsilon m}^{'} \|^2 + (2-\delta) \int_0^t \| u_{\varepsilon m}^{''} \|^2 ds \le M_1$$

So,

$$\left(K^{\frac{1}{2}} u_{\varepsilon m}^{'}\right) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \tag{2.26}$$

$$(\sqrt{\varepsilon}\, u_{\varepsilon m}^{''}) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \tag{2.27}$$

$$(u_{\varepsilon m}^{'}) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)) \tag{2.28}$$

$$(u_{\varepsilon m}^{''}) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \tag{2.29}$$

2.1.3 Limits of the Approximated Solutions

From the estimates (2.17)-(2.20) and (2.26)-(2.29), there exists a subsequence of $(u_{\varepsilon m})$, which we still denote by $(u_{\varepsilon m})$, such that:

$$u_{\varepsilon m} \to u \quad \text{weakly - star in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{2.30}$$

$$u_{\varepsilon m}^{'} \to u^{'} \quad \text{weakly - star in} \quad L^\infty(0,T;H_0^1(\Omega)) \tag{2.31}$$

$$u_{\varepsilon m}^{'} \to u^{'} \quad \text{weakly in} \quad L^2(0,T;H_0^1(\Omega)) \tag{2.32}$$

$$\sqrt{\varepsilon}\, u_{\varepsilon m}^{''} \to 0 \quad \text{weakly - star in} \quad L^\infty(0,T;L^2(\Omega)) \tag{2.33}$$

$$K u_{\varepsilon m}^{''} \to K u^{''} \quad \text{weakly - star in} \quad L^\infty(0,T;L^2(\Omega)) \tag{2.34}$$

By (2.18), (2.19) and compactness arguments we conclude that there exists a subsequence of $(u_{\varepsilon m})$, which we still denote by $(u_{\varepsilon m})$, such that:

$$u_{\varepsilon m} \to u \quad \text{strongly in} \quad L^2(0,t;L^2(\Omega)) = L^2(Q)\,. \tag{2.35}$$

Thus,

$$u_{\varepsilon m} \to u \quad \text{almost everywhere in} \quad Q\,.$$

whence, by (H.1) we have

$$F(u_{\varepsilon m}) \to F(u) \quad \text{almost everywhere in} \quad Q \tag{2.36}$$

Since $K \in C^1([0,T;L^\infty(\Omega))$, using (2.32) we obtain

$$(K u_{\varepsilon m}^{''}) \ \textit{is} \text{ bounded in } L^2(Q) \tag{2.37}$$

Then,

$$K u_{\varepsilon m}^{''} \to K u^{''} \quad \text{weakly in} \quad L^2(Q) \tag{2.38}$$

Taking $w = u_{\varepsilon m}(t)$ in (2.13), integrating from 0 to t and using (2.18), (2.19) and (2.37), we get

$$\int_Q F(u_{\varepsilon m}(t)) u_{\varepsilon m}(t) dx dt \le C \tag{2.39}$$

where C is a positive constant.

By (2.36), (2.39) and Strauss's theorem (see Strauss [1]) it follows that

$$F(u_{\varepsilon m}) \to F(u) \quad \text{weakly in} \quad L^1(Q) \tag{2.40}$$

Multiplying (2.13) by $\theta \in L^2(0,T)$, integrating from 0 to t and taking the limit as $m \to \infty$ and $\varepsilon \to 0$, we obtain, by (2.30)-(2.34), (2.38) and (2.40):

$$\left(\int_0^T Ku''\theta dt, \omega\right) + \left(\int_0^T -\Delta u\theta dt, w\right) + \left(\int_0^T -\Delta u'\theta dt, w\right) + \left(\int_0^T F(u)\theta dt, w\right) = 0, \qquad \forall w \in V_m .$$

Since the V_m is dense in $H_0^1(w)$, the above equation is true for all $w \in H_0^1(\Omega)$ and the proof of (2.11) is complete.

The initial conditions (2.12) are obtained from (2.30)-(2.32).

The uniqueness is trivial because F is Lipschitzian.

3. PROOF OF THEOREM 1

We first approximate u_0 by a sequence of bounded functions $(u_{0j})_{j\in N}$ in $H_0^1(\Omega)$. In fact, let's consider

$$\beta_j(s) = \begin{cases} s & \text{if } |s| \le j \\ j & \text{if } s > j \\ -j & \text{if } s < -j \end{cases}$$

it follows by Kinderlher-Stampacchia [8] that $\beta_j(u_0) = u_{0j} \in H_0^1(\Omega)$, $\forall_j \in N$, $u_{0j} \to u_0$ strongly in $H_0^1(\Omega)$ and $\| u_{0j} \| \le \| u_0 \|$.

Let $(F_\eta)_{\eta\in N}$ be a sequence of functions defined by:

$$F_\eta(s) = \begin{cases} (-\eta)\left[G\left(s-\frac{1}{\eta}\right) - G(s)\right] & \text{if } -\eta \le s \le -\frac{1}{\eta} \\ (\eta)\left[G\left(s+\frac{1}{\eta}\right) - G(s)\right] & \text{if } \frac{1}{\eta} \le s \le \eta \\ \text{linear by parts} & \text{on } -\frac{1}{\eta} \le s \le \frac{1}{\eta} \text{ with } F_\eta(0) = 0 \\ \text{appropriated constants} & \text{for } |s| \ge \eta \end{cases}$$

where

$$G(s) = \int_0^s F(\xi) d\xi .$$

It follows, by Strauss [1], Cooper-Medeiros [7] that F_η is Lipschitzian, for each $\eta \in N$, $sF_\eta(s) \ge 0$ and $F_\eta \to F$ uniformly on the bounded sets of R. If we consider $G_\eta(s) = \int_0^s F_\eta(\xi)d\xi$ we get, $G_\eta(0) = 0$ and $sG_\eta(s) \ge 0$, $\forall s \in R$, $\forall \eta \in N$.

Let $\phi_{\mu j} \in \mathcal{D}(\Omega)$ such that

$$\phi_{\mu j} \to u_{0j} \quad \text{strongly in} \quad H_0^1(\Omega) \quad \text{as} \quad \mu \to \infty \tag{3.1}$$

It follows by Theorem 2 that there exists a unique function $u_{\mu j \eta}$ satisfying the conditions:

$$u_{\mu j\eta} \in L^{\infty}(0,T;H_0^1(\Omega)) \tag{3.2}$$

$$u'_{\mu j\eta} \in L^{\infty}(0,T;H_0^1(\Omega)) \tag{3.3}$$

$$u''_{\mu j\eta} \in L^{2}(0,T;H_0^1(\Omega)) \tag{3.4}$$

$$Ku'_{\mu j\eta} - \Delta u_{\mu j\eta} - \Delta u'_{\mu j\eta} + F(u_{\mu j\eta}) = 0 \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \tag{3.5}$$

$$u_{\mu j\eta}(0) = \phi_{\mu j}\,, \qquad u'_{\mu j\eta}(0) = 0 \tag{3.6}$$

We now prove that $u_{\mu j\eta}$ converges to u and u is the solution of Theorem 1. Taking the inner product of (3.5) by $u'_{\mu j\eta}$ and integrating from 0 to $t \le T$, we have:

$$(K, u'^2_{\mu j\eta}) + \| u_{\mu j\eta}\|^2 + 2\int_\Omega G_\eta(u_{\mu j\eta})dx + 2\int_0^t \| u'_{\mu j\eta}\|^2 ds$$

$$\le \| \phi_{\mu j}\|^2 + 2\int_\Omega G_\eta(\phi_{\mu j})dx + \int_0^T [\delta\, | u'_{\mu j\eta}|^2 + C(\delta)(K, u'^2_{\mu j\eta})]\; ds\,. \tag{3.7}$$

Since u_{0j} is bounded in Ω, fixing j, we obtain:

$$F_\eta(u_{0j}(x)) \to F(u_{0j}(x)) \quad \text{uniformly in} \quad \Omega \quad \text{as} \quad \eta \to \infty, \tag{3.8}$$

$$\int_\Omega G_\eta(\phi_{\mu j})dx \to \int_\Omega G_\eta(u_{0j})dx \quad \text{if} \quad \mu \to +\infty\,. \tag{3.9}$$

and

$$(G_\eta(u_{0j}(x)) \to G(u_{0j}(x)) \quad \text{uniformly in} \quad \Omega \quad \text{as} \quad \eta \to \infty\,. \tag{3.10}$$

Whence, there exists a subsequence $(G_{\eta j})_{j \in \mathbb{N}}$ of $(G_\eta)_{\eta \in \mathbb{N}}$, which we still denote by $(G_j)_{j \in \mathbb{N}}$, such that

$$\int_\Omega | G_j(u_{0j}) - G(u_{0j})\,|\, dx \to 0 \quad \text{if} \quad j \to \infty\,. \tag{3.11}$$

Moreover, $G(u_{0j}) \to G(u_0)$ a.e. in Ω and $G(u_{0j}) \le G(u_0)$. Since $G(u_0) \in L^1(\Omega)$, by the Lebesgue's dominated convergence theorem we get

$$\int_\Omega | G(u_{0j}) - G(u_0)\,|\, dx \to 0 \quad \text{as} \quad j \to \infty, \tag{3.12}$$

Thus, by (3.11) and (3.12), it follows that

$$\int_\Omega G_j(u_{0j})dx \to \int_\Omega G(u_0)dx \quad \text{as} \quad j \to \infty \tag{3.13}$$

By (3.7), (3.9), (3.13) and Gronwall's inequality, we have

$$(K, u'^2_{\mu jj}) + \| u_{\mu jj}\|^2 + 2\int_\Omega G_j(u_{\mu jj}dx + (2 - C\delta)\int_0^t \| u'_{\mu jj}\|^2 dx \le C\,, \tag{3.14}$$

where C is a positive constant independent of μ, j and t.

Then, there exists a subsequence of $(u_{\mu jj})_{\mu \in \mathbb{N}}$, which we denote by $(u_{\mu j})_{\mu \in \mathbb{N}}$, and functions u_j and u such that

$$\begin{vmatrix} K^{1/2}u'_{\mu j} \to K^{1/2}u'_j & \text{weakly - star in} & L^\infty(0,T;L^2(\Omega)) \\ u_{\mu j} \to u_j & \text{weakly - star in} & L^\infty(0,T;H_0^1(\Omega)) \\ u'_{\mu j} \to u'_j & \text{weakly in} & L^2(0,T;H_0^1(\Omega)) \end{vmatrix} \tag{3.15}$$

as $\mu \to \infty$, and

$$\begin{vmatrix} K^{1/2}u'_j \to K^{1/2}u' & \text{weakly - star in} & L^\infty(0,t;L^2(\Omega)) \\ u_j \to u & \text{weakly - star} & L^2(0,T;H_1^1(\Omega)) \\ u'_j \to u' & \text{weakly in} & L^2(0,T;H_0^1(\Omega)) \end{vmatrix} \tag{3.16}$$

as $j \to \infty$.

Moreover, by (H.2) and $K^{1/2}u'_{\mu jj} \in L^\infty(0,T;L^2(\Omega))$ if follows that:

$$Ku'_{\mu jj} \in L^\infty(0,T;L^2(\Omega)) \tag{3.17}$$

and

$$Ku'_{\mu jj} \to Ku'_j \text{ weakly - star in } L^\infty(0,T;L^2(\Omega)) \tag{3.18}$$

as $\mu \to \infty$, and

$$Ku'_j \to Ku' \text{ weakly - star in } L^\infty(0,T;L^2(\Omega)) \tag{3.19}$$

as $j \to \infty$.

By (H.2), (H.3), (3.3) and (3.4) we get

$$(Ku')' \in L^2(Q). \tag{3.20}$$

So, by (3.18) and (3.19) we have that $Ku'_{\mu jj}$ is weakly continuous of $[0,T]$ in $L^2(\Omega)$. Moreover, $(Ku'_{\mu jj})(T)$ is bounded in $L^2(\Omega)$.

Multiplying (3.5) by $u_{\mu jj}(t)$ and integrating from 0 to T, we obtain

$$\int_0^T (F_j(u_{\mu jj}), u_{\mu jj})dt \le \int_0^T \| u_{\mu jj}\|^2 dt + \int_0^T \left|\left(\frac{\partial K}{\partial t}u'_{\mu jj}, u_{\mu jj}\right)\right| dt + \int_0^T \left|\left(\frac{\partial K}{\partial t}u'_{\mu jj}, u'_{\mu jj}\right)\right| dt$$

$$+ \int_0^T |a(u'_{\mu jj}, u_{\mu jj})|\, dt + |((Ku'_{\mu jj})(T), u_{\mu jj}(T))| + |((Ku'_{\mu jj})(0), u_{\mu jj}(0))|. \tag{3.21}$$

Using (H.2), (H.3) and a priori estimates, it follows that

$$\int_Q F_j(u_{\mu jj})u_{\mu jj} dxdt \le C, \tag{3.22}$$

C positive constant independent of μ, j and t.

Just as in Theorem 1, we prove that:

$$F_j(u_{\mu jj}) \to F(u_j) \text{ a.e. in } Q \text{ as } \mu \to \infty \tag{3.23}$$

whence by (3.22), (3.23) and Strauss's theorem (see Strauss [1]), we have

$$F_j(u_{\mu jj}) \to F_j(\mu_j) \text{ weakly in } L^1(Q) \text{ as } \mu \to \infty. \tag{3.24}$$

Also, by (H.3) and (3.14) it follows that

$$(K'u'_{\mu jj}) \text{ is bounded in } L^2(Q). \tag{3.25}$$

So

$$K'u'_{\mu jj} \to K'u'_j \text{ weakly in } L^2(Q) \text{ as } j \to \infty \tag{3.26}$$

and

$$K'u'_j \to K'u' \text{ weakly in } L^2(Q) \text{ as } j \to \infty. \tag{3.27}$$

Multiplying (3.5) by $w = v\theta$ with $v \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}(0,T)$, integrating from 0 to T, taking the limit as $\mu \to \infty$, and using (3.15), (3.16), (3.18), (3.24) and (3.26) we get

$$\frac{d}{dt}(Ku'_j, v) - (K'u_j, v) + a(u_j, v) + a(u'_j, v) + (F_j(u_j), v) = 0 \quad \forall v \in H_0^1(\Omega) \text{ in } \mathcal{D}'(0,T). \tag{3.28}$$

$$u_j(0) = u_{0j} \text{ and } (Ku'_j)(0) = 0. \tag{3.29}$$

Moreover, by (3.24), it follows that:

$$F_j(u_j) \to F(u) \text{ weakly in } L^1(Q). \tag{3.30}$$

Taking the limit in (3.28) as $j \to \infty$ and using (3.16), (3.19), (3.27) and (3.30) we prove (2.1)-(2.5) in theorem 1.

It's not difficult to check that $u(0) = u_0$ and $(Ku')(0) = 0$.

REMARK. Replacing (H.2) by (H.2)' $K \in C^1([0,T]:L^\infty(\Omega))$ with $K(x,0) \geq \alpha > 0$,

$$K(x,t) \geq 0, \quad (x,t) \in Q.$$

we get with the same arguments

THEOREM 3. Under hypotheses (H.1), (H.2)', (H.3) if $G(s) = \int_0^s F(\xi)d\xi$ and $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega), G(u_0) \in L^1(\Omega)$, then there exists a function $u : [0,T] \to L^2(\Omega)$ such that

$$u \in L^\infty(0,T;H_0^1(\Omega))$$

$$u' \in L^2(0,T;H_0^1(\Omega))$$

$$\sqrt{K}\,u' \in L^\infty(0,T;L^2(\Omega))$$

$$K'u' \in L^2(0,T;H_0^1(\Omega))$$

$$Ku'' - \Delta u - \Delta u' + F(u) = 0 \text{ in the weak sense in } Q$$

$$u(0) = u_0$$

$$u'(0) = u_1$$

ACKNOWLEDGEMENT. This research was completed while the second author was visiting LNCC/CNPq in a Post-Doctoral Program during 1989.

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