A REMARK ON THE WEIGHTED AVERAGES FOR SUPERADDITIVE PROCESSES

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ABSTRACT. A decompostion of a superadditive process into a difference of an additive
and a positive purely superadditive process is obtained. This result is used to prove
an ergodic theorem for weighted averages of superadditive processes.

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1. INTRODUCTION. Let (X, μ) be a probability space and let $L_p(X, ,\mu)$, $1 \leq p \leq \infty$ be

the classical Banach space of real valued functions f with $\int |f|^p d\mu = ||f||_p^p < \infty$.

Let $T:L_p \to L_p$ be a linear operator. A family of L_p functions $F = \{F_n\}_{n \geq 1}$ is called a

T-superadditive process if

$$F_{n+m} \geq F_n + T^n F_m, \text{ for all } n, m \geq 1, \tag{1.1}$$

and F is called a T-additive process if equality holds in (1.1). Notice that if we

let $f_i = F_{i+1} - F_i$, for $i \geq 0$ we have $F_n = \sum_{i=0}^{n-1} f_i$, where $F_0 \equiv 0$, for all $n \geq 1$. Consider

a sequence $A = \{a_n\}_{n \geq 0}$ of complex numbers and a T-(super) additive process F. We

define a family of L_p-functions $(F,A) = \{a_n f_n\}_{n \geq 0}$, and set $S_n(F,A) = \sum_{i=0}^{n-1} a_n f_n$. If A is

the constant sequence $1=(1,1,\ldots.)$, then $S_n(F,A) = F_n$.

In the following, we observe that the weighted and subsequential ergodic theorems for T-superadditive processes are direct consequences of their T-additive counterparts.

2. THE DECOMPOSITION OF F.

In this section a decomposition of a T-superadditive process F into a difference of a T-additive process G and a positive, purely T-superadditive process H (that is, H is a positive T-superadditive process that does not dominate any non-zero positive T-additive process) is obtained.

CASE p=1. Let T be a positive Dunford-Schwartz operator i.e., T is an L_1-contraction with $||T||_\infty \leqslant 1$. We will also assume that T is Markovian, that

is $\int Tf \, d\mu = \int f \, d\mu$. In this case, if $\sup\limits_{n>1} \frac{1}{n} ||F_n||_1 < \infty$, then the decomposition result is

obtained by M.A. Akcoglu and L. Sucheston [1]. Namely, they obtained that for all $n \geqslant 1$,

$$F_n = G_n - H_n \qquad\qquad\qquad\qquad (2.1)$$

where $G_n = \sum\limits_{i=0}^{n-1} T^i \delta$ for some $\delta \varepsilon L_1$, and $H_n = \sum\limits_{i=0}^{n-1} h_i$ with $h_i = f_i - T^i \delta$. Using this

result they showed that $\lim\limits_{n \to \infty} \frac{1}{n} F_n$ exists a.e., and moreover it is a consequence of

the same result that $\lim\limits_{n \to \infty} \frac{1}{n} H_n = 0$ a.e.

CASE $1 < p < \infty$. In this case we let T be a positive L_p-contraction and F a T-

superadditive process with $\lim\limits_{n \to \infty} \inf ||\frac{1}{n} \sum\limits_{i=0}^{n} (F_i - TF_{i-1})||_p < \infty$. Under these

conditions B. Hachem [2] showed that $\lim\limits_{n \to \infty} \frac{1}{n} F_n$ exists a.e. by reducing the problem to

a problem in an appropriate L_1-space and employing Akcoglu-Sucheston's result in case p=1 above. Here we observe that the same technique can be applied to yield to a decomposition result similar to (2.1).

Using a result of M.A. Akcoglu and L. Sucheston [3] one can decompose X uniquely into disjoint union of sets E and E^c where:

(i) E is the support of a T-invariant function $h \varepsilon L_p^+$, and supp $g \subset E$ for all T-invariant $g \varepsilon L_p^+$.

(ii) $L_p(E)$ and $L_p(E^c)$ are invariant subspaces for T.
Then the following results are obtained [2,3]:

$$1_{E^c} \left(\frac{F_n}{n}\right) \to 0 \text{ a.e., so one can assume that } X=E. \tag{2.2}$$

The operator $P:L_p(m) \to L_p(m)$ defined as $Pf = \frac{T(fh)}{h}$, $f \epsilon L_p(m)$, is a

positive $L_p(m)$-contraction and $P1=1$, where $m=h^p \cdot \mu$. $\tag{2.3}$

In particular, $\int Pfdm = \int fdm$ for all $f \epsilon L_p^+(m)$. So P can be extended to a Markovian

operator on $L_1(m)$. Consequently $F'=\{h^{-1}F_n\}$ is a bounded P-superadditive process [2]

in $L_1(m)$. Now by applying the Akcoglu-Sucheston's result [1] we can decompose F'

into a difference of a P-additive process and a positive, purely P-superadditive

process as $h^{-1}F_n=G_n-H_n$, $n \geqslant 1$. Also we see that:

$$\lim_{n \to \infty} \frac{1}{n} (h^{-1}F_n) \text{ exists m-a.e. and } \lim_{n \to \infty} \frac{1}{n} H_n = 0 \text{ m-a.e.,} \tag{2.4}$$

so $F_n = hG_n-hH_n$, and that $G=\{hG_n\}$ and $H=\{hH_n\}$ are T-additive and T-superadditive

processes respectively by (2.3). Consequently (2.4) gives that

$$\lim_{n \to \infty} \frac{1}{n} F_n \text{ exists } \mu\text{-a.e. X and}$$

$$\lim_{n \to \infty} \frac{1}{n} H_n = 0 \ \mu\text{-a.e. X} \tag{2.5}$$

by (2.2) and (2.3).

3. WEIGHTED AVERAGES.

Given a linear operator T on L_p, $1 \leqslant p \leqslant \infty$, and a sequence $A=\{a_n\}_{n \geqslant 0}$ of complex

numbers if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k f \text{ exists a.e.}$$

for all $f \epsilon L_p$, then we say that A is a good weight for T [4], or (A,T) is Birkhoff [5].

R. Sato [6] showed that the uniform sequences are good for $1 \leqslant p \leqslant \infty$. C. Ryll-
Nardzewski [7] proved that the bounded Besicovitch sequences are good for T induced by
a measure preserving transformation $\phi:X \to X$ by $Tf(x)=f(\phi(x))$ for any $f \epsilon L_1$. This
result combined with the remarkable theorem of J. Baxter and J. Olsen [5, Theorem
2.19] imply that bounded Besicovitch sequences are good for Dunford-Schwartz
operators.

Now we observe the following: Let T be an operator on L_p and F be a T-
superadditive process. If $F_n=G_n-H_n$, then for any sequence A

$$S_n(F,A) = S_n(G,A) - S_n(H,A) \tag{3.1}$$

Also

Also

$$0 < \lim_{n \to \infty} \sup \frac{1}{n} \left| S_n(A,H) \right| < M \cdot \lim_{n \to \infty} \sup \frac{1}{n} H_n$$

where $M = \sup_{n \geq 1} \left| a_n \right|$. Therefore if $\lim_{n \to \infty} \frac{1}{n} H_n = 0$ a.e., $\lim_{n \to \infty} \frac{1}{n} S_n(A,H)=0$ a.e. for

any bounded sequence A. We summarize this discussion as

THEOREM 3.2.   Let T be a positive Dunford-Schwartz operator on L_p, $1 < p < \infty$, and F be a T-superadditive process. Assume also that

(i) T is Markovian and $\sup_{n \geq 1} \left| \left| \frac{1}{n} F_n \right| \right|_1 < \infty$ when p=1,

(ii) $\lim_{n \to \infty} \inf \left| \left| \frac{1}{n} \sum_{i=1}^{n} (F_i - TF_{i-1}) \right| \right|_p < \infty$ $(F_0 = 0)$ when $1 < p < \infty$.

If A is a bounded sequence such that (A,T) is Birkhoff, then

$$\lim_{n \to \infty} \frac{1}{n} S_n (F,A) \text{ exists a.e.} \qquad\qquad (3.3)$$

REMARK 3.4.   The limit in (3.3) exists a.e. when A is a uniform sequence or a bounded Besicovitch sequence of $A \in \overline{W}_4$ [5].   In particular the subsequence theorem [5, 4] is valid for superadditive processes.

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