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ON THE LOWER SEMI-CONTINUITY OF THE SET VALUED METRIC PROJECTION

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ABSTRACT. The lower semi-continuity of best approximation operators from Banach lattices on to closed ideals is investigated. Also the existence of best approximation to sub-function modules of function modules is proved. The order intersection properties of cells are studied and used to prove the above results.

KEY WORDS AND PHRASES. Lower semi-continuity, metric projections, continuous selections, finite order intersection properties for cells, Banach lattices, function modules.

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1. INTRODUCTION AND DEFINITIONS.

During the last 20 years a series of papers have been concerned with continuity of the set valued metric projection from normed linear space on to proximinal linear subspace. Throughout this paper we deal with approximation of elements of the Banach lattice E by elements of a closed ideal G. For $x \in E$ we shall denote by $d(x,G) = \inf_{g \in G} \|x - g\|$ the distance from x to G. Every $g_o \in G$ for which $\|x - g_o\| = d(x,G)$ is called a best approximation of x in G. We shall denote by

$$P_G(x) = \left\{ g \in G : \|x - g\| = d(x,G) \right\} \tag{1.1}$$

the set of all best approximation of x by elements of G. The set valued mapping $P : E \longrightarrow 2^G$, which associates with each element x of E its (possibly empty) set of nearest elements in G, is called the metric projection of E on to G (or the metric projection of E associated with G).

In recent years, there has been considerable interest in continuous mapping $s : E \longrightarrow G$ with the property that $s(x) \in P_G(x)$ for every $x \in E$. Such a mapping, if exist, is called a continuous selection for the metric projection P_G. The available results on continuous selections for the metric projection P_G deal primarily with there existence, which follows directly from the lower semi-continuity of P_G according to a result of E. Michael [8].

For set valued mappings, various concepts of continuity are defined as follows.

DEFINITION 1.1.

(I) The metric projection P is lower semi-continuous (l.s.c.) if the set $\left\{ x \in E : P_G(x) \cap U \neq \emptyset \right\}$ is open for each open set U in G.

(II) The metric projection P_G is upper semi-continuous (u.s.c.) if the set $\left\{ x \in E : P_G(x) \cap C \neq \emptyset \right\}$ is closed for each closed set C in G.

(III) Finally, P_G is continuous (in the Hausdorff metric topology) if $x_n \longrightarrow x$ implies

$$\max \left\{ \sup \left\{ d(g, P_G(x_n)) : g \in P_G(x) \right\}, \sup \left\{ d(h, P_G(x)) : h \in P_G(x_n) \right\} \right\} \longrightarrow 0.$$

Continuity in Hausdorff metric topology can be easily shown to imply (l.s.c.). If $P_G(x)$ is compact for each x in E then the Hausdorff metric topology implies (u.s.c.). Finally, if G is boundedly compact (G intersects every closed sphere in a compact set) then P_G is always (u.s.c.) and Hausdorff metric topology is equivalent to (l.s.c.).

The metric projection is (l.s.c.) or (u.s.c.) only for restricted class of subspaces. For example, I. Singer [12] has proved that the metric projection associated with an approximatively compact subset G of a normed linear space E is (u.s.c.). Hence, in particular P_G is (u.s.c.) if G is a linear subspace of finite dimension. But even if G is a linear subspace of finite dimension P_G may fail to be (l.s.c.) as A. J. Lazar, P. P. Morris and D. E. Wulbert have shown in [7].

A subspace G is proximinal if $P_G(x) \neq \emptyset$ for each $x \in E$.

DEFINITION 1.2. A normed linear lattice is a normed linear space which is also a vector lattice, in which the order and the norm arerelated as follows : $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. If the space is complete, it is called Banach lattice.

DEFINITION 1.3. A Banach lattice E has the order intersection property if whenever elements x, y and two collections $\left\{ B(x_a, r_a) \right\}_{a \in A}$, $\left\{ B(y_b, s_b) \right\}_{b \in B}$ of cells are given in E satisfying

(1) $x_a \leq x \leq y \leq y_b$ for each a,b

(2) $B(x_a, r_a) \cap B(y_b, s_b) \neq \emptyset$ for each a,b

(3) $x \in \bigcap_a B(x_a, r_a)$ and $y \in \bigcap_b B(y_b, s_b)$

then

$$[x,y] \cap \left[\bigcap_a B(x_a, r_a) \right] \cap \left[\bigcap_b B(y_b, s_b) \right] \neq \emptyset .$$

A Banach lattice E has the f.o.i.p. (finite order intersection property) if the above property holds when the index sets A and B are finite. Also : It is known that f.o.i.p., the splitting property and f.o.i.p. in the case $|A| = |B| = 1$ are equivalent .

We now list some examples of Banach lattices with the f.o.i.p.

(1) If E is an (AL)-space, then it has the f.o.i.p.

(2) Every injective Banach lattice (and any closed ideal of it) has the f.o.i.p.

(3) The space $\mathcal{C}(X)$ has the f.o.i.p. if and only if X is Stonian.

(4) Any (AM)-space has the f.o.i.p.

For the proofs and general treatment of injective Banach lattices and Banach lattices that have the f.o.i.p., we refer the reader to D. Cartwright [3] .

The following fundamental properties of meet, join and the absolute value will be used freely in the sequel

(1) $x + y = x \vee y + x \wedge y$.

(2) $|x + y| \leq |x| + |y|$.

(3) $\big||x| - |y|\big| \leq |x - y|$.

(4) $|x - (x \wedge y)| = |(x \vee y) - y|$.

(5) $|x| \wedge |y| = 0$ if and only if $|x| \vee |y| = |x| + |y|$.

(6) $x \wedge (y + z) \leq (x \wedge y) + (x \wedge z)$ for all $x,y,z \geq 0$.

(7) $|x - y| = x \vee y - x \wedge y$.

We will prove the following results:

Let E be a Banach lattice with the finite order intersection property, G a closed ideal of E. For each x in E define

$$P_G^0(x) = \left\{ g \in P_G(x) : g^{\mp} \leq x^{\mp}, \text{ and } g^{\mp} \in P_G(x^{\mp}) \right\}. \tag{1.2}$$

Then the set valued mapping P_G^0 is lower semi-continuous.

Let E_∇ be a function module and G_∇ a sub-$\mathcal{C}(T)$-module of E_∇. If for each t in T, E_t is a Banach lattice with the f.o.i.p. and the fiber G_t is an ideal in E_t (where $G_t = \left\{\gamma(t) : \gamma \in G_\nabla\right\}$). Then G_∇ is proximinal.

MOTIVATION. It has been shown in [11] that closed ideals in injective Banach lattices are always proximinal and the metric projections associated with ideals are always (l.s.c.). These results and the fact that injective Banach lattices have the splitting property lead us to think about the above results do hold not only in injective Banach lattices but also in Banach lattices that have the f.o.i.p. The existence of best approximation to ideals in such spaces has been proved (see F. A. Sejeeni [11]).

2. METRIC PROJECTIONS.

In order to prove the results stated in the introduction, we need the following partial results which perhaps are interesting in themselves.

PROPOSITION 2.1. Let E be a Banach lattice, G a proximinal ideal in E. Then for each positive element x of E, the following hold;

(1) $d(x,G) = d(x \vee g, G)$ for each positive g in G .

(2) $x \wedge g \in P_G(x)$ and $g \in P_G(x \vee g)$ for each positive $g \in P_G(x)$.

(3) $(f \vee g) \wedge x \in P_G(x)$ $\forall f, g \geq 0$ such that $f \in G$ and $g \in P_G(x)$.

PROOF. (1) Let g_o respectively (h_o) be an arbitrary but fixed element of $P_G(x)$ ($P_G(x \vee g)$). Write $x = x \vee g + x \wedge g - g$. Then

$$d(x,G) = \|x - g_o\| = \|x\vee g + (x\wedge g - g - g_o)\| \geq d(x\vee g,G) \tag{2.1}$$

(since $-(x\wedge g - g - g_o) \in G$). Similarly,

$$d(x\vee g,G) = \|(x\vee g) - h_o\| = \|x - [(x\wedge g) - g + h_o]\|$$
$$\geq \|x - g_o\| = d(x,G) \tag{2.2}$$

(since $[(x \wedge g) - g + h_o]$ in G) and the result follows from (2.1) and (2.2) together.

(2) $|x - (x \wedge g)| = x - (x \wedge g) \le x \vee g - x \wedge g = |x - g|$ and then $\|x - (x \wedge g)\| \le \|x - g\|$ which in turn implies that $x \wedge g \in P_G(x)$ (since $x \wedge g \in G$). Similarly,

$|(x \vee g) - g| = |x - (x \wedge g)|$ implies $\|(x \vee g) - g\| = \|x - (x \wedge g)\|$
$= \|x - g\| = d(x \vee g, G)$.

(3) $|x - (f \vee g) \wedge x| \le |x - g \wedge x|$ (since $g \wedge x \le (f \vee g) \wedge x \le x$) implies $\|x - (f \vee g) \wedge x\| \le \|x - g \wedge x\| = d(x,G)$, hence $\|x - (f \vee g) \wedge x\| = d(x,G)$.

PROPOSITION 2.2. Let E be a Banach lattice, G a closed ideal in E and α an element of E. If β is an element of G and ϵ is a positive real number such that $\|\alpha - \beta\| \le r + \epsilon$ (where $r = d(\alpha,G)$), then there is an element γ in G such that $\gamma^+ \le \alpha^+$, $\gamma^- \le \alpha^-$ and $\|\alpha - \gamma\| \le r + \epsilon$.

PROOF. Define $\gamma^+ = \alpha^+ \wedge |\beta|$, $\gamma^- = \alpha^- \wedge |\beta|$ and $\gamma = \gamma^+ - \gamma^-$. Then,

$$|\alpha - \gamma| = (\alpha^+ - \gamma^+) + (\alpha^- - \gamma^-) \quad (\text{since } (\alpha^+ - \gamma^+) \wedge (\alpha^- - \gamma^-) = 0)$$
$$= |\alpha| - (\alpha^+ \wedge |\beta| + \alpha^- \wedge |\beta|)$$
$$\le |\alpha| - (\alpha^+ + \alpha^-) \wedge |\beta| = |\alpha| - |\alpha| \wedge |\beta|$$
$$\le |\alpha| \vee |\beta| - |\alpha| \wedge |\beta| = \big||\alpha| - |\beta|\big| \le |\alpha - \beta|.$$

Thus, we get $\|\alpha - \gamma\| \le \|\alpha - \beta\| \le r + \epsilon$.

PROPOSITION 2.3. Let E be a Banach lattice with the f.o.i.p., G a closed ideal of E and α an element of E. If β is an element of G such that $\beta^+ \le \alpha^+$, $\beta^- \le \alpha^-$ and $\|\alpha - \beta\| \le r + \epsilon$, then there is an element γ in G such that $\gamma^+ \le \alpha^+$, $\gamma^- \le \alpha^-$, $\|\alpha - \gamma\| \le r$ and $\|\gamma - \beta\| < \epsilon$.

PROOF. Let θ be a positive best a approximation of $|\alpha|$ such that $\theta \le |\alpha|$ (for the existence of θ see F. A. Sejeeni [11]). Now,

(1) $|\beta| \le |\beta| \le |\beta| \vee \theta \le |\alpha|$

(2) $B(|\beta|,\epsilon) \cap B(|\alpha|,r) \ne \emptyset$ (since $\big\||\alpha| - |\beta|\big\| \le r + \epsilon$)

(3) $|\beta| \in B(|\beta|,\epsilon)$ and $|\beta| \vee \theta \in B(|\alpha|,r)$.

Then there exist a σ in $[|\beta|,|\beta| \vee \theta] \cap B(|\beta|,\epsilon) \cap B(|\alpha|,r)$. Define $\gamma^+ = \sigma \wedge \alpha^+$ and $\gamma^- = \sigma \wedge \alpha^-$ ($\gamma^+ \wedge \gamma^- = 0$) and note that,
$\sigma = \sigma \wedge |\alpha| = \sigma \wedge (\alpha^+ + \alpha^-) \le \sigma \wedge \alpha^+ + \sigma \wedge \alpha^- = \gamma^+ + \gamma^- = \gamma^+ \vee \gamma^-$. Also:
$\gamma^+ \vee \gamma^- = (\sigma \wedge \alpha^+) \vee (\sigma \wedge \alpha^-) \le (\sigma \wedge |\alpha|) \vee (\sigma \wedge |\alpha|) = \sigma \vee \sigma = \sigma$. Hence, we have that $\sigma = \gamma^+ + \gamma^-$. We will show that $\gamma = \gamma^+ - \gamma^-$ is the desired element of G. To see this, note that,

$|\alpha - \gamma| = (\alpha^+ - \gamma^+) + (\alpha^- - \gamma^-) = |\alpha| - |\gamma| = \big||\alpha| - |\gamma|\big|$ implies $\|\alpha - \gamma\| = \big\||\alpha| - |\gamma|\big\| \le r$. For the second inequality, note that

$\beta^+ \le \alpha^+ \wedge |\beta|$, $\beta^- \le \alpha^- \wedge |\beta|$ and $|\beta| \le \sigma$ implies that $\beta^+ \le \gamma^+$ and $\beta^- \le \gamma^-$.

$|\gamma - \beta| = (\gamma^+ - \beta^+) + (\gamma^- - \beta^-) = |\gamma| - |\beta| = \big||\gamma| - |\beta|\big|$ implies $\|\gamma - \beta\| = \big\||\gamma| - |\beta|\big\| = \|\sigma - |\beta|\| < \epsilon$.

PROPOSITION 2.4. Let E be a Banach lattice with the f.o.i.p. and G a closed ideal in E. Then for each x in E, the set $P^0_G(x)$ is a nonempty closed subset of G.

PROOF. Let g_1 respectively (g_2) in $P_G(x^+)$ $(P_G(x^-))$ be such that $0 \le g_1 \le x^+$ $(0 \le g_2 \le x^-)$. Let h in $P_G(|x|)$ be such that $g_1 + g_2 \le h \le |x|$. Put $g^{\mp} = x^{\mp} \wedge h$ and $g = g^+ - g^-$ ($g^{\mp} \in P_G(x^{\mp})$ and $|g| \in P_G(|x|)$). Now, let f be an arbitrary element of G . Then we have $\|x - f\| \ge \||x| - |f|\| \ge \||x| - |g|\| = \|x - g\|$ (since $|x - g| = ||x| - |g||$) hence $g \in P_G^0(x)$. To see it is closed we may assume with out loss of generality that $P_G^0(x)$ is infinite. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence in $P_G^0(x)$ converging to g, then $g_n^{\mp} \longrightarrow g^{\mp}$, and $g_n^{\mp} \wedge x^{\mp}$ $(= g_n^{\mp}) \longrightarrow g^{\mp} \wedge x^{\mp}$. Thus, we have $g^+ \le x^+$ and $g^- \le x^-$. $\|x - g\| \le \|x - g_n\| + \|g_n - g\| \longrightarrow d(x,G)$ implies that $g \in P_G^0(x)$.

REMARK. In the following example we will show that it is not always true that if $g \in P_G(x)$, then $g^{\mp} \in P_G(x^{\mp})$

EXAMPLE. Consider the Banach lattice $E = C([0,\pi])$, and the ideal $G = \{g \in E : g|_{[\pi/2,\ \pi]} = 0\}$. Let $x \in E$ be defined by $x(t) = \max\{-(\sin(t)),0\} = (\sin(t))^-$. Then the element $g \in G$, defined by $g(t) = -\max\{(\sin(t)),0\} = -(\sin(t))^+$ belongs to $P_G(x)$ and yet $g^- = -g$ does not belong to $P_G(x^-) = \{0\}$ (since $x^- = 0 \in G$) .

PROPOSITION 2.5. The set valued mapping $P : E \longrightarrow 2^G$ is (l.s.c.) if and only if for each sequence $\{x_n\}$ in E converging to x and foreach $g \in P(x)$, there is a sequence $\{g_n\}$ in G with $g_n \in P(x_n)$ and g_n converges to g .

PROOF. Assume that P is (l.s.c.), $\{x_n\}$ a sequence in E converging to x and $g \in P(x)$. The set $U_k = B(g,2^{-k}) \cap G$ is open in G ($k \in \mathbb{N}$). Then by (l.s.c.) of P the set $\mathcal{U}_k = \{ y \in E : P(y) \cap U_k \neq \emptyset \}$ is a neighborhood of x. Hence, there exists an integer $N_k \in \mathbb{N}$ such that $x_n \in \mathcal{U}_k$, and then, $P(x_n) \cap U_k \neq \emptyset$ for each $n \ge N_k$. Now, We can select a sequence $\{g_n\}$ such that $g_n \in P(x_n)$ and $\|g_n - g\| \le 2^{-k}$ $(n \ge N_k)$.

Now, assume that P is not (l.s.c.) , then for some open set U in G the set $\mathcal{U} = \{ y \in E : P(y) \cap U \neq \emptyset \}$ is not open. Let $x \in \mathcal{U}$ be such that, each neighborhood V of x intersects $\mathcal{U}^c$ in a nonempty set ($\mathcal{U}^c$ is the complement of $\mathcal{U}$). Let, $g \in P(x) \cap U$, and for each $n \in \mathbb{N}$, pick x_n in $B(x,2^{-n}) \cap \mathcal{U}^c$. The sequence $\{x_n\}$ converges to x, but it is impossible for any sequence $\{g_n\}$ with $g_n \in P(x_n)$ to converge to g (since U is a neighborhood of g and $g_n \notin U$, $n \in \mathbb{N}$.

THEOREM 2.6. Let E be a Banach lattice with the f.o.i.p. and G a closed ideal in E. Then the set valued mapping $P_G^0 : E \longrightarrow 2^G$ defined by $P_G^0(x) = \{g \in P_G(x) : g^{\mp} \le x^{\mp},\ g^{\mp} \in P_G(x^{\mp})\}$ is always (l.s.c.).

PROOF. First, we will show that the result holds for positive elements.

For, let $\{x_n\}$ be a positive sequence in E converging to x $(x \ge 0)$, and $g \in P_G^0(x)$. For each $n \in \mathbb{N}$, let $h_n \in P_G^0(x_n)$. If ε is a positive real number then, we have

(1) $g \wedge x_n \le g \wedge x_n \le (g \vee h_n) \wedge x_n \le x_n$

(2) $B(g \wedge x_n,\varepsilon) \cap B(x_n,r_n) \neq \emptyset$

$\Big($since $\|x_n - g \wedge x_n\| \le \|x_n - x\| + \|x - g\| + \|g - g \wedge x_n\|$

$$< \varepsilon/3 + r + \varepsilon/3 < \varepsilon/3 + (r_n + \varepsilon/3) + \varepsilon/3 = r_n + \varepsilon$$

$\forall\ n \ge N_\varepsilon$ where $r_n = d(x_n,G)$ and $r = d(x,G)\Big)$

(3) $g \wedge x_n \in B(g \wedge x_n,\varepsilon)$ and $(g \vee h_n) \wedge x_n \in B(x_n,r_n)$.

Then for each $n \ge N_\varepsilon$ there is a g_n in $[g \wedge x_n, (g \vee h_n) \wedge x_n] \cap B(g \wedge x_n,\varepsilon) \cap B(x_n,r_n)$. Now, for $n \in \mathbb{N}$ take $\varepsilon = \varepsilon_n = 3 \max\left\{ \|x_n - x\|, |r_n - r|, \|g - g \wedge x_n\| \right\}$. Thus we can select a sequence $\{g_n\}$ such that $g_n \in P^0_G(x_n)$ and $\|g_n - (g \wedge x_n)\| \le \varepsilon_n$. The sequence $\{g_n\}$ is the desired sequence (since $\|g_n - g\| \le \|g_n - (g \wedge x_n)\| + \|(g \wedge x_n) - g\| \longrightarrow 0$).

Now, let $\{x_n\}$ be an arbitrary sequence in E converging to x and g an arbitrary element of $P^0_G(x)$. Then by the above there are positive sequences $\{f_n\}$, $\{h_n\}$ and $\{k_n\}$ in G such that $f_n \in P^0_G(x_n^+)$, $h_n \in P^0_G(x_n^-)$, $k_n \in P^0_G(|x_n|)$, $f_n \longrightarrow g^+$, $h_n \longrightarrow g^-$ and $k_n \longrightarrow |g|$. We may assume without loss of generality that $f_n + h_n \le k_n$ (otherwise set $k_n = k_n \vee (f_n + h_n)$). Now set $g_n^+ = x_n^+ \wedge k_n \ge f_n$, $g_n^- = x_n^- \wedge k_n \ge h_n$ $\left(g_n^\mp \in P_G(x_n^\mp) \right)$ and $g_n = g_n^+ - g_n^-$ (it is obvious that $g_n \longrightarrow g$). To complete the proof, we will show that $g_n \in P_G(x_n)$. To see this, let y be an arbitrary element of G, then $\|x_n - y\| \ge \big\||x_n| - |y|\big\| \ge \big\||x_n| - k_n\big\|$ (the first inequality holds since $|x_n - y| \ge \big||x_n| - |y|\big|$ while the second one holds because $k_n \in P^0_G(|x_n|)$). But, $|x_n| \ge |g_n| = g_n^+ + g_n^- = x_n^+ \wedge k_n + x_n^- \wedge k_n \ge |x_n| \wedge k_n = k_n$, then $\|x_n - y\| \ge \big\||x_n| - |y|\big\| \ge \big\||x_n| - |g_n|\big\| = \|x_n - g_n\|$ which implies that $\|x_n - g_n\| = d(x_n,G)$ i.e. , $g_n \in P_G(x_n)$.

3. FUNCTION MODULES.

DEFINITION 3.1. Let A be a Banach algebra with a norm $\|\cdot\|_A$, and let E be a Banach space. We say that E is a Banach A-module if

(i) E is a left module over A in the usual algebraic sense;

(ii) there is a positive constant k such that $\|ax\| \le k\,\|a\|_A\,\|x\|$ for all $a \in A$, $x \in E$.

Let T be a nonvoid compact Hausdorff space $(E_t)_{t \in T}$ a family of Banach spaces. The product $\prod_{t\in T} E_t$ can be thought of as a space of functions on T where the values of the functions at different points lie (possibly) in different spaces. We will restrict our attention to the subspace

$$\prod_{t\in T}^{\infty} E_t = \left\{\alpha \in \prod_{t\in T} E_t : \|\alpha\|_\infty = \sup_{t\in T} \|\alpha(t)\|_t < \infty\right\}$$

(where $\|\cdot\|_t$ is the norm on the Banach space E_t).

DEFINITION 3.2. A function module is a triple $\left(T, (E_t)_{t\in T}, E_\infty\right)$, where T is a nonvoid compact Hausdorff space (*called the base space*), $(E_t)_{t\in T}$ a family of Banach spaces (*the component spaces*) and E_∞ a closed subspace of $\prod_{t\in T}^{\infty} E_t$ such that the following are satisfied:

(1) $\mathbf{E}_\infty$ is a $C(T)$-module (where $C(T)$ is the Banach algebra of all continuous scalar valued functions on T) $(f.\alpha)(t) = f(t).\alpha(t)$ $f \in C(T)$, $\alpha \in \mathbf{E}_\infty$.

(2) For every $\alpha \in \mathbf{E}_\infty$, the map $t \longmapsto \|\alpha(t)\|_t$ is upper semi-continuous.

(3) $E_t = \overline{\left\{ \alpha(t) : \alpha \in \mathbf{E}_\infty \right\}}$ for every $t \in T$.

(4) $\overline{\left\{ t : t \in T, E \neq \{0\} \right\}} = T$.

A sub-function module is a subspace which is a $C(T)$-module. A function module of Banach lattices is a function module such that the components E_t are Banach lattices and $\mathbf{E}_\infty$ is closed under the the lattice operations $\wedge$ and $\vee$ which are defined pointwise $((\alpha \vee \beta)(t) = \alpha(t) \vee \beta(t))$.

DEFINITION 3.3. Let $\mathbf{E}_\infty$ be a function module in $\prod_{t\in T}^{\infty} E_t$, $\mathbf{G}_\infty$ a sub-function module of $\mathbf{E}_\infty$ and α an element of $\mathbf{E}_\infty$. Then,

(i) the element γ of $\mathbf{G}_\infty$ is global best approximation of α if

$$\|\alpha - \gamma\|_\infty = \sup_{t\in T} \|\alpha(t) - \gamma(t)\|_t = \inf \left\{ \|\alpha - \beta\|_\infty : \beta \in \mathbf{G}_\infty \right\}.$$

(ii) the element γ is a local best approximation of α if

$$\|\alpha(t) - \gamma(t)\|_t = \inf \left\{ \|\alpha(t) - g\|_t : g \in G_t \right\} \text{ i.e., } \gamma_t \in \mathbf{P}_{G_t}(\alpha(t)) \quad (t \in T).$$

LEMMA 3.4. Let $\mathbf{E}_\infty$ be a function module of Banach lattices and $\mathbf{G}_\infty$ a sub-function module of $\mathbf{E}_\infty$ such that the fiber G_t is an ideal in E_t. Then for each α in $\mathbf{E}_\infty$ and for each positive real numbers ε there is an element γ in $\mathbf{G}_\infty$ such that $\|\alpha - \gamma\|_\infty \le r + \varepsilon$, $(\alpha(t))^+ \le (\gamma(t))^+$ and $(\alpha(t))^- \le (\gamma(t))^-$ for each $t \in T$ (where $r = d(\alpha,G)$).

PROOF. Let β in $\mathbf{G}_\infty$ be such that $\|\alpha - \beta\|_\infty \le r + \varepsilon$. For $t \in T$ define $\gamma_1(t) = (\alpha(t))^+ \wedge |\beta(t)|$ and $\gamma_2(t) = (\alpha(t))^- \wedge |\beta(t)|$. Then $\gamma_i \in \prod_{t\in T}^{\infty} G_t = \mathbf{G}_\infty$ $i = 1, 2$. Let $\gamma = \gamma_1 - \gamma_2$, then for each $t \in T$ we have

$$\begin{aligned}
|\alpha(t) - \gamma(t)| &= \left|\left((\alpha(t))^+ - (\alpha(t))^-\right) - \left(\gamma_1(t) - \gamma_2(t)\right)\right| \\
&= \left|\left((\alpha(t))^+ - \gamma_1(t)\right) - \left((\alpha(t))^- - \gamma_2(t)\right)\right| \\
&= \left((\alpha(t))^+ - \gamma_1(t)\right) + \left((\alpha(t))^- - \gamma_2(t)\right) \\
&= |\alpha(t)| - |\gamma(t)| \\
&\le |\alpha(t)| \vee |\beta(t)| - \left((\alpha(t))^+ \wedge |\beta(t)| + (\alpha(t))^- \wedge |\beta(t)|\right) \\
&\le |\alpha(t)| \vee |\beta(t)| - |\alpha(t)| \wedge |\beta(t)| \\
&= \Big| |\alpha(t)| - |\beta(t)| \Big| \le |\alpha(t) - \beta(t)|.
\end{aligned}$$

Hence $\|\alpha(t) - \gamma(t)\|_t \le \|\alpha(t) - \beta(t)\|_t \le r + \varepsilon$.

THEOREM 3.5. Let $\mathbf{E}_\infty$ be a function module of Banach lattices and $\mathbf{G}_\infty$ a sub-function module of $\mathbf{E}_\infty$. If for each $t \in T$ the space E_t has the f.o.i.p. and the fiber G_t is an ideal of E_t, then $\mathbf{G}_\infty$ is proximinal.

PROOF. Let α be an element of $\mathbf{E}_\infty$, ε a positive real number and $r = d(\alpha,G)$. Then by lemma 3.4 there is a γ in $\mathbf{G}_\infty$ such that $\|\alpha - \gamma\| \le r + \varepsilon$ and $(\gamma(t))^{\mp} \le (\alpha(t))^{\mp}$ for each $t \in T$. Now, by proposition 2.3 there is a g_t in G_t such that

$$\|\alpha(t) - g_t\|_t \le r \quad \text{and} \quad \|g_t - \gamma(t)\|_t < \varepsilon . \tag{3.1}$$

Put $f_t = 1/2(\gamma(t) + g_t)$, then

$$\|\alpha(t) - f_t\|_t < r + \varepsilon/2 \quad \text{and} \quad \|f_t - \gamma(t)\|_t < \varepsilon/2 . \tag{3.2}$$

Let $\gamma_t \in \mathbb{G}_\infty$ be such that $\gamma_t(t) = f_t$ and U_t a neighborhood of t such that for each $s \in U_t$ we have

$$\|\alpha(s) - \gamma_t(s)\|_s < r + \varepsilon/2 \quad \text{and} \quad \|\gamma(s) - \gamma_t(s)\|_s < \varepsilon/2. \tag{3.3}$$

The collection $\left\{ U_t \mid t \in T \right\}$ forms an open covering of T. Let $t_1, \ldots, t_n$ in T be such that $T = \bigcup_{i=1}^{n} U_{t_i}$ and $\{f_i\}_{i=1}^{n}$ the partition of unity subordinate to $\{U_{t_i}\}_{i=1}^{n}$. Set $\gamma_o = \sum_{i=1}^{n} f_i \cdot \gamma_{t_i}$ then for $t \in T$

$$\begin{aligned} \|\alpha(t) - \gamma_o(t)\| &= \Big\| \sum_{i=1}^{n} f_i(t) \cdot \alpha(t) - \sum_{i=1}^{n} f_i(t) \cdot \gamma_{t_i}(t) \Big\| \\ &= \Big\| \sum_{i=1}^{n} f_i(t) \cdot (\alpha(t) - \gamma_{t_i}(t)) \Big\| \\ &\le \sum_{i=1}^{n} |f_i(t)| \cdot \|(\alpha(t) - \gamma_{t_i}(t))\| \\ &\le r + \varepsilon/2 . \end{aligned}$$

Similarly $\|\gamma - \gamma_o\| \le \varepsilon/2$.

$$\|\alpha - \gamma_o\|_\infty \le r + \varepsilon/2 \quad \text{and} \quad \|\gamma - \gamma_o\|_\infty \le \varepsilon/2 . \tag{3.4}$$

Taking $\varepsilon = 2^{-n}$ ($n \in \mathbb{N}$) we can construct a sequence $\{\gamma_n\}$ satisfying

$$\|\alpha - \gamma_n\| \le r + 2^{-n} \quad \text{and} \quad \|\gamma_n - \gamma_{n+1}\| \le 2^{-n}. \tag{3.5}$$

The second inequality of (3.5) implies that $\{\gamma_n\}$ is Cauchy, and then it has a limit γ in $\mathbb{G}_\infty$. The first inequality of (3.5) implies that $\gamma \in P_{\mathbb{G}_\infty} G(\alpha)$.

PROPOSITION 3.6. Let T be a compact Hausdorff space, E a Banach lattice with the f.o.i.p. and G a closed ideal of E. Then $C(T,G)$ contains local and global best approximation for each f in $C(T,E)$.

PROOF. The existence of global best approximation follows from theorem 3.5 and the fact that the space $C(T,G)$ is an ideal in the Banach lattice $C(T,E)$. For local best approximation , we define $P : T \longrightarrow 2^G$ by $P(t) = P^0_G(f(t))$ (where $P^0_G(f(t))$ is the set of all best approximation of f(t) in G satisfying (1.2)). Then set valued mapping P is (l.s.c.) since $P = (P^0_G \circ f)$, f is continuous and $P^0_G$is (l.s.c.). Let s be a continuous selection of P, then $g = s \circ f$ is desired element of G .

Finally, we conclude this paper with the observation that in some cases global best approximation always exists and yet the set valued mapping P defined above admits no continuous selection at all as we will see in the following.

EXAMPLE. Let $E_\infty = C([0,1],\mathbb{R})$ (the space of all continuous real valued functions on $[0,1]$ with $\|f\| = \sup_{t\in T}|f(t)|$), G_∞ the space of all f in E_∞ which vanish on $[0,1/2]$. If we take $f \in E_\infty$ to be the constant function $f(t) = 1$ $(t \in T)$. Then we have the following :

$$G_t = \begin{cases} \{0\} & \text{for } t \in [0,1/2] \\ \mathbb{R} & \text{for } t \in (1/2,1] \end{cases} \quad \text{and} \quad P(t) = \begin{cases} \{0\} & \text{if } t \in [0,1/2] \\ \{1\} & \text{if } t \in (1/2,1] \end{cases}.$$

Thus P is a single valued discontinuous function on $[0,1]$ which admits no continuous selection at all.

REFERENCES

1. Behrends, E. M-structure and the Banach-Stone Theorem, Lecture Notes in Mathematics, # 736, Springer Verlag, NewYork 1979.
2. Birkhoff, G., Lattice Theory, Amer. Math. Soc. Colloquium Publ.Vol. 25, Providence, R. I., 1940.
3. Cartwright, D., Extensions of positive operators between Banach lattices, Memoirs, Amer. Math. Soc. 164 (1975) PP 1-48.
4. Fakhoury,H.,Existence d'une projection continue de Meilleure approximation dans certains espaces de Banach,J.Math. Pures et Appl. 53 (1974) PP 1-16.
5. Gierz, G., Injective Banach lattices with strong order unit, Pacific J. of Math. Vol. 110 No. 2 (1984) PP 297-305
6. Haydon, R., Injective Banach lattices, Math. Z. 156 (1977).
7. Lazar, A.J., Morris, P.D., Wulbert, D.E., Continuous selection for metric projections, J. Func. Anal. 3 (1967) PP 193-216.
8. Michael,E., Continuous selection Ann. Math. 63 (1954) PP361-382
9. Rieffel, M.A., Induced Banach representation of Banach algebra and locally compact groups, J. Func. Anal. 1 (1967) PP 443-491
10. Schaefer, H.H., Banach lattices and positive operators, Grundlehren der Math. Band 215, Springer Verlag, NewYork 1974.
11. Sejeeni, F.A., Best approximation in C(T)-modules, Dissertation U.C.R. Riverside Dce. 1986.
12. Singer,I., Best approx. in normed linear spaces by elements of linear subspaces, Grundlehren der Math., Band 171, Springer Verlag, NewYork 1970.
13. Weinstein, S.E., Approximation of functions of several variables: product Chebychev approximation, J. Approx. Theory 2 (1969).