Internat. J. Math. & Math. Sci.
VOL. 14 NO. 4 (1991) 731-736

REMARKS ON QUASILINEAR EVOLUTIONS EQUATIONS

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(Received June 6, 1990 and in revised form July 21, 1990)

ABSTRACT. In this paper we study the existence result of classical solutions for the quasilinear equation $u_{tt}-\Delta u-M(\int_\Omega|\nabla u|^2dx)\Delta u_{tt} = f$, with initial data $u(0)=u_0$, $u_t(0)=u_1$ and homogeneous boundary conditions.

KEY WORDS.- Partial diferential equation, quasilinear evolution equation, boundary problem.
AMS(MOS): Subject classification, 35B65, 35A05.

1. INTRODUCTION: Let Ω be an open and bounded set of $\mathbb{R}^n$, with smooth boundary Γ. Let's denote by Q the cylinder $Q = \Omega\times]0,T[$ and by Σ its lateral boundary. Our notations and function spaces are standart and follows the same pattern as Lions's book [2].

Ebihara et al [1] was proved that there exist only one classical solution for a semilinear model, given by following initial-boundary value problem

$$u_{tt} - \Delta u - M(\int_\Omega |\nabla u|^2dx)\Delta u_{tt} = f \quad \text{in } Q \tag{1.1}$$

$$u(0) = u_0,\ u_t(0) = u_1 \quad \text{in } \Omega \tag{1.2}$$

$$u(x,t) = 0 \quad in\ \Sigma \tag{1.3}$$

when the following hypotheses hold:

(i) $M(\lambda) \in C^1(0,+\infty)$, *and there exist positive constants* α, ρ *such that the following inequality is valid:*

$$M(\lambda) \geq \alpha\sqrt{\lambda} + \rho, \ \forall\ \lambda \in [0,+\infty[$$

(ii) *There exists a non negative function* $\beta(\lambda)$ *satisfying:*

$$\left|\frac{d}{d\lambda}M(\lambda)\right|\sqrt{\lambda} \leq \beta(\lambda)M(\lambda) \qquad \forall\ \lambda \geq 0$$

(iii) *The initial datas are such that:*

$$u_0,\ u_1 \in D(A^{(l+1)/2}), \quad l \geq 1$$

$$f,\ \frac{d}{dt}f \in C(0,T;D(A^{l/2})), \quad l \geq 1$$

Where $A = -\Delta$ and for $D(A^s)$ we are denoting the domain of the operator A^s. The main result of this paper is to prove the existence result of classical solutions for system (1.1)-(1.3) when

H1. *M is a continuos function such that:* $M(\lambda) \geq m_0 > 0$

H2. $f \in C(0,T;D(A^{l/2}))$, $l \geq 2$ *and* $u_0,\ u_1 \in D(A^{(l+1)/2})$, $l \geq 2$

2. THE MAIN RESULT: Let's denote by $w_1, \ldots, w_m$ and by $\lambda_1, \ldots, \lambda_m$ the m firts *orthonormal* eigen functions and eigen values of the Laplacian respectively. Let's denote by V_m the finite dimensional vector space generated by the firts m eigen functions and by P_m the projector operator on V_m, that is:

$$P_m v = \sum_{l=1}^{m}\left(\int_\Omega v(x)w_l(x)dx\right)w_l$$

It is easy to see that $A^s P_m = P_m A^s$ in $D(A^s)$. Moreover we have that

$$\int_\Omega |P_m w|^2 dx \leq \int_\Omega |w|^2 dx \tag{2.1}$$

Then the aproximated problem is defined as follows.

$$u_{tt}^{(m)} - \Delta u^{(m)} - M\left(\int_\Omega |\nabla u^{(m)}|^2 dx\right)\Delta u_{tt}^{(m)} = f_m \tag{2.2}$$

$$u^{(m)}(0) = u_0^m,\ u_t^{(m)}(0) = u_1^m \quad in\ \Omega$$

where

$$u^{(m)}(t) = \sum_{l=1}^{m} g^{l,m}(t)w_l, \quad u_0^m = P_m u_0, \quad u_1^m = P_m u_1$$

Before to prove the main result of this paper we will show the following Lemmas:

LEMMA 2.1.- *Let's suppose that* $v,\ v_t,\ v_{tt} \in C(0,T;L^2(\Omega))$ *and*

$$\int_\Omega |v_{tt}(x,t)|^2 dx \leq a + b\int_\Omega |v(x,t)|^2 dx$$

Then we have:

$$\int_\Omega |v(x,t)|^2 dx \le (a+2b\int_\Omega |v(x,0)|^2dx+4bt^2\int_\Omega |v_t(x,0)|^2dx)e^{4bt^4}$$

PROOF.- Since

$$v(x,t) = \int_0^t v_t(x,\xi)d\xi + v(x,0) \quad a.\ e. \quad in \quad x$$

we have:

$$|v(x,t)| \le \sqrt{t}(\int_0^t |v_t(x,\xi)|^2 d\xi)^{1/2} + |v(x,0)|$$

From where it follows

$$\int_\Omega |v(x,t)|^2 dx \le 2t\int_0^t\int_\Omega |v_t(x,\xi)|^2 dx d\xi + 2\int_\Omega |v(x,0)|^2 dx$$

Applying the relation above to v_t we have:

$$\int_\Omega |v_t(x,t)|^2 dx \le 2t\int_0^t\int_\Omega |v_{tt}(x,\xi)|^2 dx d\xi + 2\int_\Omega |v_t(x,0)|^2 dx$$

From the two last inequalities we conclude:

$$\int_\Omega |v(x,t)|^2 dx \le$$

$$2\int_\Omega |v(x,0)|^2 dx + 4t^2\int_\Omega |v_t(x,0)|^2 dx + 4t^3\int_0^t\int_\Omega |v_{tt}(x,\xi)|^2 dx d\xi$$

Finally, from the hypotheses, the last inequality and Gronwall's inequality the result of Lemma 2.1 follows □

LEMMA 2.2.- *Let suppose that $w \in C([0,T];L^2(\Omega))$, then we have that*

$$P_m w \to w \quad strong\ in\ C([0,T];L^2(\Omega))$$

PROOF.- By the pointwise convergence of $P_m w$ in t, it's sufficient to show that $P_m w$ is a Cauchy sequence in $C([0,T];L^2(\Omega))$. Let's take $\varepsilon>0$, by the continuity of w we have that there exist $\delta > 0$ such that

$$|t - s| < \delta \quad \Rightarrow \quad \int_\Omega |w(x,t) - w(x,s)|^2 dx < \frac{\varepsilon}{3} \tag{2.3}$$

By the compacity of $[0,T]$, there exist s_1, s_2, ... s_N, satisfying

$$[0,T] \subset \bigcup_{i=1}^{N}]s_i-\delta, s_i+\delta[$$

and from the pointwise convergence of $P_m w$ we conclude that there exists a positive number N such that

$$\int_\Omega |P_m w(\cdot,s_i) - P_\mu w(\cdot,s_i)|^2 dx < \frac{\varepsilon}{3}, \quad \forall\ m,\ \mu \ge N,\ i = 1,\dots,N \tag{2.4}$$

Finally by (2.1), (2.3), (2.4) and the following inequality

$$[\int_\Omega |P_m w(x,t)-P_\mu w(x,t)|^2 dx]^{1/2} \le$$

$$[\int_\Omega |P_m(w(x,t)-w(x,s_i))|^2 dx]^{1/2} + [\int_\Omega |P_m w(x,s_i)-P_\mu w(x,s_i)|^2 dx]^{1/2} +$$

$$+ [\int_\Omega |P_\mu(w(x,s_i)-w(x,t))|^2 dx]^{1/2}$$

the result of Lemma 2.2 follows □

THEOREM 2.3.- *Let's suppose that* **H1** *and* **H2** *are valid. Then there exists*

(1.1), (1.2) and (1.3). Remains to show that u is a classical solution. Let's note that $u^{(m)}$ belongs to $C^2(0,T;D(A^{(l+1)/2}))$ for all $m \in \mathbb{N}$, then in order to prove that $u \in C^2(0,T;C^k(\Omega))$, we will show that $(u_{tt}^{(m)})_{m\in\mathbb{N}}$ is a Cauchy's sequence in $L^\infty(0,T;D(A^{(l+1)/2}))$, for all $l \geq 2$. In fact let $\mu \in \mathbb{N}$, then

$$u_{tt}^{(\mu)} - \Delta u^{(\mu)} - M(\int_\Omega |\nabla u^{(\mu)}|^2 dx)\Delta u_{tt}^{(\mu)} = P_\mu f$$

From (2.2) and the above equation we have:

$$(u_{tt}^{(m)}-u_{tt}^{(\mu)}) - \Delta(u^{(m)}-u^{(\mu)}) - M(\int_\Omega |\nabla u^{(m)}|^2 dx)\Delta(u_{tt}^{(m)}-u_{tt}^{(\mu)}) = G_{m\mu}$$

where

$$G_{m\mu} = (M(\int_\Omega |\nabla u^{(m)}|^2 dx)-M(\int_\Omega |\nabla u^{(\mu)}|^2 dx))\Delta u_{tt}^{(\mu)} + P_m f - P_\mu f$$

Multiplying the system above by $A^l(u_{tt}^{(m)}-u_{tt}^{(\mu)})$ and integrating in Ω we have

$$m_0\int_\Omega |A^{\frac{l+1}{2}}(u_{tt}^{(m)}-u_{tt}^{(\mu)})|^2 dx \leq$$

$$\int_\Omega |A(u^{(m)}-u^{(\mu)})A^l(u_{tt}^{(m)}-u_{tt}^{(\mu)})|dx + \int_\Omega |G_{m\mu}A^l(u_{tt}^{(m)}-u_{tt}^{(\mu)})|dx$$

From which it follows that:

$$\frac{1}{2}m_0^2\int_\Omega |A^{\frac{l+1}{2}}(u_{tt}^{(m)}-u_{tt}^{(\mu)})|^2 dx \leq \int_\Omega |A^{\frac{l+1}{2}}(u^{(m)}-u^{(\mu)})|^2 dx + \int_\Omega |A^{\frac{l}{2}}G_{m\mu}|^2 dx$$

From Lemma (3.1) and the last inequality we have

$$\frac{1}{2}m_0^2\int_\Omega |A^{\frac{l+1}{2}}(u_{tt}^{(m)}-u_{tt}^{(\mu)})|^2 dx \leq$$

$$(\int_\Omega |A^{\frac{l}{2}}G_{m\mu}|^2 dx + 2\int_\Omega |A^{\frac{l+1}{2}}(u_0^m-u_0^\mu)|^2 dx + 4t^2\int_\Omega |A^{\frac{l+1}{2}}(u_1^m-u_1^\mu)|^2 dx)Exp(\frac{8}{m_0^2}\ t^4)$$

Finally from Lemma 2.2 and since u_0, $u_1 \in D(A^{(l+1)/2})$ we have that

$$A^{l/2}G_{m\mu} \to 0 \text{ as } m,\ \mu \to +\infty \ \ \textit{strongly in } C([0,T];L^2(\Omega))$$

Then we have that $(u_{tt}^{(m)})$ a Cauchy sequence in $L^\infty(0,T;D(A^{(l+1)/2}))$ and the proof is now complete □

REMARK 2.4.- UNIQUENESS: If M is locally Lipschitz, then we have uniqueness. In fact, let u and v be two solutions, putting $w = u-v$ we have

$$w_{tt} - \Delta w - M(\int_\Omega |\nabla u|^2 dx)\Delta w_{tt} = (M(\int_\Omega |\nabla u|^2 dx) - M(\int_\Omega |\nabla v|^2 dx))\Delta v_{tt}$$

Multiplying by Δw_{tt} applying H1 and the Lipschitz condition on M we have that there exists a positive constant c_1 such that:

$$m_0\int_\Omega |\Delta w_{tt}|^2 dx \leq \int_\Omega |\Delta w \Delta w_{tt}|dx + c_1(\int_\Omega |\Delta w|^2 dx)^{1/2}(\int_\Omega |\Delta w_{tt}|^2 dx)^{1/2}$$

only one classic solution of system (1.1), (1.2) and (1.3)

PROOF.- Since $D(A^{(l+1)/2}) \subset H^{l+1}(\Omega) \subset C^k(\tilde{\Omega})$ if $l+1 > \frac{n}{2} + k$, it's sufficient to show that there exists a solution of system (1.1), (1.2) and (1.3) satisfying $u \in C^2([0,T];D(A^{(l+1)/2}))$. In order to prove it let's multiply (2.2) by $A^l u_{tt}^{(m)}$ and integrating in Ω we have:

$$\int_\Omega |A^{\frac{l}{2}} u_{tt}^{(m)}|^2 dx + M(\int_\Omega |\nabla u^{(m)}|^2 dx)\int_\Omega |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx =$$

$$- \int_\Omega A u^{(m)} A^l u_{tt}^{(m)} dx + \int_\Omega f_m A^l u_{tt}^{(m)} dx$$

By H1 and H2 the last equality becomes:

$$m_0 \int_\Omega |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx \le \int_\Omega |A^{\frac{l+1}{2}} u^{(m)} A^{\frac{l+1}{2}} u_{tt}^{(m)}| dx + \int_\Omega |[A^{\frac{l}{2}} f_m] A^{\frac{l}{2}} u_{tt}^{(m)}| dx$$

from where it follows that:

$$\frac{1}{2} m_0^2 \int_\Omega |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx \le \frac{1}{\lambda_1^2} \int_\Omega |[A^{\frac{l}{2}} f_m]|^2 dx + \int_\Omega |A^{\frac{l+1}{2}} u^{(m)}|^2 dx$$

By Lemma 2.1 and the above inequality we obtain:

$$\frac{1}{2} m_0^2 \int_\Omega |A^{\frac{l+1}{2}} u_{tt}^{(m)}(x,t)|^2 dx \le \tag{2.5}$$

$$(\int_\Omega |[A^{\frac{l}{2}} f_m]|^2 dx + 2\int_\Omega |A^{\frac{l+1}{2}} u_0^m|^2 dx + 4t^2 \int_\Omega |A^{\frac{l+1}{2}} u_1^m|^2 dx) Exp(\frac{8}{m_0^2} t^4)$$

From (2.5) and since:

$$\int_\Omega |A^{\frac{l+1}{2}} u_t^{(m)}(x,t)|^2 dx \le 2t \int_\Omega |A^{\frac{l+1}{2}} u_{tt}^{(m)}(x,t)|^2 dx + 2\int_\Omega |A^{\frac{l+1}{2}} u_1^m|^2 dx$$

$$\int_\Omega |A^{\frac{l+1}{2}} u^{(m)}(x,t)|^2 dx \le 2t \int_\Omega |A^{\frac{l+1}{2}} u_t^{(m)}(x,t)|^2 dx + 2\int_\Omega |A^{\frac{l+1}{2}} u_0^m|^2 dx$$

we conclude that there exists a subsequence of $(u^{(m)})_{m \in \mathbb{N}}$, which we still denoting of the same way and a function $u \in L^\infty(0,T;D(A^{(l+1)/2}))$, satisfying

$$u^{(m)} \to u \quad \textit{weak star in } L^\infty(0,T;D(A^{(l+1)/2})) \quad \textit{as } m \to \infty$$

$$u_t^{(m)} \to u_t \quad \textit{weak star in } L^\infty(0,T;D(A^{(l+1)/2})) \quad \textit{as } m \to \infty$$

$$u_{tt}^{(m)} \to u_{tt} \quad \textit{weak star in } L^\infty(0,T;D(A^{(l+1)/2})) \quad \textit{as } m \to \infty$$

From the last convergences and the Lions-Aubin's theorem (see Lions's [2], theorem 5.1, chap 1) we conclude in particular that:

$$u^{(m)} \to u \quad \textit{strongly in } C([0,T];H_0^1(\Omega)) \quad \textit{as } m \to \infty$$

By standard methods we can prove that u is a strong solution of system from where it follows that there exists c_2 such that:

$$\int_\Omega |\Delta w_{tt}|^2 dx \le c_2 \int_\Omega |\Delta w|^2 dx$$

By Lemma 2.1, since $w(x,0) = w_t(x,0) = 0$, we obtain that $\Delta w = 0$, and from this it follows that $w = 0$, that is $u = v$ □

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