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A NON-UNIQUENESS THEOREM IN THE THEORY OF VORONOI SETS

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ABSTRACT. It is shown that two distinct, bounded, open subsets of $\mathbb{R}^2$ may possess the same Voronoi set.

KEY WORDS AND PHRASES. *Voronoi set.*
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1. INTRODUCTION

Let $\{D_i\}_{0\le i\le n}$ be a finite collection of non-empty, bounded, open and simply connected subsets of $\mathbb{R}^2$ which satisfy $D_i \subset D_0$, $D_i \ne D_0$, $1 \le i \le n$ and $D_i \cap D_j = \emptyset$, $1 \le i < j \le n$. Then if we define $\Omega = D_0 \setminus \bigcup_{i=1}^{n} \overline{D}_i$, Ω is a non-empty, bounded, open and connected subset of $\mathbb{R}^2$ with boundary $\partial\Omega = \bigcup_{i=0}^{n} \partial D_i$. (Loosely speaking, Ω is a domain D_0 containing "obstacles" D_i, $1 \le i \le n$.) The the following definition of the Voronoi diagram Vor(Ω) of Ω is taken from [1].

For any $(x,y) \in \Omega$, define Near(x,y) as the set of points in $\partial\Omega$ closest to (x,y). ("Closest to" is, of course, defined in terms of ordinary Euclidean distance in the plane.) Since $\partial\Omega$ is closed, Near(x,y) is always non-empty.
The *Voronoi diagram* Vor(Ω) of Ω is then defined to be the set of points

$$\{(x,y) \in \Omega : \text{Near}(x,y) \text{ contains more than one point}\}.$$

Vor(Ω) is used in [1] in connection with motion planning problems.

Clearly given the sets $\{D_i\}$, Vor(Ω) is unique. However, here we take the opposite point of view and consider the construction of the sets $\{D_i\}$ from a given Voronoi diagram.

A preliminary question that one might ask is: could it be possible for two collections $\{D_i\}$ and $\{D_i'\}$ to have the same Voronoi diagrams? It is easy to see that the answer is yes: for $0 < \varepsilon < 1$ let

$$D_0^\varepsilon = \{(x,y) \mid x^2+y^2 < (1+\varepsilon)^2\} \quad \text{and}$$
$$D_1^\varepsilon = \{(x,y) \mid x^2+y^2 < (1-\varepsilon)^2\}.$$

Then if $\Omega^{\varepsilon} = D_0 \backslash \overline{D}_1$, $Vor(\Omega^{\varepsilon})$ is the unit circle, centre the origin, whatever the value of ε might be.

A more subtle question is the following: Suppose $D_0 = D_0'$, then is it possible for two different collections $\{D_i\}$ and $\{D_i'\}$ to have the same Voronoi diagram? Informally, what we are asking is whether, given a fixed domain D_0, it is possible to arrange two different sets of obstacles within D_0, both of which produce the same Voronoi diagram.) We show the answer is again in the affirmative.

2. THE EXAMPLE

Let

$$D_0 = \{(x,y) \mid |x| < 4, \ |y| < 4\}$$

$$D_1 = \{(x,y) \mid |x| < 3, \ 1 < y < 3\}$$

$$D_2 = \{(x,y) \mid |x| < 3, \ -3 < y < -1\}.$$

Then Ω and $Vor(\Omega)$ (where $\Omega = D_0 \backslash \overline{D}_1 \cup \overline{D}_2$) are depicted in Figure 1. Note in particular that $Vor(\Omega)$ contains the line segment $\{(x,0) \mid |x| \leq 1\}$.

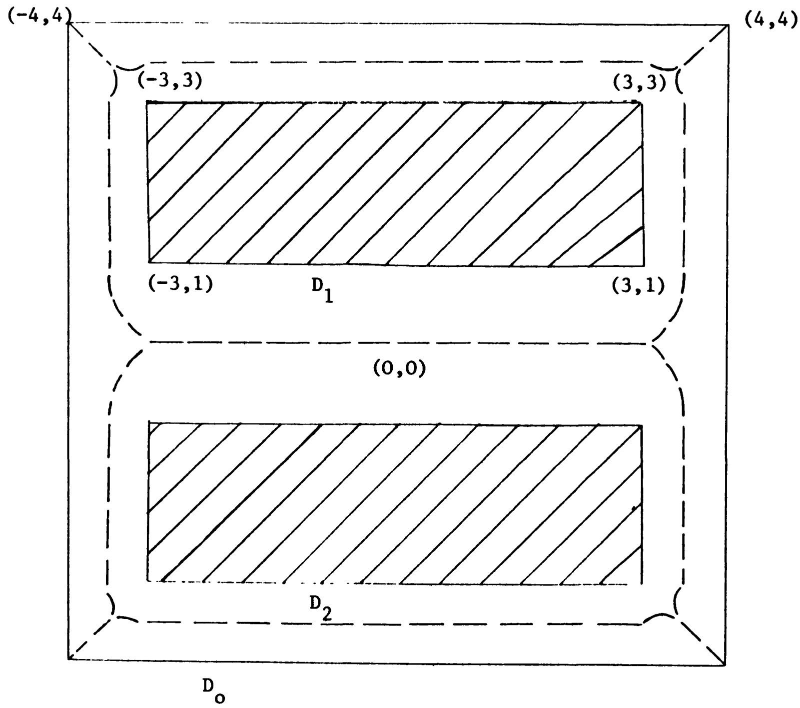

Figure 1 - $Vor(\Omega)$ is denoted by the dashed line

We modify D_1 and D_2 as follows.
Let $C = \{(x,y) \mid x^2+y^2 \le 2\}$ and put $D_1' = D_1 \backslash C$, $D_2' = D_2 \backslash C$. Then if $\Omega' = D_0 \backslash \overline{D_1'} \cup \overline{D_2'}$, $\text{Vor}(\Omega) = \text{Vor}(\Omega')$, (see Figure 2).

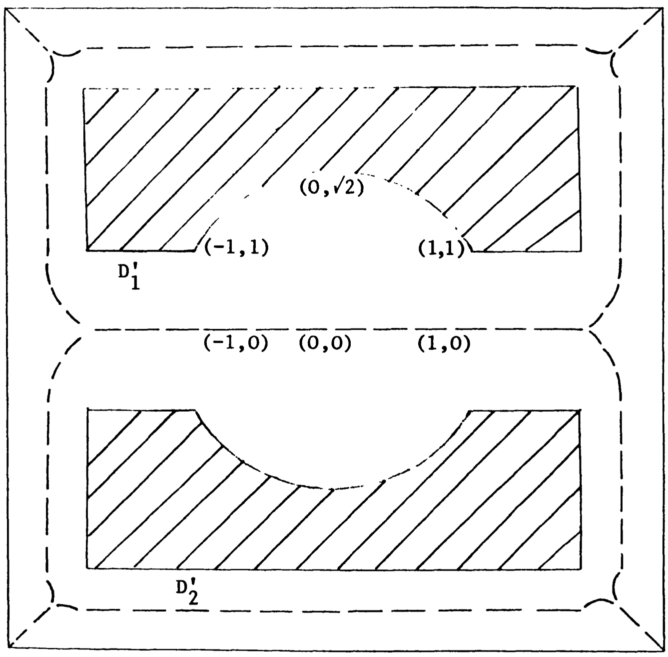

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Figure 2 - Vor(Ω') is denoted by the ashed line

To see that the Voronoi diagrams of Ω and Ω' are indeed the same first note that it suffices to consider those points (x,y) in Ω' for which $|x| \le 1$ and $|y| \le \sqrt{2}$ since for any other $(x,y) \in \Omega'$, Near(x,y) will be unchanged by the modifications made to D_1 and D_2. To begin with, consider those points *within* the triangle whose vertices are $(-1,0)$, $(0,0)$ and $(-1,1)$. It is clear that if (x,y) is such a point then Near$(x,y) = \{(-1,1)\}$ and so $(x,y) \notin \Omega'$. The same conclusion is true for the points in Ω' which lie on the straight lines joining $(-1,1)$ to $(-1,0)$ and $(-1,1)$ to $(0,0)$, (excluding the endpoints of those lines). Next consider the points $(x,0)$ where $-1 \le x < 0$. For such a point Near$(x,0) = \{(-1,1), (-1,-1)\}$ and so $(x,0) \in \text{Vor}(\Omega')$. It is also clear that $(0,0) \in \text{Vor}(\Omega')$. Now consider those points within the sector of C which has vertices $(0,0)$, $(-1,1)$ and $(0,\sqrt{2})$. If (x,y) is such a point then it is easy to see that Near(x,y) consists of the single point obtained by projecting the straight line joining $(0,0)$ to (x,y) until it intersects D_1'. The same conclusion is true for the points on the straight line between $(0,0)$ and $(0,\sqrt{2})$ (excluding the endpoints of course). The results for the remaining points in Ω' follow immediately from the symmetry of Ω'. Hence $\mathrm{Vor}(\Omega) = \mathrm{Vor}(\Omega')$.

A possible weakness of this example is that the sets D_1' and D_2' are not convex. The answer to the same question as that posed in §1 but with the additional hypothesis that all the sets in $\{D_i\}$ and $\{D_i'\}$ be convex would appear to be unknown.

REFERENCES

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