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SPACES OF COMPACT OPERATORS WHICH ARE M-IDEALS IN L(X,Y)

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ABSTRACT. Suppose X and Y are reflexive Banach spaces. If $K(X,Y)$, the space of all compact linear operaters from X to Y is an M-ideal in $L(X,Y)$, the space of all bounded linear operators from X to Y, then the second dual space $K(X,Y)^{**}$ of $K(X,Y)$ is isometrically isomorphic to $L(X,Y)$.

KEY WORDS AND PHRASES. Compact operators, M-ideal, dual space, projective tensor product.

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1. INTRODUCTION

It is well known that if X and Y are reflexive Banach spaces one of which satisfies the approximation property then the second dual space $K(X,Y)^{**}$ of $K(X,Y)$, the space of compact linear operators from X to Y, is isometrically isomorphic to $L(X,Y)$, the space of bounded linear operators from X to Y (Diestel [1, p.17]). Harmand and Lima [2] proved that if X is a reflexive Banach space and $K(X)$ is an M-ideal in $L(X)$ then $K(X)^{**}$ is isometrically isomorphic to $L(X)$.

The purpose of this paper is to generalize the result of Harmand and Lima to the case of $K(X,Y)$ and $L(X,Y)$ by modifying their proof. In Theorem 3.3 we will prove that if X and Y are reflexive Banach spaces and $K(X,Y)$ is an M-ideal in $L(X,Y)$ then $K(X,Y)^{**}$ is isometrically isomorphic to $L(X,Y)$.

2. NOTATIONS AND PRELIMINARIES.

Let X and Y be Banach spaces. $X \simeq Y$ means that X and Y are isometrically isomorphic. $L(X,Y)$ (resp. $K(X,Y)$) will denote the space of all bounded linear operators (resp. compact linear operators) from X to Y. If $X = Y$, then we simply write $L(X)$ (resp. $K(X)$). X^* will denote the dual space of X and we will write $< x, x^* >$ for the action of $x^* \in X^*$ on $x \in X$ instead of $x^*(x)$. B_X will denote the closed unit ball of X.

A closed subspace J of a Banach space X is called an L-summand if there exists a projection P on X such that $PX = J$ and $\|x\| = \|Px\| + \|x - Px\|$ for every x in X. In this case we write $X = J \oplus_1 J'$ where $J' = (I-P)X$. A closed subspace J of a Banach space X is called an M-ideal in X if J°, the annihilator of J in X^*, is an L-summand in X^*.

Let $X \hat{\otimes} Y$ be the projective tensor product of Banach spaces X and Y. If $u \in X \hat{\otimes} Y$, then there exist sequences (x_i) in X and (y_i) in Y such that $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, with $\sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < \infty$. Moreover, we have $\|u\| = \inf \sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < \infty$, the infimum being taken over all representations $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$ and $\sum_{i=1}^{\infty} \|x_i\| \, \|y_i\| < \infty$. (Diestel and Uhl [3, p. 227]).

Let Z be another Banach space and $T \in L(X,Z)$. We define $Tu = \sum_{i=1}^{\infty} x_i \otimes Ty_i$ for $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in X \hat{\otimes} Y$. Then $Tu \in X \hat{\otimes} Z$ and $\|Tu\| \le \|T\| \, \|u\|$. If $u = \sum_{i=1}^{\infty} x_i^* \otimes x_i \in X^* \hat{\otimes} X$ with $\sum_{i=1}^{\infty} \|x_i^*\| \, \|x_i\| < \infty$, the map $u \to tr(u) = \sum_{i=1}^{\infty} < x_i, x_i^* >$ defines a bounded linear functional on $X^* \hat{\otimes} X$ with norm no larger than 1.

THEOREM 2.1 (Diestel and Uhl [3], Shatten [4]). Let X and Y be Banach spaces. The Banach space $L(X,Y^*)$ is isometrically isomorphic to $(Y \hat{\otimes} X)^*$ and under this identification $T \in L(X,Y^*)$ act on $u \in Y \hat{\otimes} X$ by $< u,T > = tr(Tu)$.

THEOREM 2.2 (Feder and Sapher [5]). Let X and Y be Banach spaces. If either X^{**} or Y^* has the Radon-Nikodym Property, then map $V: Y^* \hat{\otimes} X^{**} \to K(X,Y)^*$ defined by $< T, V(u) > = tr(T^{**}u)$ for $T \in K(X,Y)$ and $u \in Y^* \hat{\otimes} X^{**}$ is a quotient map.

3. SPACES OF COMPACT OPERATORS

Harmand and Lima [2] proved that if $K(X)$ is an M-ideal in $L(X)$ then there exists a net (T_α) in $B_{K(X)}$ such that

(i) $T_\alpha x \to x$ for all $x \in X$

(ii) $T_\alpha^* x^* \to x^*$ for all $x^* \in X^*$

(iii) $\| T_\alpha - I \| \to 1$.

In the case of $K(X,Y)$ and $L(X,Y)$, we have the following analogue which also plays a key role in the proof of our main result (Theorem 3.3).

THEOREM 3.1. If X and Y are Banach spaces and $K(X,Y)$ is an M-ideal in $L(X,Y)$, then for each T in $B_{L(X,Y)}$ there is a net (T_α) in $B_{K(X,Y)}$ such that

(i) $T_\alpha x \to Tx$ for all $x \in X$

(ii) $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$.

PROOF. Suppose $K(X,Y)$ is an M-ideal in $L(X,Y)$. Then we can write $L(X,Y)^* = K(X,Y)^\circ \oplus_1 J$ for some subspace J of $L(X,Y)^*$.

The map $\phi \to \phi + K(X,Y)^\circ$ defines an isometry from J onto $L(X,Y)^*/K(X,Y)^\circ$ and the map $\phi + K(X,Y)^\circ \to \phi|_{K(X,Y)}$ defines an isometry from $L(X,Y)^*/K(X,Y)^\circ$ onto $K(X,Y)^*$ (Rudin [6, p.91]). Hence the map $\phi \to \phi|_{K(X,Y)}$ gives an isometry from J onto $K(X,Y)^*$.

Let Q be the projection on $L(X,Y)^*$ with the range J. Then $\phi \in L(X,Y)^*$ is in the range of Q if and only if the restriction of ϕ to $K(X,Y)$ has the same norm as ϕ. If $T \in L(X,Y) \subseteq L(X,Y)^{**}$ with $\| T \| \le 1$, then for $\phi \in K(X,Y)^\circ$ we have $(Q^*T)\phi = TQ(\phi) = 0$ thus $Q^*T \in K(X,Y)^{\circ\,\circ} = J^* = K(X,Y)^{**}$. Since $Q^*T \in K(X,Y)^{**}$ and $\| Q^*T \| \le 1$, by the Goldstein's theorem there is a net (T_α) in $B_{K(X,Y)}$ such that

$$T_\alpha \to Q^*T \text{ in the weak}^*\text{-topology on } J^* = K(X,Y)^{**}.$$

We claim that $T_\alpha x \to Tx$ for all $x \in X$ and $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$. For $x^{**} \in X^{**}$ and $y^* \in Y^*$, define $\phi_{x^{**} \otimes y^*} \in L(X,Y)^*$ by

$$< A, \phi_{x^{**} \otimes y^*} > = < A^* y^*, x^{**} > .$$

Then we can easily see that $\phi_{x^{**} \otimes y^*} \in J = K(X,Y)^*$ and hence

$$< T_\alpha^* y^*, x^{**} > \to < T^* y^*, x^{**} > .$$

By the weak*- compactness of $B_{X^{**}}$ we get that

$$T_\alpha^* \to T^* y^* \text{ for all } y^* \in Y^*.$$

Similarly, for $y^* \in Y^*$ and $x \in X$ the functional $\psi_{y^* \otimes x}$ on $L(X,Y)$ defined by $< A, \psi_{y^* \otimes x} > = < Ax, y^* >$ for $A \in L(X,Y)$ is in the range of Q and hence $T_\alpha x \to Tx$ for all $x \in X$.

The following proposition is essentially due to Harmand and Lima [2] who treated a special case $X = Y$.

PROPOSITION 3.2. Let X and Y be Banach spaces and V the map defined in Theorem 2.2. If $K(X,Y)$ is an M-ideal in $L(X,Y)$, then $T^{**} \in (kerV)^\circ$ for every $T \in L(X,Y)$.

PROOF. Recall that by Theorem 2.1 we have $(Y^* \hat{\otimes} X^{**})^* \simeq L(X^{**}, Y^{**})$ and under this identification $S \in L(X^{**}, Y^{**})$ acts on $u \in Y^* \hat{\otimes} X^{**}$ by $< u, S > = tr(Su)$.

Let $T \in L(X,Y)$, $\| T \| \le 1$. By Theorem 3.1 there is a net (T_α) in $B_{K(X,Y)}$ such that $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$. Let $u = \sum_{i=1}^{\infty} y_i^* \otimes x_i^{**} \in kerV$ with $\sum_{i=1}^{\infty} \| y_i^* \| \, \| x_i^{**} \| < \infty$. We may assume that $\| x_i^{**} \| \le 1$ for all i and

$\| y_i^* \| \to 0$. Then we get

$$\begin{aligned} 0 &= \langle T_\alpha, V(u) \rangle \\ &= tr(T_\alpha^{**} u) \\ &= \sum_{i=1}^{\infty} \langle y^*, T_\alpha^{**} x_i^{**} \rangle \\ &= \sum_{i=1}^{\infty} \langle T_\alpha^* y_i^*, x_i^{**} \rangle \\ &\to \sum_{i=1}^{\infty} \langle T^* y_i^*, x_i^{**} \rangle \\ &= tr(T^{**} u) \\ &= \langle u, T^{**} \rangle . \end{aligned}$$

Thus $T^{**} \in (kerV)^\circ$.

THEOREM 3.3. If X and Y are reflexive Banach spaces and $K(X,Y)$ is an M-ideal in $L(X,Y)$ then $K(X,Y)^{**}$ is isometrically isomorphic to $L(X,Y)$.

PROOF. Since X and Y are reflexive, X and Y^* have the Radon-Nikodym property and hence by Theorem 2.2 the map $V: Y^* \hat{\otimes} X^{**} \to K(X,Y)^*$ defined by

$$\langle T, V(U) \rangle = tr(T^{**}u) \quad \text{for } u \in Y^* \hat{\otimes} X^{**}, \ T \in K(X,Y)$$

is a quotient map. Thus $V^*: K(X,Y)^{**} \to (Y^* \hat{\otimes} X^{**})^*$ is an isometry with the range $(kerV)^\circ$ and hence we have

$$\begin{aligned} K(X,Y)^{**} &\simeq (KerV)^* \\ &\subseteq (Y^* \hat{\otimes} X^{**})^* \\ &\simeq L(X^{**}, Y^{**}) \\ &= L(X,Y). \end{aligned}$$

Since X and Y are reflexive, $T = T^{**}$ for all $T \in L(X,Y)$ and by Proposition 3.2 $(Y^* \hat{\otimes} X^{**})^* \subseteq (KerV)^\circ$. Thus $K(X,Y)^{**} \simeq L(X,Y)$.

Recall that for $1 \le p \le \infty$ the l_p-sum $(\Sigma X_n)_p$ of a sequence of (X_n) of Banach spaces is the Banach space of all sequences (x_n) with $x_n \in X_n$ and with the norm $\| (x_n) \| = (\Sigma \| x_n \|^p)^{1/p} < \infty$.

COROLLARY 3.4. Suppose X and Y are closed subspaces of $(\Sigma X_n)_p$ and $(\Sigma Y_n)_q$ ($1 < p \le q \le \infty$, $dimX_n < \infty$, $dimY_n < \infty$), respectively. If $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology, then $K(X,Y)^{**} \simeq L(X,Y)$.

PROOF. X and Y are reflexive and $K(X,Y)$ is an M-ideal in $L(X,Y)$ (Cho [7]).

REMARK. If X and Y are as in Corollary 3.4 and either X or Y satisfies the compact approximation property, then $K(X,Y)$ is dense in $L(X,Y)$ in the strong operator topology [7] and hence $K(X,Y)^{**} \simeq L(X,Y)$.

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