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A NOTE ON THE SUPPORT OF RIGHT INVARIANT MEASURES

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ABSTRACT. A regular measure μ on a locally compact topological semigroup is called right invariant if $\mu(Kx) = \mu(K)$ for every compact K and x in its support. It is shown that this condition implies a property reminiscent of the right cancellation law. This is used to generalize a theorem of A. Mukherjea and the author (with a new proof) to the effect that the support of an r*-invariant measure is a left group iff the measure is right invariant on its support.

KEY WORDS AND PHRASES. Topological semigroup, left group, right invariant (Borel) measure, r*-invariant measure, support of a Borel measure, locally compact semigroup.

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1. INTRODUCTION.

In what follows S will denote a T_2, locally compact topological semigroup (jointly continuous multiplication) and μ a positive <u>regular</u> (Radon) measure on the Borel σ-algebra of S with support $F = \{s \in S;$ for every open $V \supset s$, $\mu(V) > 0\}$, as in [1] and [2]. We shall use the notation $Bx^{-1} = t_x^{-1}(B) = \{s \in S; sx \in B\}$, t_x denoting the right continuous translation $s \to sx$. The measure is called <u>r*-invariant on S</u> if $\mu(Bx^{-1}) = \mu(B)$ for all Borel B and x in S. Such measures received considerable attention in the past in connection with the (still unsolved) conjecture of L.N. Argabright (Proc. Amer. Math. Soc. <u>17</u> (1966), 377-382) that their support is a <u>left group</u> i.e., F is <u>left simple</u> (Fx =F for all x in F) and right cancellative (equivalently , if it is left simple and contains an idempotent element). The measure μ is called <u>right invariant on its support</u> if

$$\mu(Kx) = \mu(K) \text{ for every compact } K \subset F \text{ and every } x \in F \tag{1.1}$$

In [1] A. Mukherjea and the author proved the "rather tight" result

THEOREM 1. The support of an r*-invariant measure on S is a left group iff the measure is right invariant on its support.

Professor Mukherjea in a meeting at University of South Florida asked the questions (i) whether the "intriguing" condition (1.1) (introduced by himself) implies some sort of right cancellation on F in view of the fact proven by Rigelhof [3] that (1.1) plus that the t_x's, $x \in F$, are open maps, imply right cancellation on F. (ii) Whether Theorem 1 (above) can be substantially generalized. In this note we show: As for question (i) indeed there is a generalized" right cancellation on S (See

Lemma 1, below) but as for question (ii), Theorem 1 cannot substantially be generalized except that we may only assume that $\mu(Bx^{-1}) \geq \mu(B)$ for every Borel B and every $x \in F$. (Unlike condition (1.1), no extra generality is obtained whether we assume $B \subset S$ or $B \subset F$). Moreover, our proof (although patented on that of [1]) does not use the functional analytic apparatus of [1] since it uses a version of cancellation from the intrinsic properties of the measure.

2.

We begin by showing in what sense S is pre- right cancellable mod F.

LEMMA 1. Let μ be right invariant on its support (i.e., μ satisfies (1.1)). Then

(i) If for $f_1, f_2, f_3 \in F$, $f_1f_2 = f_3f_2$, then $ff_1 = ff_3$ for every $f \in F$ that is, we can cancel on the right by premultiplying by any element of the support.

(ii) If F is also a right ideal of S, then for $s_1, s_3 \in S$, $f_2 \in F$, the equation $s_1f_2 = s_3f_2$ implies $fs_1 = fs_3$ for all $f \in \overline{FF}$ = closure(FF) and in particular for any idempotent element $e \in F$.

PROOF. We shall argue by contradiction as in Rigelhof [3, p. 175] . We prove (ii): (The proof of (i) is done similarly). Assume $s_1f_2 = s_3f_2$ but $fs_1 \neq fs_3$ so that we can find disjoint compact neighborhoods U and V respectively of these two distinct points (with f some point in $\overline{FF}$). Now $Us_1^{-1} \cap Vs_3^{-1}$ must contain a compact neighborhood W of f which in turn must contain a right translate of some compact neighborhood of the form $K\varphi$ for some $\varphi \in F$ ($f \in \overline{FF}$), i.e.,

$K\varphi \subset W \subset Us_1^{-1} \cap Vs_3^{-1}$, so that

$$\mu(K) + \mu(K) = \mu(K\varphi s_1) + \mu(K\varphi s_3) = \mu(K\varphi s_1 \cup K\varphi s_3)f_2) = \mu(K \cup K)\varphi s_1 f_2) = \mu(K),$$

which is a contradiction.

COROLLARY 1. Let μ satisfy (1.1). Then

(i) For any pair of idempotents $e_1, e_2 \in F$, we have $e_1e_2 = e_1$ so that the idempotents in F form a left-zero subsemigroup of F.

(ii) For any idempotent $e \in F$, eF is right cancellable.

(iii) If $zyzyx = zyx$ for $x,y,z \in F$, then zy is idempotent.

PROOF. (i): It follows since $e_1e_2 = e_1e_2e_2$ and by Lemma 1 we may cancel e_2 by premultiplying by e_1 and use the fact that e_1 is idempotent. (ii): Similarly by the above Lemma. (iii): First cancel x by premultiplying by y and then cancel zy by premultiplying by z and obtain $zyzy = zy$.

Now we are ready to give the generalization of Theorem 1 as follows:

THEOREM 2. Suppose μ satisfies

$$\mu(Bf^{-1}) \geq \mu(B) \quad \text{for every Borel } B \text{ and every } f \in F \tag{2.1}$$

Then F is a left group iff μ satisfies (1.1).

PROOF. Clearly (1.1) plus inner regularity of μ imply $\mu(Bf^{-1}) \leq \mu(B)$ for all Borel B and $f \in F$ so that we have $(Bf^{-1}) = \mu(B)$ for every $f \in F$ and Borel B. Also (2.1) implies that $\overline{Ff} = F$ for all $f \in F$. In the proof of Theorem 1 in [1], we produced an idempotent e in Fa, for $a \in F$, so that $Fe = \overline{Fe} = F \subset Fa$ and so $Fa = F$ for all $a \in F$ (cf. [1],p. 974). Now, the same proof goes through without any difficulty except that instead of the right cacellation on Fa, $a \in F$, we use Corollary 1 (iii) above.

We give next a result summarizing certain important conditions on F and μ that are equivalent to F being a left group.

COROLLARY 2. For a locally compact second countable semigroup S admitting an r*-invariant measure μ, these are equivalent:

(i) F is right cancellable

(ii) μ is right invariant on its support, i.e., satisfies (1.1))

(iii) S is pre- right cancellative with respect to F, i.e., $s_1s_2 = s_3s_2$ with $s_1,s_2,s_3 \in F$, implies $fs_1 = fs_3$ for all $f \in F$.

(iv) F is a left group.

(v) F has the right translations t_f closed for all $f \in F$.

(vi) F has the right translations open and μ satisfies (1.1).

REMARK. It is not known to our knowledge if (v) and (iv) are equivalent in the absence of second countability.

PROOF. Most of these follow from Theorem 1 or Theorem 2. Note that right cancellation implies that t_f are one-to-one and for compact K, $Kxx^{-1} \cap F = Kx$,so that right invariance on its support follows from r*-invariance, so (i) => (ii) => (iii) => (iv) (cf. Theorem 2). Since F is metrizable being regular the technique in [4] for producing an idempotent in Fx applies and thus F becomes a left group. By the result of Rigelhof, (vi) implies (i).([3]p. 175). For (iii), see our Lemma 2, below.

REMARK. The following will show the "tightness" of the conditions of Theorem 2. It is well known that a property that "melds" naturally (at first sight) with condition (1.1) is that of lower r*-invariance, i.e., that $\mu(Bx^{-1}) \leq \mu(B)$ for all Borel $B \subset S$ and $x \in S$, for <u>it</u> and (1.1) are equivalent to the condition (cf.[2] and [5, p. 92])

$\mu(Kx) \geq \mu(K)$ for all compact $K \subset S$ and $x \in S$ <u>with</u> (2.2)
this inequality becoming equality whenever K and x <u>are in F</u>.

This condition (2.2) implies that F is a right ideal and $Fe = F$ for every idempotent $e \in S$, but these are not enough to make Theorem 2 valid, for the example of $[0 , \infty)$ with addition and Lebesgue measure shows that μ is not r*-invariant (it does not satisfy (2.1) of Theorem 2). However this S is pre- right cancellative as the following Lemma generally indicates.

LEMMA 2. Suppose μ satisfies (2.2) and suppose that $s_1s_2 = s_3s_2$ for $s_1,s_2,s_3 \in S$. Then $fs_1 = fs_3$ for all $f \in \overline{FF}$. If moreover $s_1s_2 \in F$, then $fs_1 = fs_3$ for all $f \in F$.

PROOF. Suppose first that $s_1s_2 \notin F$. Then the second equality in the proof of Lemma 1 (ii) with f_2 replaced by s_2 becomes less or equal and the last remains equality and thus a contradiction obtains. Next assume that $s_1s_2 = s_3s_2$ and $s_1s_2 \in F$. Again, as before (See proof of Lemma 1) there are disjoint compact neighborhoods

U, V of fs_1 , fs_3 respectively, such that the intersection of Us_1^{-1} and Vs_3^{-1} contain a compact neighborhood W of f (we use $W \cap F$ instead of W). Then we have again the inequality

$$\mu(W) + \mu(W) \leq \mu(Ws_1) + \mu(Ws_3) \leq \mu(Ws_1 \cup Ws_3)s_2) = \mu(W \cup W)s_1s_2) = \mu(W),$$

again a contradiction.

REMARK. The most difficult part in problems involving the nature of the support F is producing an idempotent element in Fx or in F itself. For this, it would be intersting to have a "survey paper" giving all known methods for producing an idempotent in the presence of measure and/or topological invariance conditions. Apart for having some compact subsemigroup or a compact fiber $xx^{-1} \neq \emptyset$ or a two-sided version of (2.2) and a subsemigroup of positive finite inner measure, we know only the technique in [1] which is in some sense an adoptation of a method of Gelbaum and Kalisch (Canad. J. of Math. 4 (1952), 396-40?), and the technique of [4] which needs metrizability !. (For example, when the t_x are closed mappings, can the "ontoness" of the t_x be used to prove that the operator $\pi_s f(x) = f(xs)$ on $L_2(S, \mu)$ is onto in the non-second countable case ? (that will suffice to prove that the support of an r*-invariant measure is a left group when the t_x's are closed).

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