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BERNOULLI TRIALS AND PERMUTATION STATISTICS

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Abstract Several coin-tossing games are surveyed which, in a natural way, give rise to "statistically" induced probability measures on the set of permutations of $\{1, 2, ..., n\}$ and on sets of multipermutations. The distributions of a general class of random variables known as binary tree statistics are also given.

KEY WORDS AND PHRASES. Permutation statistics,probability measure, binary free statistics and multi-permutations.

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1. Introduction The distributions of permutation statistics have long been considered relative to the equiprobable measure $1/n!$ on the set of permutations of $\{1, 2, ..., n\}$. For instance, among many such results to be found in David and Barton [DB, 150-183] or in Bender [B], it has been established that the classic permutation statistic known as the descent number is asymptotically normal. More recently, Diaconis [D1, 128] has verified that the inversion number is likewise asymtotically normal relative to the equiprobable measure. Additional results of this type may be found in [Cr, D, H].

Some work has also been done concerning distributions of permutation statistics relative to probability measures which are not uniform ([D2]). It is this general vein of research to which this paper belongs: Motivated by the results of Moritz and Williams [MW], several coin-tossing games are presented which lead to natural non-uniform probability measures that are induced by permutation statistics.

Specifically, in sections 3 through 6, three coin-tossing games are described which give rise to measures respectively induced by the Mahonian statistics known as the comajor index, the major index, and the inversion number. In sections 8 and 9, two more coin-tossing games and their associated measures are given which are based on permutation decompositions. Throughout, and particularly in section 7, the distributions of random variables known as binary tree statistics are also presented. In the final section, some natural directions for further research are indicated.

2. Mahonian and Binary Tree Statistics For the most part, the measures and random variables considered arise in connection with what are respectively known as Mahonian and binary tree statistics. Before defining these two classes of statistics, a number of preliminaries are needed.

For a sequence $j_1, j_2, \ldots, j_n$ of non-negative integers, the symbol $1^{j_1}\, 2^{j_2} \ldots n^{j_n}$ will be used to denote the multiset consisting of j_1 ones, j_2 twos, ..., and j_n n's. A sequentially ordered list σ of length $j \equiv (j_1 + j_2 + \ldots + j_n)$ and of the form

$$\sigma = \sigma(1)\,\sigma(2) \ldots \sigma(j) \tag{2.1}$$

in which, for $1 \le k \le n$, k appears exactly j_k times will be referred to as a multipermutation of $1^{j_1}\, 2^{j_2} \ldots n^{j_n}$. The symbol $\mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$ will denote the set of such multipermutations. For simplicity, $\mathfrak{L}[n]$ will signify the set of permutations $\mathfrak{L}[1^1 2^1 \ldots n^1]$.

In order to define a Mahonian statistic, it is convenient to introduce the notion of q-analogs. The q-analog of a non-negative integer, the q-factorial, the q-binomial coefficient, and the q-multinomial coefficient are respectively defined to be

(2.2) (i) $(m)_q \equiv 1 + q + q^2 + \ldots + q^{m-1}$ (ii) $(m)_q! = (1)_q (2)_q \ldots (m)_q$

(iii) $\binom{m}{k}_q \equiv \dfrac{(m)_q!}{(k)_q!(m-k)_q!}$ (iv) $\binom{j}{j_1\, j_2 \ldots j_n}_q \equiv \dfrac{(j)_q!}{(j_1)_q!\,(j_2)_q! \ldots (j_n)_q!}$

where $j \equiv (j_1 + j_2 + \ldots + j_n)$ and $(0)_q! \equiv 1$. A statistic $s : \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}] \to \mathbb{R}$ (reals) is then said to be Mahonian if

$$\sum_{\sigma} q^{s(\sigma)} = \binom{j}{j_1\, j_2 \ldots j_n}_q \tag{2.3}$$

where the sum is over all σ in $\mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$. In the permutation case, the right-hand side of (2.3) reduces to $(n)_q!$.

The two classic examples of Mahonian statistics are the inversion number and the major index. Let $|A|$ denote the cardinality of a set A and respectively define the descent set and the descent number of a multipermutation $\sigma \in \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$ of length j by

(2.4) (i) $\mathcal{D}es\,\sigma \equiv \{k : 1 \le k \le j-1,\ \sigma(k) > \sigma(k+1)\}$ (ii) $des\,\sigma \equiv |\mathcal{D}es\,\sigma|$.

Then the inversion number and the major index of σ are defined to be

(2.5) (i) $inv\,\sigma \equiv |\{(k,l) : 1 \le k < l \le j,\ \sigma(k) > \sigma(l)\}|$ (ii) $maj\,\sigma \equiv \sum_{k \in \mathcal{D}es\,\sigma} k$

As an example, for $\sigma = 2\,1\,2\,2\,1\,3\,1 \in \mathcal{L}[1^3 2^3 3^1]$, it is easy to see that $\mathcal{D}es\,\sigma = \{1, 4, 6\}$, $des\,\sigma = 3$, $maj\,\sigma = 1 + 4 + 6 = 11$, and $inv\,\sigma = 3 + 0 + 2 + 2 + 0 + 1 + 0 = 8$. The fact that both maj and inv are Mahonian on $\mathcal{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$ was first established by MacMahon [M1].

For simplicity, the discussion of binary tree statistics is restricted to the case of permutations. For a set C containing $(n + 1)$ integers, let $\mathcal{L}[C]$ denote the set of permutations on C. For $\sigma \in \mathcal{L}[C]$, let k be such that $\sigma(k + 1)$ is the minimum of C and define

(2.6) (i) $A \equiv \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$ (ii) $B \equiv \{\sigma(k + 2), \sigma(k + 3), \ldots, \sigma(n + 1)\}$.

Then the rooted binary planar tree decomposition of σ is defined to be the factorization

(2.7) $$\sigma = \alpha\, m\, \beta$$

where $\alpha \equiv \sigma(1)\sigma(2)\ldots\sigma(k) \in \mathcal{L}[A]$, m is the minimum element of C, and $\beta \equiv \sigma(k + 2)\sigma(k + 3) \ldots \sigma(n + 1) \in \mathcal{L}[B]$. The mapping

(2.8) $$\sigma \to (A, B, \alpha, \beta)$$

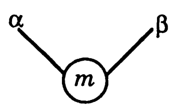
from $\mathcal{L}[C]$ to the set of 4-tuples (A, B, α, β) satisfying the conditions

(2.9) (i) $A \cup B = C \setminus \{m\}$ (ii) $A \cap B = \phi$

(iii) $\alpha \in \mathcal{L}[A]$ (iv) $\beta \in \mathcal{L}[B]$

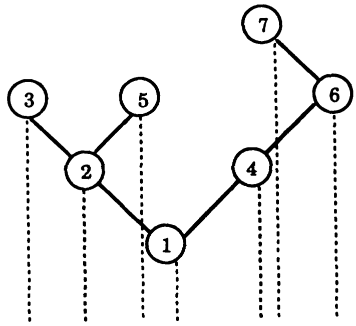
is a bijection.

If one views the factorization $\sigma = \alpha\, m\, \beta$ geometrically as



and iterates, then the result is a rooted binary planar tree with increasing labels. For example, the binary tree associated with $\sigma = 3\,2\,5\,1\,4\,7\,6 \in \mathcal{L}[7]$ is

(2.10)



$\sigma = 3 \quad 2 \quad 5 \quad 1 \quad 4 \; 7 \quad 6$

As indicated in (2.10), σ is easily recovered from its binary tree by simply projecting onto a horizontal axis.

Finally, as essentially defined in [R5], a real valued map s on the set of permutations of integers is said to be a binary tree statistic if for all σ factorized as in (2.7) there exist constants $a, b, c \in \mathbb{R}$ and a function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that

(2.11) $$s(\sigma) = a\, s(\alpha) + b\, s(\beta) + c\, I(A,B) + f\,(|A|, |B|)$$

where $\mathbb{N} \equiv \{0, 1, 2, \ldots\}$, $s(\phi) = 0$, and $I(A,B)$ denotes the number of inversions from set A to B, that is, $I(A,B) \equiv |\{(k,l) \in A \times B : k > l\}|$.

Although seemingly remote, many classic permutation statistics actually satisfy (2.11). For one, the descent number of a permutation $\sigma \in \mathcal{L}[n]$ satisfies the following recurrence relationship

(2.12) $$\mathit{des}\,\sigma = \mathit{des}\,\alpha + \mathit{des}\,\beta + \chi(|A| \geq 1)$$

where χ("statement") is defined to be 1 if "statement" is true and 0 otherwise. As a second example, it is easily verified that the inversion number of a permutation σ satisfies the identity

(2.13) $$\mathit{inv}\,\sigma = \mathit{inv}\,\alpha + \mathit{inv}\,\beta + I(A,B) + |A|.$$

Several examples of binary tree statistics are listed in Table 1 (which is partially reproduced from [R5]).

TABLE 1

Examples of Binary Tree Statistics		
Name	References	Identity relative to $\sigma = \alpha\, m\, \beta$
1. length		$len\ \sigma = len\ \alpha + len\ \beta + 1$
2. descents	CS,DB,FS	$des\ \sigma = des\ \alpha + des\ \beta + \chi(\lvert A\rvert \geq 1)$
3. rises	C1	$ris\ \sigma = ris\ \alpha + ris\ \beta + \chi(\lvert A\rvert = 0)$
4. inversions	M1, S	$inv\ \sigma = inv\ \alpha + inv\ \beta + I(A,B) + \lvert A\rvert$
5. 312 patterns	R4	$312(\sigma) = 312(\alpha) + 312(\beta) + I(A,B)$
6. 213 patterns	R4	$213(\sigma) = 213(\alpha) + 213(\beta) - I(A,B) + \lvert A\rvert\,\lvert B\rvert$
7. left lower records	C1,DB	$\ell r(\sigma) = \ell r(\alpha) + 1$
8. right lower records	C1,DB	$rr(\sigma) = rr(\beta) + 1$
9. troughs	FV	$tr(\sigma) = tr(\alpha) + tr(\beta) + \chi(\lvert A\rvert \geq 1)\,\chi(\lvert B\rvert \geq 1)$
10. peaks (leaves)	FV	$p(\sigma) = p(\alpha) + p(\beta) + \chi(\lvert A\rvert = 0)\,\chi(\lvert B\rvert = 0)$

Essentially, the only permutation statistic considered in this paper which is not a binary tree statistic is the major index. However, it does satisfy the identity

(2.14)
$$maj\ \sigma = maj\ \alpha + maj\ \beta + \lvert A\rvert + (\lvert A\rvert + 1)\ des\ \beta$$

relative to the factorization in (2.7).

3. A Generalization of Moritz' and Williams' Game In solving a problem associated with a simple coin-tossing game, Moritz and Williams [MW] discovered a "statistically" induced measure on $\mathcal{L}[n]$. However, as their results readily extend to the setting of multipermutations, the following generalized versions of their game and problem are considered.

(3.1) Multipermutation Extension of Moritz' and Williams' Game. Players 1, 2, ..., n respectively begin with $j_1, j_2, \ldots, j_n$ lives. In turn, a coin is passed from player to player which, when tossed, lands heads up with probability p and tails up with probability $q = 1 - p$. Upon receiving the coin, player k attempts to toss a string of consecutive tails equal in length to his/her remaining number of lives. If successful, player k passes the coin to player $(k + 1)$. If not successful, player k loses a life and tries again to toss a string of consecutive tails equal in length to his/her remaining number of lives. Player k continues to toss until succeeding. In the situation that player k has no remaining lives, then player k of course achieves success after zero tosses. The game ends when all lives have been lost.

<u>Problem</u>. For $\sigma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$, determine the probability that the players lose their lives in the order specified by σ (that is, $\sigma(\ell)$ is the player who loses the ℓ^{th} life of the game).

As an illustration of this game, suppose players 1, 2, 3 respectively have $j_1 = 1$, $j_2 = 3$, and $j_3 = 2$ lives. Then, a typical flipping sequence (FS) together with the resulting multipermutation of "death" are as displayed below

(3.2)
$$\begin{array}{l}
\text{FS} = \text{T} : \text{T T H T H T} : \text{T T} \;\Big|\; \text{H} : \text{T} : \text{T H T} \;\Big|\; \phi : \text{H} : \text{T} \;\Big|\; \phi : \phi : \text{H} \\
\sigma = \quad 2 \quad 2 \quad {*}1 \quad 3 \quad {*} \quad 2 \quad 3
\end{array}$$

where asterisks highlight descents in σ, colons in FS indicate when the coin is being passed to the next player, and bars demark sweeps through the tossing order (i.e., the first sweep through the tossing order 1, 2, 3 is T : T T H T H T : T T with player 2 losing two lives, the second sweep through the order is H : T : T H T with players 1 and 3 each losing one life, and so on).

To solve the problem stated in (3.1), it is a relatively easy matter to adapt Moritz' and Williams' proof for the permutation case. The first step is to extend their norm on permutations to the multipermutation setting: For $\sigma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$, let MFS($\sigma$) denote the "minimal (i.e. shortest) flipping sequence" for which the game results in σ. Then the norm of σ is defined to be

(3.3)
$$cm\,\sigma \equiv \text{"the number of tails occurring in MFS}(\sigma)\text{."}$$

For instance, the multipermutation σ of (3.2) together with its associated minimal flipping sequence are given below in (3.4). Thus, $cm\,(221323) = 6$.

(3.4)
$$\begin{array}{l}
\text{MFS}(\sigma) = \text{T} : \text{H H T} : \text{T T} \;\Big|\; \text{H} : \text{T} : \text{H T} \;\Big|\; \phi : \text{H} : \text{H} \\
\sigma = \quad 2\ 2 \quad {*}\,1 \quad 3 \quad {*} \quad 2 \quad 3
\end{array}$$

The solution to the problem of (3.1) may now be stated and proven: For $\sigma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$, the probability that the game of (3.1) ends in σ is given by

(3.5) $$P_{cm}(\sigma) = q^{cm\,\sigma} \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1}$$

where $j \equiv j_1 + j_2 + \ldots + j_n$. The proof proceeds by induction. Clearly, the formula for P_{cm} is true when $j = 1$. Then, to carry out the induction step, begin by observing that the first heads in a flipping sequence that results in $\sigma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$ either occurs

(3.6) (i) after the initial sweep through the tossing order, or

(ii) during the initial sweep.

In case (i), the game may be considered as simply being restarted after the initial consecutive string of j tails has been tossed.

In case (ii), some player, say k, tosses the first heads on the ℓ^{th} toss of the game where $j_1 + j_2 + \ldots + j_{k-1} < \ell \le j_1 + j_2 + \ldots + j_k$. In this case, players 1, 2, ..., k, ..., n may be viewed as starting a new game in which the distribution of lives is $j_1, j_2, \ldots, j_k - 1, \ldots, j_n$ and in which the tossing order is $k, k+1, \ldots, n, 1, 2, \ldots, k-1$. Corresponding to this new tossing order, sequentially rename the players as $k = \underline{1}$, $k+1 = \underline{2}$, ..., $k-1 = \underline{n}$. Since the flipping sequence for the new game is obtained by lopping off the initial ℓ tosses from the corresponding flipping sequence of σ, the result of the new game is a multipermutation $\gamma \in \mathcal{L}[\underline{1}^{j_k - 1}\, \underline{2}^{j_{k+1}} \ldots \underline{n}^{j_{k-1}}]$ which satisfies the property

(3.7) $$cm\,\sigma = j_1 + j_2 + \ldots + j_{k-1} + cm\,\gamma .$$

To complete the proof of (3.5), first note that the two cases of (3.6) together imply that

(3.8) $$P_{cm}(\sigma) = q^j P_{cm}(\sigma) + \sum_{\ell} q^{\ell - 1}\, p\, P_{cm}(\gamma)$$

where the index ℓ runs from $(j_1 + j_2 + \ldots + j_{k-1} + 1)$ to $(j_1 + j_2 + \ldots + j_k)$. Then, inductively assuming that formula (3.5) holds for γ, it follows from (3.7) and (3.8) that

(3.9) $$P_{cm}(\sigma) = \frac{1-q}{1-q^j} \sum_{\ell} q^{\ell - 1 + cm\gamma} \binom{j-1}{j_1\, j_2 \cdots j_k - 1 \cdots j_n}_q^{-1}$$

$$= \frac{q^{cm\,\sigma}(j_k)_q}{(j)_q} \binom{j-1}{j_1\, j_2 \cdots j_k - 1 \cdots j_n}_q^{-1} = q^{cm\,\sigma} \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1} .$$

Thus, (3.5) is true for all integers $j \ge 1$.

In closing this section, it is worth noting that the measure P_{cm} reduces to the usual equiprobable measure on $\mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$ when $q = 1$. Thus, P_{cm} is a q-analog of the equiprobable measure.

4. The Comajor Index Unknowingly, Moritz and Williams actually proved that their norm is Mahonian on $\mathcal{L}[n]$. Essentially their argument can be used to verify that the norm is also Mahonian on $\mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$: Since P_{cm} is a measure and because of (3.5), one has that

$$1 = \sum_{\sigma} P_{cm}(\sigma) = \sum_{\sigma} q^{cm\sigma} \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1}$$

where both sums are over all $\sigma \in \mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$. But this immediately implies that

$$\sum_{\sigma} q^{cm\sigma} = \binom{j}{j_1\, j_2 \cdots j_n}_q .$$

However, it turns out that the norm is not a new Mahonian statistic. In fact, it is nothing more than a slightly disguised variation of the major index. Known as the comajor index [DF], this variation is defined for $\sigma \in \mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$ by

$$\text{(4.1)} \qquad comaj\, \sigma \equiv \sum_{k \in \mathcal{D}es\, \sigma} (j - k)$$

where $j \equiv j_1 + j_2 + \dots + j_n$. Thus, in contrast, the major index sums descent indices "relative to the left-hand side of σ" and the comajor index sums descent indices "relative to the right-hand side."

In order to verify that the norm and the comajor index are indeed equal, one begins by reconsidering the game of (3.1) and the example of (3.4). It is not difficult to see that for any multipermutation $\sigma \in \mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$ and its associated $MFS(\sigma)$ that the following facts hold:

(4.2) (i) There is a natural one-to-one correspondence between bars and asterisks.

(ii) The contribution to $comaj\,\sigma$ made by the k^{th} descent in σ is equal to the number of tails between the $(k-1)^{\text{st}}$ and k^{th} bars in MFS (σ).

It then immediately follows that $comaj\,\sigma = cm\,\sigma$ (which explains the usage of cm in denoting the norm). Incidentally, the preceding argument was first given for the permutation case in [RT].

Turning our attention to random variables, example (3.4) suggests a very natural one: Let $\mathbf{X}: \mathfrak{L}[1^{j_1} 2^{j_2} \dots n^{j_n}] \to \mathbb{R}$ be the number of bars that occur in the MFS associated to a multipermutation. Thus, by (4.2i), $\mathbf{X}(\sigma) = des\,\sigma$.

Now, although the measure P_{cm} is relatively new, it turns out that the literature on permutation and multipermutation statistics is full of methods and results which are of significance to the study of the descent number relative to the q-measure P_{cm}. However, being neither immediately interested in nor aware of q-measures, researchers have not presented results in q-probabilistic settings. Thus, there is usually some degree of work involved in extracting desired results.

As an example on the level of permutations, consider the probability generating function for descents relative to the measure P_{cm} on $\mathfrak{L}[n]$, that is,

$$(4.3) \qquad C_n(t,q) \equiv \sum_{\sigma \in \mathfrak{L}[n]} t^{des\,\sigma}\, P_{cm}(\sigma)\,.$$

From the methods of [R4], it is possible to derive a recurrence relationship for $C_n(t,q)$: Begin by observing that, even though not a binary tree statistic, the comajor index satisfies the identity

$$(4.4) \qquad cm\,\sigma = cm\,\alpha + cm\,\beta + (|B|+1)\,\chi(|A| \geq 1) + (|B|+1)\,des\,\alpha$$

for any permutation σ factorized as in (2.7). In view of (2.8) and (2.12), it then follows that $C_{n+1}(t,q)$ is equal to

$$\frac{1}{(n+1)_q!} \sum_{k=0}^{n} \sum_{|A|=k} \sum_{\alpha \in \mathfrak{L}[A]} \sum_{\beta \in \mathfrak{L}[B]} (t\,q^{|B|+1})^{\chi(|A| \geq 1)}\, (tq^{|B|+1})^{des\,\alpha}\, q^{cm\,\alpha} t^{\,des\,\beta} q^{cm\,\beta}.$$

Taking into account the natural correspondence between $\mathfrak{L}[A]$ and $\mathfrak{L}[\,|A|\,]$ and then by regrouping terms, one is led to the recurrence relationship

$$(4.5) \qquad (n+1)_q C_{n+1}(t,q) = C_n(t,q) + t \sum_{k=1}^{n} q^{n-k+1} \binom{n}{k} \binom{n}{k}_q^{-1} C_k(tq^{n-k+1}, q)\, C_{n-k}(t,q)$$

where $C_0(t,q) \equiv 1$.

Besides several more identities on the level of permutations, an explicit formula for the distribution of des relative to P_{cm} on $\mathcal{L}[1^{j_1}\, 2^{j_2} \dots n^{j_n}]$ could be given at this point. However, it is more convenient to present these identities at the end of the next section.

5. The Major Index Game As *comaj* and *maj* are closely related, it is possible to alter (3.1) in such a way so as to give rise to a measure induced by the major index. A "$\mathcal{Maj}$" version of (3.1) is given in (5.1).

(5.1) The $\mathcal{Maj}$ Game. Players 1, 2, ..., n respectively begin with $j_n, j_{n-1}, \dots, j_1$ lives. In turn, a coin is passed from player to player which, when tossed, lands heads up with probability p and tails up with probability $q = 1 - p$. Upon receiving the coin, player k attempts to toss a string of consecutive tails equal in length to the number of lives remaining to player $(n + 1 - k)$. If successful, player k passes the coin to player $(k + 1)$. If not, player $(n - k + 1)$ loses a life and player k reattempts the task of tossing a string of consecutive tails equal in length to the now diminished number of lives remaining to player $(n + 1 - k)$. Player k continues tossing until succeeding. In the event that player $(n + 1 - k)$ has no remaining lives, player k of course succeeds after zero tosses. The game ends when all lives have been lost.

Problem. For $\theta \in \mathcal{L}[1^{j_n}\, 2^{j_{n-1}} \dots n^{j_1}]$, determine the probability that the players go out in the reverse order specified by θ (that is, for $j \equiv j_1 + j_2 + \dots + j_n$, $\theta(j + 1 - \ell)$ is the player who loses the ℓ^{th} life of the game).

The probability $P_m(\theta)$ that $\theta \in \mathcal{L}[1^{j_n}\, 2^{j_{n-1}} \dots n^{j_1}]$ is the outcome of the game in (5.1) is given by

$$(5.2) \qquad P_m(\theta) = q^{maj\,\theta} \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1} .$$

Formula (5.2) may of course be derived by appropriately modifying the proof of (3.5). However, a bijective proof is given here which lays bare the explicit relationship between the outcomes σ and θ of the games of (3.1) and (5.1).

To begin with, note that if S is any sequence of tosses such that σ results when the rules of (3.1) are applied and θ results when the rules of (5.1) are applied, then $\sigma = cr\theta$ where the reversal r and the complement c of a multipermutation $\gamma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \dots n^{j_n}]$ are respectively defined to be

$$(5.3) \qquad \text{(i)} \quad r\gamma \equiv \gamma(j)\gamma(j-1)\dots\gamma(1)$$

$$\text{(ii)} \quad c\gamma \equiv (n+1-\gamma(1))(n+1-\gamma(2))\dots(n+1-\gamma(j)).$$

Since the probability of any such S occurring is independent of the game being played, it follows that $P_m(\theta) = P_{cm}(\sigma)$. Moreover,

$$cm\,\sigma = cm(cr\theta) = \sum_{k=1}^{j-1} (j-k)\ \chi(n+1-\theta(j+1-k) > n+1-\theta(j-k))$$

$$= \sum_{\ell=1}^{j-1} \ell\ \chi(\theta(\ell+1) < \theta(\ell)) \ = \ maj\,\theta.$$

Thus, from (3.5), one has that

$$P_m(\theta) = P_{cm}(\sigma) = \frac{q^{cm(cr\theta)}}{\binom{j}{j_1\, j_2 \cdots j_n}_q} = \frac{q^{maj\,\theta}}{\binom{j}{j_1\, j_2 \cdots j_n}_q} ,$$

which establishes formula (5.2).

Since $des\,\theta = des\,(cr\theta)$, the descent number is again a natural random variable to consider. In fact, as maj is a classic Mahonian statistic, the literature on permutation and multipermutation statistics contains a substantial amount of information relating the descent number and the major index. From MacMahon's work [M2, Vol. 2, p. 211], one can derive that the probability generating function for des on multipermutations is

$$(5.4) \qquad \sum_{\theta} t^{des\,\theta}\, P_m(\theta) = \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1} (t:q)_{n+1} \sum_{k=0} t^k \prod_{\ell=1}^{n} \binom{k+j_\ell}{j_\ell}_q$$

where the sum is over all $\theta \in \mathcal{L}[1^{j_n}\, 2^{j_{n-1}} \ldots n^{j_1}]$ and $(t:q)_{n+1} \equiv (1-t)(1-tq)\ldots(1-tq^n)$. In the case of permutations, if one defines

$$(5.5) \quad \text{(i)} \ \ M_n(t,q) \equiv \sum_{\sigma \in \mathcal{L}[n]} t^{des\,\sigma}\, P_m(\sigma) \qquad \text{(ii)} \ \ M_{n,k}(q) \equiv \sum_{\sigma \in \mathcal{L}[n]} \chi(des\,\theta = k)\, P_m(\theta) ,$$

then, from (2.8) and [C2], it may be verified that

$$(5.6) \quad \text{(i)} \quad (n+1)_q\, M_{n+1}(t,q) = M_n(tq,q) + t \sum_{k=1}^{n} q^k \binom{n}{k}\binom{n}{k}_q^{-1} M_k(t,q)\, M_{n-k}(tq^{k+1},q)$$

$$\text{(ii)} \quad (n+1)_q\, M_{n+1,k}(q) = (k+1)_q\, M_{n,k}(q) + q^k (n+1-k)_q\, M_{n,k-1}(q)$$

where $M_0(t,q) \equiv 1$, $M_{0,0}(q) \equiv 1$ and $M_{0,k}(q) \equiv 0$ for $k \geq 1$.

It is now convenient to unveil the additional identities for the distribution of the descent number relative to P_{cm} which were alluded to at the end of section 4. By first using the bijection

$$cr : \mathfrak{L}[1^{j_n}\, 2^{j_{n-1}} \ldots\, n^{j_1}] \to \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$$

and then by noting that, if $\sigma = cr\theta$, then $des\, \sigma = des\,(cr\theta) = des\, \theta$ and $P_{cm}(\sigma) = P_m(\theta)$, it follows that

$$\sum_{\theta} t^{des\, \theta}\, P_m(\theta) = \sum_{\sigma} t^{des\, \sigma}\, P_{cm}(\sigma)$$

where the first sum is over all $\theta \in \mathfrak{L}[1^{j_n}\, 2^{j_{n-1}} \ldots\, n^{j_1}]$ and the second is over $\sigma \in \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$. Thus, the right-hand side of (5.4) gives the probability generating function for des relative to P_{cm} on $\mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$ and the identities of (5.6) remain valid if P_m is replaced by P_{cm} in (5.5).

6. The Inversion Game Since the inversion number is a classic Mahonian statistic, it is natural to question whether or not there exists a game that leads to a measure on $\mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$ which is induced by the inversion number. The answer to this query is yes and such a game is described in (6.1).

(6.1) The *Inv* Game. Players 1, 2, ..., n respectively begin with $j_1, j_2, \ldots, j_n$ lives. Beginning with player 1, a coin is passed from player to player which, when tossed, lands heads up with probability p and tails up with probability $q = 1 - p$. Upon receiving the coin, player k makes a single attempt at tossing a string of consecutive tails equal in length to his/her remaining number of lives. If player k is successful, then the coin is passed to player $(k + 1)$. If player k tosses a heads, then player k loses a life, the coin is immediately passed back to player 1, and play resumes. The game ends when all lives have been lost.

Problem. For $\sigma \in \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$, determine the probability that the players lose their lives in the order specified by σ.

To prove that the probability of $\sigma \in \mathfrak{L}[1^{j_1}\, 2^{j_2} \ldots\, n^{j_n}]$ being the outcome of (6.1) is given by the formula

$$P_i(\sigma) = q^{inv\, \sigma} \binom{j}{j_1\, j_2 \cdots j_n}_q^{-1}, \tag{6.2}$$

it is insightful to consider a specific example: Using the symbol $mfs(\sigma)$ to denote the minimal flipping sequence for which the *Inv* game results in σ, the $mfs(\sigma)$ for the multipermutation $\sigma = 2\,2\,1\,3\,2\,3 \in \mathcal{L}[1^1\, 2^3 3^2]$ is displayed below

(6.3)
$$\begin{array}{rcccccccccccccccccc} mfs\,(\sigma) = & T & : & H & \Big| & T & : & H & \Big| & H & \Big| & \phi : T & : & H & \Big| & \phi : & H & \Big| & \phi : \phi : H \\ & & & \vdots & & & & \vdots & & \vdots & & & & \vdots & & & \vdots & & \vdots \\ \sigma = & & & 2 & & & & 2 & & 1 & & & & 3 & & & 2 & & 3 \end{array}$$

where the colons indicate instances at which the coin is being passed from player k to player $(k+1)$ and the bars indicate when the coin is being passed back to player 1. The important insight to gain from (6.3) is that the contribution made to the inversion number by $\sigma(k)$, $1 \le k \le j-1$, is equal to the number of tails between the $(k-1)^{\mathrm{st}}$ and k^{th} bars of the associated $mfs(\sigma)$. Thus,

(6.4)
$$inv\ \sigma = \text{"the number of tails in } mfs(\sigma)\text{."}$$

With minor modifications, the induction proof of (3.5) may now be recast so as to establish (6.2): Clearly, (6.2) is true for $j = 1$. To carry out the induction step, note that the first heads to occur in a flipping sequence which results in σ either takes place

(6.5) (i) after the j^{th} toss, or (ii) on or before the j^{th} toss.

In case (i), the game may be considered as being restarted after the first j consecutive tails have been tossed.

In case (ii), some player, say k, tosses the first heads on the ℓ^{th} toss of the game where $j_1 + j_2 + \ldots + j_{k-1} < \ell \le j_1 + j_2 + \ldots + j_k$. Player k loses a life, passes the coin back to player 1, and a new game is started which results in a multipermutation $\gamma \in \mathcal{L}[1^{j_1}\, 2^{j_2} \ldots k^{j_k - 1} \ldots n^{j_n}]$. Moreover, γ satisfies the property that

(6.6)
$$inv\,\sigma = j_1 + j_2 + \ldots + j_{k-1} + inv\,\gamma\ .$$

Together, the cases of (6.5) imply that

(6.7)
$$P_i(\sigma) = q^j P_i(\sigma) + \sum_{\ell} q^{\ell - 1} p\, P_i(\gamma)$$

where the index ℓ runs from $(j_1 + j_2 + \dots + j_{k-1} + 1)$ to $(j_1 + j_2 + \dots + j_k)$. Using a calculation analogous to the one of (3.9), formula (6.2) then follows by induction.

As for natural random variables, the number of bars in $mfs(\sigma)$, which is $(j-1)$ for all $\sigma \in \mathcal{L}[1^{j_1} 2^{j_2} \dots n^{j_n}]$, is of no interest. However, although not as evident as in (3.4), the descent number of σ can be characterized in terms of $mfs(\sigma)$: Letting $T(k)$ denote the number of tails between the $(k-1)^{\text{st}}$ and k^{th} bars of $mfs(\sigma)$, one has that $\sigma(k) > \sigma(k+1)$ if and only if $T(k) > T(k+1)$.

Although the *inv* is a classic Mahonian statistic, apparently no closed or reasonable recurrence formulas are known which relate the descent and inversion numbers on the level of multipermutations. In the case of permutations though there are a number of readily available formulas: If one defines

$$(6.8) \qquad I_n(t,q) \equiv \sum_{\sigma \in \mathcal{L}[n]} t^{des\,\sigma} P_i(\sigma),$$

then, from [R1,S], one has that

$$(6.9) \qquad \sum_{n \geq 0} I_n(t,q) u^n = \frac{(1-t)\, exq[(1-t)u]}{1 - t\, exq[(1-t)u]}$$

where $exq[z] \equiv \sum_{n \geq 0} \frac{z^n}{(n)_q!}$ is a q-analog of the exponential function.

Except for a minor twist, a "binary tree" recurrence relationship for $I_n(t,q)$ can be derived in essentially the same way as the one of (4.5). First, note from [GJ, p. 98] that, for a set D of n integers,

$$(6.10) \qquad \sum_{(A,B)} q^{I(A,B)} = \binom{n}{k}_q$$

where the sum is over all ordered pairs (A,B) such that $|A| = k$, $A \cup B = D$ and $A \cap B = \phi$. Then, (2.8), (2.12) and (2.13) imply that $I_{n+1}(t,q)$ is equal to

$$\frac{1}{(n+1)_q!} \sum_{k=0}^{n} \sum_{|A|=k} \sum_{\alpha \in \mathcal{L}[A]} \sum_{\beta \in \mathcal{L}[B]} t^{\chi(|A| \geq 1)} q^{|A|} q^{I(A,B)} t^{des\,\alpha} q^{inv\,\alpha} t^{des\,\beta} q^{inv\,\beta}.$$

Then regrouping and using (6.10) gives the identity

$$(6.11) \qquad (n+1)_q I_{n+1}(t,q) = I_n(t,q) + t \sum_{k=1}^{n} q^k I_k(t,q) I_{n-k}(t,q)$$

where $I_0(t,q) \equiv 1$.

7. Distributions of Binary Tree Statistics on $\mathfrak{L}[n]$ In the case of permutations, the bijection $\sigma \to (A, B, \alpha, \beta)$ of (2.8) may be used to obtain the distributions of any binary tree statistic. The recurrence relationships for two such distributions relative to P_i and P_m are now presented. For a binary tree statistic s as defined in (2.11), let

$$(7.1) \quad \text{(i)} \quad I_n(z,q) \equiv \sum_{\sigma \in \mathfrak{L}[n]} z^{s(\sigma)} P_i(\sigma) \qquad \text{(ii)} \quad M_n(z,t,q) \equiv \sum_{\sigma \in \mathfrak{L}[n]} z^{s(\sigma)} t^{des\,\sigma} P_m(\sigma)$$

where, for technical reasons, an extra parameter involving the descent number has been included in the *maj* setting. Then, from (2.8), (2.11) through (2.14), and (6.10), it is not too difficult to verify that

$$(7.2) \quad \text{(i)} \quad (n+1)_q I_{n+1}(z,q) = \sum_{k=0}^{n} z^{f(k,n-k)} q^k \binom{n}{k}_{qz^c} \binom{n}{k}_q^{-1} I_k(z^a,q)\, I_{n-k}(z^b,q)$$

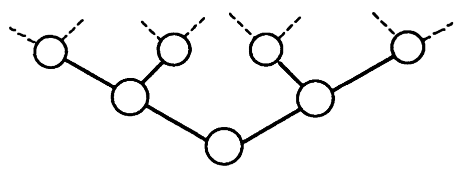
$$\text{(ii)} \quad (n+1)_q M_{n+1}(z,t,q) = \sum_{k=0}^{n} z^{f(k,n-k)} t^{\chi(|A|\geq 1)} q^k \binom{n}{k}_{z^c} \binom{n}{k}_q^{-1} M_k(z^a,t,q) M_{n-k}(z^b,tq^{k+1},q)$$

with the initial conditions $I_0(z,q) \equiv 1$ and $M_0(z,q) \equiv 1$.

8. A Binary Tree Measure on $\mathfrak{L}[n]$ In many instances, the derivation of an identity involving permutation statistics is based on some permutation decomposition. For instance, the binary tree decomposition $\sigma \to (A, B, \alpha, \beta)$ is the underlying basis for the identities of (7.2). With this in mind, it becomes natural to consider whether or not there is a measure which is "compatible" with a given decomposition. In this vein, a game is presented in this section which is "compatible" with the binary tree decomposition. A second example of a decomposition based game is given in section 9.

The playing board for the "binary tree" game, as sketched in (8.1), is an infinite binary tree with each node being empty and having two ascendant nodes.

(8.1)



The rules of the binary tree game and its associated problem are as follows:

(8.2) The Binary Tree Game. In turn, players 1, 2, ..., n approach the root of (8.1) and seek out an empty node to occupy. Player k does not begin searching until after player $(k - 1)$ has located and occupied a node. Each player's search is governed by a coin which, when tossed, comes up heads with probability p and tails with probability $q = (1 - p)$. Whenever an occupied node is encountered, the coin is to be tossed: If the coin lands heads (tails) up, then the search must proceed to the right (left) ascendant of the occupied node. Upon encountering an empty node, a player must occupy it. After all players have occupied nodes, the players are then ranked according to the order that results when the players are projected onto a horizontal axis.

Problem. For $\sigma \in \mathfrak{L}[n]$, determine the probability that the players are ranked according to σ.

As an example of this game, the result of the sequence of tosses

$$S = \phi : \mathrm{T} : \mathrm{T\,T} : \mathrm{H} : \mathrm{T\,H} : \mathrm{H\,H} : \mathrm{H\,H\,T} \tag{8.3}$$

is the permutation σ of (2.10). Note that the subsequence of S lying between the $(k-1)^{\mathrm{st}}$ and k^{th} colons corresponds to player k's search.

To determine the probability $P_{bt}(\sigma)$ that the game of (8.1) ends in $\sigma \in \mathfrak{L}[n]$, begin by observing that the flipping sequence S associated to σ is unique. Then, if one defines

(8.4) (i) $T(\sigma) \equiv$ "the number of tails in S" (ii) $H(\sigma) \equiv$ "the number of heads in S,"

it trivially follows that

$$P_{bt}(\sigma) = q^{T(\sigma)}\, p^{H(\sigma)} \, . \tag{8.5}$$

As is easily verified, neither T nor H is Mahonian. Moreover, for no choice of the value of q will P_{bt} reduce to the usual equiprobable measure $1/n!$ on $\mathfrak{L}[n]$.

Although (8.5) is straightforward enough, there are a couple of alternate ways of calculating $P_{bt}(\sigma)$. Relative to (2.8), we have

$$P_{bt}(\sigma) = q^{|A|}\, p^{|B|}\, P_{bt}(\alpha)\, P_{bt}(\beta) \tag{8.6}$$

where $P_{bt}(\phi) \equiv 1$. The second way, which is independent of both S and the binary tree decomposition, relies on the fact that T and H can both be expressed as linear combinations of known permutation statistics; namely, the inversion number and the number of *312* patterns (see Table 1).

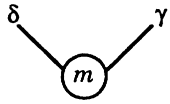
In order to observe these linear combinations, one needs to first return to a previous characterization of a *312* pattern as given in [R4]: An ordered triple (i, j, l) is said to be a *312* pattern in a permutation $\sigma \in \mathfrak{L}[n]$ if

(8.7) (i) $1 \le i < j < l \le n$ (ii) $\sigma(j) < \sigma(l) < \sigma(i)$

(iii) $\sigma(j)$ = mininimum of the set $\{ \sigma(i), \sigma(i+1), \ldots, \sigma(l) \}$.

Then, $312(\sigma)$ is defined to be the number of 312 patterns in σ.

Now, if player k, upon reaching the node occupied by m as diagrammed below,



tosses tails, then player k must proceed to δ and thereby create a total of (1 + length of γ) inversions and a total of (length of γ) 312 patterns. Thus,

$$T(\sigma) = inv\,\sigma - 312(\sigma).$$

Analogous reasoning may be used to verify that $H(\sigma) = inv(r\sigma) - 312(r\sigma)$ where r is as defined in (5.3i).

As a primary advantage of considering the binary tree measure, the probability generating function for a binary tree statistic relative to P_{bt} satisfies a recurrence relationship which, although similar to, is much simpler than those of (7.2). Let

$$(8.8) \qquad T_n(z,q) \equiv \sum_{\sigma \in \mathcal{L}[n]} z^{s(\sigma)} P_{bt}(\sigma)$$

where s is a binary tree statistic as defined in (2.11). Then, once again using (2.8) together with (6.10) and (8.6), one is led to the identity

$$(8.9) \qquad T_{n+1}(z,q) = \sum_{k=0}^{n} z^{f(k,n-k)} q^k (1-q)^{n-k} \binom{n}{k}_{z^c} T_k(z^{a'},q) T_{n-k}(z^b,q)$$

where $T_0(z,q) \equiv 1$.

9. THE r-MAJOR INDEX GAME A second example of a game which is compatible with a decomposition arises in connection with the statistic known as the r-major index. For simplicity, this game will only be presented in the context of permutations.

As in [R2], for an integer $r \ge 1$, the r-descent set and r-descent number of a permutation $\sigma \in \mathcal{L}[n]$ are respectively defined to be

$$(9.1) \quad (i) \quad r\mathcal{D}es\,\sigma \equiv \{k : \sigma(k) \ge \sigma(k+1) + r, 1 \le k \le n-1\} \qquad (ii) \quad rdes\,\sigma \equiv |r\mathcal{D}es\,\sigma| .$$

The r-major index is then defined to be

$$(9.2) \qquad rmaj\,\sigma \equiv |\{(k,\ell) : 1 \le k < \ell \le n, \sigma(k) > \sigma(\ell) > \sigma(k) - r\}| + \sum_{k \in r\mathcal{D}es\,\sigma} k .$$

For instance, if $r = 2$ and $\sigma = 47836215 \in \mathcal{L}[8]$, then $r\mathcal{D}es\,\sigma = \{3, 5\}$, $rdes\,\sigma = 2$, and $rmaj\,\sigma = 5 + (3 + 5) = 13$.

In view of (2.5), the r-major index is seen to be a weighted mixture between the major index and the inversion number. In fact, $rmaj$ reduces to maj when $r = 1$ and to inv when $r = \infty$. Moreover, as established in [R2], the r-major index is Mahonian on $\mathcal{L}[n]$, that is,

$$\sum_{\sigma \in \mathcal{L}[n]} q^{rmaj\,\sigma} = (n)_q! \, . \tag{9.3}$$

The decomposition employed in [R2] to establish (9.3) is based on observing the effect that "inserting" n into a permutation $\gamma \in \mathcal{L}[n-1]$ has on the r-major index. To tabulate this effect, the n possible insertion positions in $\gamma = \gamma(1)\gamma(2) \ldots \gamma(n-1)$ are labeled as follows: Using labels $0, 1, \ldots, (n-1)$ in order, first scan γ from right to left and label the positions that, upon insertion of n, will **not** result in the creation of a new r-descent. Then, scanning back from left to right, label the remaining positions. As an example, for $\gamma = 4736215 \in \mathcal{L}[7]$ and $r = 2$, the top and bottom rows of the display

$$\tag{9.4} \begin{array}{cccccccccccccccc} & \bullet & & 3 & & 2 & & \bullet & & 1 & & \bullet & & \bullet & & 0 \\ & & & \downarrow & & \downarrow & & & & \downarrow & & & & & & \downarrow \\ \gamma = & & 4 & & 7 & & 3 & & 6 & & 2 & & 1 & & 5 & \\ & \uparrow & & & & & & \uparrow & & & & \uparrow & & \uparrow & & \\ & 4 & & \bullet & & \bullet & & 5 & & \bullet & & 6 & & 7 & & \end{array}$$

indicate the labels as distributed by the two scans.

As given in [R2], the key facts concerning this insertion procedure may be summed up as follows: If $\Gamma(\gamma, \ell)$ denotes the permutation that results when n is inserted into position ℓ of γ, then

(9.5) (i) $\Gamma : \mathcal{L}[n-1] \times \{0, 1, \ldots, n-1\} \to \mathcal{L}[n]$ is a bijection and

(ii) $rmaj\ \Gamma(\gamma, \ell) = \ell + rmaj\,\gamma$.

For instance, if γ is the permutation of (9.4), then $\Gamma(\gamma,2) = 47836215$ and $rmaj\,\Gamma(\gamma,2) = 13 = 2 + rmaj\,\gamma$.

A game which is compatible with the "insertion" decomposition Γ may now be stated:

(9.6) The *rMaj* Game. Players 1, 2,..., n are to be ranked in a linear order. For $1 < k \leq n$, assume that players 1, 2, ..., $(k-1)$ have been ranked according to a permutation $\gamma \in \mathcal{L}[k-1]$. Player k then determines his/her ranking relative to players 1, 2,..., $(k-1)$ by flipping a coin until heads occurs: If the heads occurs on the $(mk + \ell + 1)^{\text{th}}$ toss where $0 \leq \ell \leq k-1$, then player k is inserted into position ℓ of γ. The coin is then passed to player $(k+1)$.

Problem. If the probability of heads is p and of tails is $q = 1-p$, then what is the probability that the players will be ranked according to a given permutation $\sigma \in \mathcal{L}[n]$?

To establish that the solution to the problem of (9.6) is given by the formula

$$P_r(\sigma) = \frac{q^{rmaj\,\sigma}}{(n)_q!} \tag{9.7}$$

for $\sigma \in \mathcal{L}[n]$, first note that it is clearly true for $n = 1$. Then, for $n \geq 2$, assume that (9.7) holds for all $\gamma \in \mathcal{L}[n-1]$. By (9.5i), any $\sigma \in \mathcal{L}[n]$ is of the form $\sigma = \Gamma(\gamma, \ell)$ for some $\gamma \in \mathcal{L}[n-1]$ and $0 \leq \ell \leq n-1$. It then follows that

$$P_r(\sigma) = (q^{\ell} + q^{\ell+n} + q^{\ell+2n} + \dots)\; p\; P_r(\gamma) = \frac{q^{\ell}(1-q)}{(1-q^n)} \bullet \frac{q^{rmaj\,\gamma}}{(n-1)_q!} = \frac{q^{rmaj\,\sigma}}{(n)_q!} .$$

Thus, (9.7) holds for all $n \geq 1$.

As for random variables, there is one that is very natural: For $\sigma \in \mathcal{L}[n]$ and $1 \leq k \leq n-1$, suppose that $\gamma_k \in \mathcal{L}[k]$ and $0 \leq \ell_k \leq k$ are such that

$$\sigma = \Gamma(\gamma_{n-1}, \ell_{n-1}),\ \gamma_{n-1} = \Gamma(\gamma_{n-2}, \ell_{n-2}), \dots, \gamma_2 = \Gamma(\gamma_1, \ell_1) \tag{9.8}$$

where $\gamma_1 \equiv 1$. Then define

$$\mathfrak{X}_k(\sigma) \equiv \chi(\ell_k \text{ is a "bottom" row label}) \tag{9.9}$$

where top and bottom row labels are as exemplified in (9.4). It then follows that

$$rdes\,\sigma = \mathfrak{X}_1(\sigma) + \mathfrak{X}_2(\sigma) + \dots + \mathfrak{X}_{n-1}(\sigma) . \tag{9.10}$$

Unfortunately though, the random variables $\{\mathfrak{X}_k\}_{k \geq 1}$ are not independent.

Using the properties of Γ as listed in (9.5), the distribution of *rdes* relative to P_r defined by

$$(9.11) \qquad R_{n,k}(q) \equiv \sum_{\sigma \in \mathcal{L}[n]} \chi(rdes\,\sigma = k)\, P_r(\sigma)$$

may be easily verified to satisfy the recurrence relationship

$$(9.12) \qquad (n+1)_q\, R_{n+1,\,k}(q) = (r+k)_q\; R_{n,\,k}(q) \;+\; q^{k+r-1}(n+2-k-r)_q\, R_{n,\,k-1}(q)$$

where $R_{r,\,0}(q) = (r)_q!$. Of course, (9.12) reduces to (5.6ii) when r = 1.

In closing this section, it is noted that the game of (9.6) and the measure of (9.7) readily extend to the set of multipermutations $\mathcal{L}[1^{j_1}\, 2^{j_2} \ldots n^{j_n}]$. The relevant insertion procedure and identities may be found in [R3] .

10. CONCLUDING REMARKS The asymptotic results mentioned in the introduction suggest a host of tantalizing problems. In view of the fact that the descent number, the inversion number, and the number of records (see Table 1) are all known to be asymptotically normal relative to the equiprobable measure on $\mathcal{L}[n]$, it is only natural to consider the following questions:

(10.1) (i) What class of binary tree statistics are asymptotically normal relative to the equiprobable measure on $\mathcal{L}[n]$? What about relative to the measures P_{cm}, P_m, P_i, and P_{bt} ?

(ii) Is *rdes* asymptotically normal relative to the measure P_r on $\mathcal{L}[n]$? What about on multipermutations?

(iii) Which statistics on multipermutations are asymptotically normal relative to P_{cm}, P_m, or P_i ?

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