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ON k-IDEALS OF SEMIRINGS

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ABSTRACT. Certain types of ring congruences on an additive inverse semiring are characterized with the help of full k-ideals. It is also shown that the set of all full k-ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular.

KEY WORDS AND PHRASES. Semiring, inverse semiring, k-ideals and ring congruence.
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1. PRELIMINARIES. A semiring is a system consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that

(i) S together with addition is a semigroup;

(ii) S together with multiplication is a semigroup; and

(iii) $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in S$.

A semiring S is said to be additively commutative if $a + b = b + a$ for all $a, b \in S$. A left (right) ideal of a semiring S is non-empty subset I of S such that

i) $a + b \in I$ for all $a, b \in I$; and

ii) $ra \in I (ar \in I)$ for all $r \in S$ and $a \in I$.

An ideal of a semiring S is a non-empty subset I of S such that I is both a left and right ideal of S.

Henriksen [1] defined a more restricted class of ideals in a semiring, which he called k-ideals.

A left k-ideal I of a semiring S is a left ideal such that if $a \in I$ and $x \in S$ and if either $a + x \in I$ or $x + a \in I$, then $x \in I$.

Right k-ideal of a semiring is defined dually. A non-empty subset I of a semiring S is called a k-ideal if it is both a left k-ideal and a right k-ideal.

A semiring S is said to be additively regular if for each $a \in S$, there exists an element $b \in S$ such that $a = a + b + a$. If in addition, the element b is unique and satisfies $b = b + a + b$, then S is called an additively inverse semiring. In an additively inverse semiring the unique inverse b of an element a is usually denoted by a'. Karvellas [2] proved the following result:

Let S be an additively inverse semiring. Then

i) $x = (x')'$, $(x+y)' = y' + x'$, $(xy)' = x'y = y'x$ and $xy = x'y'$ for all $x, y \in S$.

ii) $E^+ = \{x \in S: x + x = x\}$ is an additively commutative semilattice and an ideal of S.

2. FULL k-IDEALS. In this section S denotes an additively inverse semiring in which addition is commutative and E^+ denotes the set of all additive idempotents of S.

A left k-ideal A of S is said to be full if $E^+ \subseteq A$. A right k-ideal of S is defined dually.

A non-empty subset I of S is called a full k-ideal if it is both left and a right full k-ideal.

EXAMPLE 1. In a ring every ring ideal is a full k-ideal.

EXAMPLE 2. In a distributive lattice with more than two elements, a proper ideal is a k-ideal but not a full k-ideal.

EXAMPLE 3. $Z \times Z^p = \{(a,b): a, b$ are integers and $b > 0\}$. Define

$$(a,b) + (c,d) = (a+c, \text{l.c.m of } b,d) \text{ and } (a,b)(c,d) = (ac, h.c.f. \text{ of } b,d).$$

Then $Z \times Z^p$ becomes an additively inverse semiring in which addition is commutative.

Let $A = \{(a,b) \in Z \times Z^p: a = 0, b \in Z^p\}$. Then A is a full k-ideal of $Z \times Z^p$.

LEMMA 2.1. Every k-ideal of S is an additively inverse subsemiring of S.

PROOF. Let I be a k-ideal of S. Clearly I is a subsemiring of S. Let $a \in I$. Then

$$a + (a' + a) = a \in I.$$

Since I is a k-ideal, it follows $a' + a \in I$. Again this implies that $a' \in I$. Hence the lemma.

LEMMA 2.2. Let A be an ideal of S. Then

$$\overline{A} = \{a \in S: a + x \in A \text{ for some } x \in A\} \text{ is a } k\text{-ideal of } S.$$

PROOF. Let $a, b \in \overline{A}$. The $a + x, b + y \in A$ for some $x, y \in A$. Now

$$a + x + b + y = (a+b) + (x+y) \in A.$$

As $x + y \in A, a + b \in \overline{A}$. Next let $r \in S, ra + rx = r(a+x) \in A$.
As $rx \in A$, $ra \in \overline{A}$. Similarly, $ar \in \overline{A}$. As a result $\overline{A}$ is an ideal of S. Next, let c and $c + d \in \overline{A}$. Then there exists x and y in A such that $c + x \in A$ and $c + d + y \in A$.
Now

$$d + (c + x + y) = (c + d + y) + x \in A \text{ and } c + x + y \in A.$$

Hence $d \in \overline{A}$ and $\overline{A}$ is a k-ideal of S. Since $a + a' \in A$ for all $a \in A$, it follows that $A \subseteq \overline{A}$.

COROLLARY. Let A be an ideal of S. Then $\overline{A} = A$ iff A is a k-ideal.

LEMMA 2.3. Let A and B be two full k-ideals of S, then $\overline{A+B}$ is a full k-ideal of S such that

$$A \subseteq \overline{A+B} \text{ and } B \subseteq \overline{A+B}.$$

PROOF. It can be shown that $A + B$ is an ideal of S. Then from Lemma 2.2, we find $\overline{A+B}$ is a k-ideal and $A + B \subseteq \overline{A+B}$. Now $E^+ \subseteq A, B$. Hence $E^+ \subseteq A + B \subseteq \overline{A+B}$. This implies that $\overline{A+B}$ is a full k-ideal. Let $a \in A$. Then

$$a = a + a' + a = a + (a' + a) \in A + B \text{ as } a' + a \in E^+ \subseteq B.$$

Hence $A \subseteq \overline{A+B}$ and similarly $B \subseteq \overline{A+B}$.

THEOREM 2.4. If $I(S)$ denotes the set of all full k-ideals of S, then $I(S)$ is a complete lattice which is also modular.

PROOF. We first note that $I(S)$ is a partially ordered set with respect to usual set inclusion. Let $A, B \in I(S)$. Then $A \cap B \in I(S)$ and from Lemma 2.3, $\overline{A+B} \in I(S)$. Define $A \wedge B = A \cap B$ and $A \vee B = \overline{A+B}$. Let $C \in I(S)$ such that $A, B \subseteq C$. Then $A+B \subseteq C$ and $\overline{A+B} \subseteq \overline{C}$. But $\overline{C} = C$. Hence $\overline{A+B} \subseteq C$. As a result $\overline{A+B}$ is the l.u.b. of A, B. Thus we find that $I(S)$ is a lattice. Now E^+ is an ideal of S. Hence $\overline{E}^+ \in I(S)$ and also $S \in I(S)$; consequently $I(S)$ is a complete lattice. Next suppose that $A, B, C \in I(S)$ such that

$$A \wedge B = A \wedge C \text{ and } A \vee B = A \vee C \text{ and } B \subseteq C.$$

Let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A+B}$. Hence there exists $a+b \in A+B$ such that $x+a+b = a_1+b_1$ for some $a_1 \in A$, $b_1 \in B$.
Then

$$x+a+a'+b = a_1+b_1+a'.$$

Now $x \in C, a+a' \in C$ and $b \in B \subseteq C$. Hence $a_1+b_1+a' \in C$. But $b_1 \in C$. Consequently, $a_1+a' \in C \cap A = C \cap B$. Hence $a_1+a' \in B$. So from $x+a+b = a_1+b_1$ we find that $x+a+a'+b = a_1+a'+b \in B$. But $(a+a')+b \in B$ and B is a k-ideal. Hence $x \in B$ and $B = C$. This proves that $I(S)$ is a modular lattice.

3. RING CONGRUENCES.

A congruence ρ on a semiring S is called a ring congruence if the quotient semiring S/ρ is a ring.

In this section we assume S is an additively inverse semiring in which addition is commutative. We want to characterize those ring congruences on S such that - $(a\rho) = a'\rho$ where a' denotes the inverse of a in S and $-(a\rho)$ denotes the additive inverse of $a\rho$ in the ring S/ρ.

THEOREM 3.1. Let A be a full k-ideal of S. Then the relation

$\rho_A = \{(a,b) \in S \times S : a+b' \in A\}$ is a ring congruence on S such that $-(a\rho_A) = a'\rho_A$.

PROOF. Since $a+a' \in E^+ \subseteq A$ for all $a \in S$, it follows that ρ_A is reflexive. Let $a+b' \in A$. Now from Lemma 2.1, we find that $(a+b')' \in A$. Then $b+a' = (b')'+a' = (a+b')' \in A$. Hence ρ_A is symmetric. Let $a+b' \in A$ and $b+c' \in A$. Then $a+b+b'+c' \in A$. Also $b+b' \in E^+ \subseteq A$. Since A is a k-ideal, we find that $a+c' \in A$. Hence ρ_A is an equivalence relation. Let $(a,b) \in \rho_A$ and $c \in S$. Then $a+b' \in A$. Since

$$(c+a)+(c+b)' = c+a+b'+c' = (a+b')+(c+c') \in A, ca+(cb)' = ca+cb' = c(a+b') \in A,$$

$$ac+(bc)' = ac+b'c = (a+b')c \in A,$$

it follows that ρ_A is a congruence on S. So we obtain the quotient semiring where addition and multiplication are defined by

$$a\rho_A + b\rho_A = (a+b)\rho_A \text{ and } (a\rho_A)(b\rho_A) = (ab)\rho_A.$$

Now

$$a\rho_A + b\rho_A = (a+b)\rho_A = (b+a)\rho_A = b\rho_A + a\rho_A.$$

Let $e \in E^+$ and $a \in S$. Now $(e+a)+a' = e+(a+a') \in E^+$.
We find that $(e+a)\rho_A = a\rho_A$. Then $e\rho_A + a\rho_A = a\rho_A$.

Also

$$a\rho_A + a'\rho_A = (a + a')\rho_a = e\rho_A.$$

Hence $e\rho_A$ is the zero element and $a'\rho_A$ is the negative element of $a\rho_A$ in the ring S/ρ_A.

THEOREM 3.2. Let ρ be a congruence on S such that S/ρ is a ring and $-(a\rho) = a'\rho$. Then there exists a full k-ideal A of S such that $\rho_A = \rho$.

PROOF. Let $A = \{a \in S: (a,e) \in \rho$ for some $e \in E^+\}$. Since ρ is reflexive, it follows that $E^+ \subseteq A$. Then $A \neq \phi$, since $E^+ \neq \phi$. Let $a, b \in A$. Then there exist $e, f \in E^+$ such that $(a,e) \in \rho$ and $(b,f) \in \rho$. Then $(a+b, e+f) \in \rho$. But $e + f \in E^+$. Hence $a + b \in A$. Again for any $r \in S$, $(ra, re) \in \rho$ and $(ar, er) \in \rho$. But re and er $\in E^+$. Hence A is an ideal of S.

Let $a + b \in A$ and $b \in A$. Then there exist $e, f, \in E^+$ such that $(a + b, f) \in \rho$ and $(b, e) \in \rho$. Hence $f\rho = (a + b)\rho = a\rho + b\rho = a\rho + e\rho$. But $f\rho$ and $e\rho$ are additive idempotents in the ring S/ρ. Hence $e\rho = f\rho$ is the zero element of S/ρ. As a result, $a\rho$ is the zero element of S/ρ. Then $a\rho = e\rho$. This implies $a \in A$. So we find that A is a full k-ideal of S. Consider now the congruences ρ_A and ρ. Let $(a,b) \in \rho$. Then $(a + b', b + b') \in \rho$. But $b + b' \in E^+$. Hence $a + b' \in A$ and $(a,b) \in \rho_A$. Conversely suppose that $(a,b) \in \rho_A$. Then $a + b' \in A$. Hence $(a + b', e) \in \rho$ for some $e \in E^+$. As a result, $e\rho = a\rho + b'\rho = a\rho - b\rho$ holds in the ring S/ρ. But $e\rho$ is the zero element of S/ρ. Consequently $a\rho = b\rho$. This show that $(a,b) \in \rho$ and hence $\rho_A = \rho$.

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