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FUNCTIONS STARLIKE WITH RESPECT TO OTHER POINTS

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ABSTRACT. In [7], Sakaguchi introduce the class of functions starlike with respect to symmetric points. We extend this class. For $0 \leq \beta < 1$, let $S_s^*(\beta)$ be the class of normalised analytic functions f defined in the open unit disc D such that $\mathrm{Re}\, zf'(z)/(f(z)-f(-z)) > \beta$, for some $z \in D$. In this paper, we introduce 2 other similar classes $S_c^*(\beta)$, $S_{sc}^*(\beta)$ as well as give sharp results for the real part of some function for $f \in S_s^*(\beta)$, $S_c^*(\beta)$ and $S_{sc}^*(\beta)$. The behaviour of certain integral operators are also considered.

KEY WORDS AND PHRASES. Starlike functions of order β, *functions starlike with respect to symmetrical points, close-to-convex, integral operators.*
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1. INTRODUCTION.

Let S be the class of analytic functions f, univalent in the unit disc $D = \{z : |z| < 1\}$, with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For $0 \leq \beta < 1$, denote by $S^*(\beta)$, the class of starlike functions of order β. Then $f \in S^*(\beta)$ if, and only if, for $z \in D$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta.$$

In [7], Sakaguchi introduced the class S_s^* of analytic functions f, normalised by (1.1) which are starlike with respect to symmetrical points. We begin by defining the class S_s^*, which is contained in K, the class of close-to-convex functions.

DEFINITION 1.

A function $f \in S_s^*$ if, and only if, for $z \in D$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0.$$

We now extend this definition as follows:

DEFINITION 2.

A function f with normalisations (1.1) is said to be starlike of order β, with respect to symmetric points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > \beta.$$

We denote this class by $S_s^*(\beta)$ and note that $S_s^* = S_s^*(0)$.

In the same manner, we define the following new classes of close-to-convex functions, which are generalisations of the classes in El-Ashwah and Thomas [2].

DEFINITION 3.

A function f normalised by (1.1) is said to be starlike of order β, with respect to conjugate points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\} > \beta.$$

We denote this class by $S_c^*(\beta)$.

DEFINITION 4.

A function f normalised by (1.1) is said to be starlike of order β, with respect to symmetric conjugate points if, and only if, for $z \in D$ and $0 \leq \beta < 1$,

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right\} > \beta.$$

We denote this class by $S_{sc}^*(\beta)$.

REMARK.

The class S_s^* has been studied by several authors, (eg. Wu [9] and Stankiewicz [8]). For $f \in S_s^*(\beta)$, Owa et al [4] proved that for $\frac{1}{4} \leq \beta < \frac{1}{2}$,

$$\operatorname{Re}\left\{\frac{f(z)-f(-z)}{z}\right\} > \frac{2}{3-4\beta}, \qquad z \in D.$$

2. RESULTS.

THEOREM 1.

Let $f \in S_s^*(\beta)$, then for $z = re^{i\theta} \in D$,

$$\operatorname{Re}\left\{\frac{f(z)-f(-z)}{z}\right\}^{1/(1-\beta)} \geq \frac{2^{1/(1-\beta)}}{1+r^2} > 2^{\beta/(1-\beta)}.$$

The result is sharp for f_0 given by

$$f_0(z) - f_0(-z) = 2z(1+z^2)^{\beta-1}.$$

To prove Theorem 1., we first require the following lemma.

LEMMA 1.

Let $g \in S^*(\beta)$ and be odd. Then for $z = re^{i\theta} \in D$,

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\}^{1/(1-\beta)} \geq \frac{1}{1+r^2}.$$

PROOF OF LEMMA.

Pinchuk [5] showed that if $F \in S^*(\beta)$, then for $z = re^{i\theta} \in D$,

$$\left| \left(\frac{z}{F(z)}\right)^{1/2(1-\beta)} - 1 \right| \leq r. \tag{2.1}$$

Since g is odd, we may write $[g(z)]^2 = F(z^2)$, so that (2.1) gives

$$\left| \left(\frac{z}{g(z)}\right)^{1/(1-\beta)} - 1 \right| \leq r^2,$$

where on squaring both sides, gives

$$\left(\frac{r}{|g(z)|}\right)^{2/(1-\beta)} - 2\,\mathrm{Re}\left[\left(\frac{z}{g(z)}\right)^{1/(1-\beta)}\right] + 1 \leq r^4.$$

Thus

$$2\mathrm{Re}\left[\overline{\left(\frac{g(z)}{z}\right)}^{1/(1-\beta)}\right] \geq (1-r^4)\left(\frac{|g(z)|}{r}\right)^{2/(1-\beta)} + 1$$

$$\geq \frac{1-r^4}{(1+r^2)^2} + 1,$$

where we have used the inequality [6]

$$|g(z)| \geq \frac{r}{(1+r^2)^{1-\beta}},$$

for odd starlike functions of order β.

The Lemma now follows at once.

PROOF OF THEOREM 1.

Since $f \in S^*_s(\beta)$, it follows that we may write

$$g(z) = \frac{f(z)-f(-z)}{2},$$

for g an odd starlike function of order β. An application of Lemma 1 proves the Theorem.

Results analogous to Theorem 1 can also be found for the classes $S^*_c(\beta)$ and $S^*_{sc}(\beta)$.

THEOREM 2.

Let $f \in S^*_c(\beta)$. Then for $z = re^{i\theta} \in D$,

$$\mathrm{Re}\left(\frac{f(z)+\overline{f(\bar{z})}}{z}\right)^{1/2(1-\beta)} \geq \frac{(\sqrt{2})^{1/(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for $f(z) + \overline{f(\bar{z})} = 2z(1+z)^{2(\beta-1)}$

PROOF

Since $f \in S^*_c(\beta)$, it is easy to see that, if

$$F(z) = \frac{f(z)+\overline{f(\bar{z})}}{2}$$

then $F \in S^*(\beta)$. Using the same techniques as in the proof of Lemma 1, it follows from (2.1) that

$$\operatorname{Re}\left(\frac{F(z)}{z}\right)^{1/2(1-\beta)} \geq \frac{1}{1+r}.$$

The result now follows immediately.

Similarly, we have the following result, which we state without proof.

THEOREM 3.

Let f $S^*_{sc}(\beta)$. Then for $z = re^{i\theta} \varepsilon D$,

$$\operatorname{Re}\left(\frac{f(z)-\overline{f(-\bar{z})}}{z}\right)^{1/2(1-\beta)} \geq \frac{2^{1/2(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for $f(z) - \overline{f(-\bar{z})} = 2z(1+z)^{2(\beta-1)}$

We now consider the results of some integral operators. In [1] Das and Singh, obtained analogous results of the Libera integral operator. They proved that for $f \varepsilon S^*_s(0)$, the function h given by

$$h(z) = \frac{1}{2}\int_0^z t^{-1}[f(t)-f(-t)]dt$$

also belongs to $S^*_s(0)$.

The result below generalises that of Das and Singh.

THEOREM 4.

Let $f \varepsilon S^*_s(\beta)$. Then the function H defined by

$$H(z) = \frac{a+1}{2z^a}\int_0^z t^{a-1}[f(t)-f(-t)]dt, \tag{2.2}$$

also belongs to $S^*_s(\beta)$ for $z \varepsilon D$ and $a + \beta > 0$.

We first require the following Lemma due to Miller and Mocanu [5].

LEMMA 2.

Let M and N be analytic in D with $M(0) = N(0) = 0$ and let β be any real number. If $N(z)$ maps D onto a (possibly many sheeted) region which is starlike with respect to the origin, then for $z \varepsilon D$,

$$\operatorname{Re}\frac{M'(z)}{N'(z)} > \beta \Rightarrow \operatorname{Re}\frac{M(z)}{N(z)} > \beta,$$

and

$$\operatorname{Re}\frac{M'(z)}{N'(z)} < \beta \Rightarrow \operatorname{Re}\frac{M(z)}{N(z)} < \beta.$$

PROOF OF THEOREM 4.

(2.2) gives,

$$\frac{2zH'(z)}{H(z)-H(-z)} = \frac{z^a[f(z)-f(-z)]-a\int_0^z t^{a-1}[f(t)-f(-t)]dt}{\int_0^z t^{a-1}[f(t)-f(-t)dt} = \frac{M(z)}{N(z)}, \text{ say}.$$

Note that $M(0) = N(0) = 0$ and for $f \varepsilon S^*_s(\beta)$,

$$\operatorname{Re}\left\{1+\frac{zN''(z)}{N'(z)}\right\} = a + \operatorname{Re}\left\{\frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)}\right\}$$

$$> a + \beta.$$

Thus $N(z)$ is starlike if, and only if $a > -\beta$.

Furthermore, since

$$\operatorname{Re}\frac{M'(z)}{N'(z)} = \operatorname{Re}\left\{\frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)}\right\} > \beta.$$

Lemma 2 shows that $H \in S_s^*(\beta)$.

Finally, we give the following analogous results for the classes $S_c^*(\beta)$ and $S_{sc}^*(\beta)$.

THEOREM 5.

Let $f \in S_c^*(\beta)$. Then H defined by

$$H(z) = \frac{a+1}{2z^a}\int_0^z t^{a-1}[f(t)+\overline{f(\bar{t})}]dt, \tag{2.3}$$

also belongs to $S_c^*(\beta)$ for $z \in D$ and $a+\beta > 0$.

PROOF.

Since $f \in S_c^*(\beta)$, (2.3) gives

$$\overline{\int_0^{\bar{z}} t^{a-1}[f(t)+\overline{f(\bar{t})}]dt} = \overline{2\int_0^{\bar{z}} t^{a-1}\left\{t+\sum_{n=2}^{\infty}\operatorname{Re} a_n t^n\right\}dt}$$

$$= 2\int_0^{z} t^{a-1}\left\{t+\sum_{n=2}^{\infty}\operatorname{Re} a_n t^n\right\}dt$$

$$= \int_0^z t^{a-1}[f(t)+\overline{f(\bar{t})}]dt.$$

Thus

$$\frac{2z\,H'(z)}{H(z)+\overline{H(\bar{z})}} = \frac{z^a[f(z)+\overline{f(\bar{z})}] - a\int_0^z t^{a-1}[f(t)+\overline{f(\bar{t})}]dt}{\int_0^z t^{a-1}[f(t)+\overline{f(\bar{t})}]dt}$$

$$= \frac{M(z)}{N(z)},$$

where $M(0) = N(0) = 0$ and $N \in S^*$ for $a+\beta > 0$.

On using Lemma 2 it follows that $H \in S_c^*(\beta)$.

THEOREM 6.

Let $f \in S_{sc}^*(\beta)$. Then H defined by

$$H(z) = \frac{a+1}{2z^a}\int_0^z t^{a-1}[f(t)-\overline{f(-\bar{t})}]dt, \tag{2.4}$$

also belongs to $S_{sc}^*(\beta)$ for $z \in D$ and $a+\beta > 0$.

PROOF.

For $f \in S^*_{sc}(\beta)$, (2.4) gives

$$\overline{H(-\bar{z})} = \frac{a+1}{(-z)^a} \overline{\int_0^{-\bar{z}} t^{a-1}[f(t) - \overline{f(-\bar{t})}]dt}$$

$$= \frac{a+1}{(-z)^a} \left\{ \frac{2(-z)^{a+1}}{a+1} + \sum_{n=2}^{\infty} \frac{(-z)^{n+a}}{n+a} (\overline{a_n} + (-1)^{n+1} a_n) \right\}$$

$$= \frac{-(a+1)}{z^a} \int_0^z t^{a-1}[f(t) - \overline{f(-\bar{t})}]dt.$$

As before, writing

$$\frac{2zH'(z)}{H(z) - \overline{H(-\bar{z})}} = \frac{M(z)}{N(z)},$$

one can show that $N \in S^*$ and hence using Lemma 2 the result follows.

REFERENCES

1. R.N.Das and P.Singh, On subclasses of schlicht mapping, Indian J. Pure Appl. Math., 8, (1977), 864-872.
2. R.M.El-Ashwah and D.K.Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc., 2(1), (1987), 85-100.
3. S.S.Miller and P.T.Mocanu, Second order differential inequalities in the complex plane, J. Math. Ana. Appl., 65, (1978), 289-304.
4. S.Owa, Z.Wu and F.Ren, A note on certain subclass of Sakaguchi functions, Bull. de la Royale de Liege, 57(3), (1988), 143-150.
5. B.Pinchuk, On starlike and convex functions of order β, Duke Math. Journal, 35, (1968), 721-734.
6. M.S.Robertson, On the theory of univalent functions, Ann. Math., 37, (1936), 374-408.
7. K.Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan, 11, (1959), 72-75.
8. J.Stankiewicz, Some remarks on functions starlike w.r.t. symmetric points, Ann. Univ. Marie Curie Sklodowska, 19(7), (1965), 53-59.
9. Z.Wu, On classes of Sakaguchi functions and Hadamard products, Sci. Sinica Ser. A, 30, (1987), 128-135.