Internat. J. Math. & Math. Sci.
VOL. 14 NO. 2 (1991) 339-344

A FLAG TRANSITIVE PLANE OF ORDER 49 AND ITS TRANSLATION COMPLEMENT

M.L. NARAYANA RAO

Department of Mathematics
Post Graduate College
Basheer Bagh, Hyderabad - 500 001
India

K. SATYANARAYANA

Department of Mathematics
Osmania University
Hyderabad, 500 007
India

K.M. ARJUNA RAO

Department of Mathematics
Bhavan's New Science College
Hyderabad - 500 029
India

(Received March 29, 1988)

ABSTRACT. The translation complement of the flag transitive plane of order 49 [Proc. Amer. Math. Soc. 32 (1972), 256-262] constructed by Rao is computed. It is shown that the flag transitive group itself is the translation complement and it is a solvable group of order 600.

1. INTRODUCTION.

Rao [1] has constructed a non-Desarguesian translation plane π of order 49 and exhibited a collineation group that is transitive on the distinguished points of π. In this paper we have computed the translation complement G of π and shown that the flag transitive collineation group is the translation complement of π. Further, G is a solvable group of order 600.

2. DESCRIPTION OF π AND ITS FLAG TRANSITIVE COLLINEATION GROUP.

Throughout this paper F, (a,b,c,d), det M and d.p. denote the finite field GF(7), the two by two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of M and the distinguished point respectively.

The translation plane π under study was constructed through a 1-spread set ℓ over F (see Lemma 3.1 of [1, p.258]) whose elements L_i, $1 \leq i \leq 49$, were generated by

$$L_{i+1} = (A+L_iP)^{-1}A$$

where $A = (1,0;1,4)$, $P = (5,6;0,6)$, $L_1 = (0,0;0,0)$ and $L_{25} = (2,0;5,3)$ and it may be constructed as follows:

Let $V_i=\{(x,y)|x,y \in F\times F,\ y = xL_i\}$ and $V_o= \{(0,y)|0 = (0,0),y \in F\times F\}$. The incidence structure whose points are the vectors of $F^4 = V$ and whose lines are V_i, $0 \leq i \leq 49$, and their right cosets in the additive group of V with inclusion as the incidence relation is the translation plane π.

Any nonsingular linear transformation on V induces a collineation of π if and only if the linear transformation permutes the subspaces V_i, $0 \leq i \leq 49$, among themselves. Lemma 4.1 of [1] and Theorem 1 of [2] are now used in this paper to compute the collineations of π. Rao has shown that the lienar transformations

$$R = \begin{pmatrix} A & A \\ P & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}$$

where $U = (0,3;5,0)$, $V = (3,5;0,5)$, $W = (4,5;2,5)$ and $Z = (6,0;2,1)$ on V induce collineations on π and their actions on the set of d.p.s. of π are

R:(0,1,2,...,24)(25,26,27,...,49)

S:(0,38)(1,45,24,31)(2,27,23,49)(3,34,22,42)(4,41,21,35)
(5,48,20,28)(6,30,19,46)(7,37,18,39)(8,44,17,32)
(9,26,16,25)(10,33,15,43)(11,40,14,36)(12,47,,13,29).

From the actions of R and S, it is clear that the group $G' = \langle R,S\rangle$ is transitive on the set of d.p.s of π and consequently G' is flag transitive group of π.

3. SPREAD SETS OF π AND SOME OF THEIR PROPERTIES.

We say that a spread set over F of π has a det. structure $(a_1,a_2,a_3,a_4,a_5,a_6)$ if the number of matrices of the spread set which are of determinant i is a_i, $1 \leq i \leq 6$.

It may be noted that the spread set ℓ of π was constructed by taking V_o,V_1 and V_2 as the fundamental subspaces ($x = y, y = 0$ and $y = x$ respectively) and the det. strucutre of ℓ is (9,9,6,8,8,8). We now construct another 1-spread set ℓ' from ℓ of π with the fundamental subspaces V_o,V_{38} and V_4, since the spread set ℓ is not amenable for easy computations and we study some properties of ℓ' and det. structures of certain matrix representative sets of π. This information is useful in the computation of the translation complement G of π.

Let T be a 4×4 matrix given by $T = \begin{pmatrix} I & C \\ 0 & D \end{pmatrix}$ where $C = (5,6;1,3)$, $D = (4,4;5,0)$, $I = (1,0;0,1)$ and 0 is the 2×2 zero matrix. Define for each $L_i \in \ell$, $1 \leq i \leq 49$,

$$M_i = C + L_i D$$

Let $\ell' = \{M_i \mid 1 \leq i \leq 49\}$. The matrices M_i, $1 \leq i \leq 49$ are listed in table 3.1. The entry a,b under the heading C.P. of M_i indicates that the matrix M_i has the characteristic polynomial $\lambda^2+a\lambda + b$. It may be noted that the det. structure of ℓ' is (4,12,6,12,6,8). If $A,B \in GL(2,F)$ then the det. structure of $A^{-1}\ell'B$ is

(4,12,6,12,6,8) if det $A^{-1}B = 1$

(12,4,6,12,8,6) if det $A^{-1}B = 2$

(6,6,4,8,12,12) if det $A^{-1}B = 3$

(12,12,8,4,6,6) if det $A^{-1}B = 4$

(6,8,12,6,4,12) if det $A^{-1}B = 5$

(8,6,12,6,12,4) if det $A^{-1}B = 6$

We say that the above det. structures are the allied det. structures of ℓ'.

Table 3.1

i	M_i	C.P. of M_i	i	M_i	C.P. of M_i
1	(5,6;1,3)	6,2	26	(2,5;0,6)	6,5
2	(2,3;6,3)	2,2	27	(4,2;3,6)	4,4
3	(1,5;1,0)	6,2	28	(5,5;4,3)	6,2
4	(1,0;0,1)	5,1	29	(6,2;2,0)	1,3
5	(0,2;5,4)	3,4	30	(4,5;0,5)	5,6
6	(4,0;1,5)	5,6	31	(1,6;6,5)	1,4
7	(2,4;5,6)	6,6	32	(6,6;3,1)	0,2
8	(0,1;3,2)	5,4	33	(5,4;2,1)	1,4
9	(5,2;0,2)	0,3	34	(6,1;0,3)	5,4
10	(3,3;2,5)	6,2	35	(1,3;4,6)	0,1
11	(4,6;2,2)	1,3	36	(5,3;3,0)	2,5
12	(0,4;4,5)	2,5	37	(3,2;5,3)	1,6
13	(3,5;0,4)	0,5	38	(0,0;0,0)	
14	(2,1;2,6)	6,3	39	(4,1;5,1)	2,6
15	(0,6;2,4)	3,2	40	(0,5;6,1)	6,5
16	(6,4;4,2)	6,3	41	(3,4;1,4)	0,1
17	(0,3;1,1)	6,4	42	(6,3;5,2)	6,4
18	(5,0;6,4)	5,6	43	(4,3;4,4)	6,4
19	(2,0;3,3)	2,6	44	(1,2;5,5)	1,2
20	(4,4;6,0)	3,4	45	(3,6;4,0)	4,4
21	(2,2;4,1)	4,1	46	(2,6;1,6)	6,6
22	(3,1;3,4)	0,2	47	(6,5;6,2)	6,3
23	(5,1;5,0)	2,2	48	(1,4;2,3)	3,2
24	(1,1;3,5)	1,2	49	(3,0;6,6)	5,4
25	(6,0;1,2)	6,5			

The planes associated with ℓ and ℓ' are isomorphic and the isomorphism is given by T. Without any loss of generality we take the plane associated with ℓ' as π since ℓ' is one of the spread sets of π. The collineations R and S now become the collineatons α and δ of π, where $\alpha = T^{-1}RT$, $\delta = T^{-1}ST$. The actions of α and δ on the set of d.p.s of π are same as R and S. Therefore $\langle \alpha,\delta \rangle$ is the flag transitive collineation group of π. Since δ is a collineation of π, $\delta^2 = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$ where $Q = (2,5;3,5)$, $R = (3,2;5,4)$ is also a collienation of π and its action on the set of d.p.s of π is given by

δ^2: (0)(38)(1,24)(2,23)(3,22)(4,21)(5,20)(6,19)(7,18)
(8,17)(9,16)(10,15)(11,14)(12,13)(25,26)(27,49)
(28,48)(29,47)(30,46)(31,45)(32,44)(33,43)(34,42)
(35,41)(36,40)(37,39)

LEMMA 3.1. $A^{-1}\ell' A = \ell'$, if and only if A is a scalar matrix.

PROOF. If A is a scalar matrix then the lemma follows trivially. Conversely suppose that $A^{-1}\ell' A = \{A^{-1}MA / M \in \ell'\} = \ell'$. From table 3.1. we notice that ℓ' contains M_9 and M_{13} with the characteristic polynomials $\lambda^2 + 3$ and $\lambda^2 + 5$ respectively and no other matrix of ℓ' has these polynomials as the characteristic polynomial. Therefore, we have, $A^{-1}M_9A = M_9$ and $A^{-1}M_{13}A = M_{13}$. Taking $A = (a,b;c,d)$ and solving the simultaneous equations obtained from $M_9A = AM_9$ and $M_{13}A = AM_{13}$ we get $b = c = 0$ and $a = d$. Hence the lemma.

LEMMA 3.2. Let $M_k \in \ell'$. The spread sets ℓ' and $\ell' M_k^{-1}$ are conjugate if k=4, 21 and are not conjugate otherwise.

PROOF. The first part of the lemma follows from lemma 3.1 and the collineation δ^2 when k=4 and k = 21 respectively. If ℓ' and $\ell' M_k^{-1}$ are conjugate then their det. structures must be same and this is possible if the det $M_k = 1$. Therefore ℓ' and ℓM_k^{-1} are not conjugate if det $M_k \neq 1$. The matrices of ℓ' which are of determinant 1 are M_k, k=4,21,35 and 41. If k=35 then $M_{11}M_k^{-1} \in \ell' M_k^{-1}$ and its characteristic polynomial is $\lambda^2 + 4\lambda + 3$. The spread sets ℓ' and ℓ', M_{35}^{-1} are not conjugate since ℓ' does not contain a matrix with the characteristic polynomial $\lambda^2 + 4\lambda + 3$. We reject k=41 by observing the characteristic polynomial of $M_3M_k^{-1}$ and using the same argument as in the previous case. The lemma now follows.

Let $M_k \in \ell'$. The det. structures of $\ell' - M_k = \{M - M_k | M \in \ell'\}$ are computed and are furnished in the table 3.2 for specified values of k. This information is useful in the sequel.

Table 3.2.

k	The det. structure of $\ell' - M_k$	k	The det. structure of $\ell' - M_k$
1	(9,9,6,8,8,8)	25	(6,10,8,4,14,6)
2	(6,7,11,7,11,6)	27	(6,11,10,11,4,6)
3	(7,7,8,6,10,10)	28	(5,8,5,9,12,9)
4	(6,10,8,6,10,8)	29	(6,4,14,10,8,6)
5	(13,4,5,13,8,5)	30	(12,5,6,7,11,7)
6	(8,6,10,6,10,8)	31	(7,10,6,9,5,11)
7	(5,6,11,5,10,11)	32	(12,7,11,5,6,7)
8	(7,7,10,12,6,6)	33	(8,5,5,9,9,12)
9	(6,6,10,8,9,9)	34	(5,9,8,16,6,4)
10	(2,10,8,10,8,10)	35	(16,9,4,5,6,8)
11	(9,2,10,9,8,10)	36	(11,6,10,11,6,4)
12	(10,6,7,10,8,7)	37	(10,9,11,7,5,6)

4. THE TRANSLATION COMPLEMENT G OF π.

Let G_0 be the group of all collineations of π that fix the d.p. 0; $G_{0,38}$ be the group of all collineations of π that fix the d.p.s. 0 and 38 and $G_{0,38,4}$ be the group of all collineations that fix the d.p.s 0, 38 and 4.

LEMMA 4.1. $G_{0,38,4}$ is generated by scalar collineations and it is of order 6.

PROOF. Any collineation $\sigma \in G_{0,38,4}$ is of the form $\sigma = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ for some $A \in GL(2,F)$, satisfying the condition that for every matrix $m \in \ell'$ there exists a matrix $N \in \ell'$ such that $A^{-1}{}_{MA} = N$. That is, $A^{-1}\ell' A = \ell'$. By lemma 3.1 we have $A = (a,0;0,a)$, $a \in F$, $a \neq 0$ and σ always induces a collineation of π fixing all the d.p.s. of π. Such a collineation σ is called a scalar collineation. If a is a generator of F then $G_{0,38,4} = \langle \sigma \rangle$ and it is of order 6. Hence the lemma.

LEMMA 4.2. $G_{0,38} = \langle \delta^2 \rangle$ and it is of order 12.

PROOF. Any collineation $\beta \in G_{0,38}$ is of the form $\beta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for some $A,B \in GL(2,F)$. Further A and B must satisfy the condition that for each matrix $M \in \ell'$ there exists a matrix $N \in \ell'$ such that $A^{-1}MB = N$. Taking $M = M_4$ we get a condition that $A^{-1}B \in \ell'$. Let $A^{-1}B = M_k$ for some k, $1 < k < 49$, $k \neq 38$. Then we obtain that the spread sets ℓ' and $\ell' M_k^{-1}$ are conjugate. By lemma 3.2 we have k=4 and 21. Therefore every collineation $\beta \in G_{0,38}$ either fixes the d.p. 4 or maps the d.p. 4 onto the d.p. 21 and hence β either fixes the d.p. 4 or interchanges the d.p.s 4 and 21. Since δ^2 is a collineation of $G_{0,38}$ interchanging the d.p.s 4 and 21, we have

$$G_{0,38} = G_{0,38,4} \quad G_{0,38,4}\delta^2 = \langle \sigma, \delta^2 \rangle = \langle \delta^2 \rangle$$

since $\langle \sigma \rangle \quad \langle \delta^2 \rangle$. Further, $|G_{0,38}| = 2|G_{0,38,4}| = 12$. Hence the lemma.

LEMMA 4.3. $G_0 = G_{0,38}$.

PROOF. If $\gamma \in G_0$ and maps the d.p. 38 onto the d.p.k then δ is of the form $\delta = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ for some A,B and $D \in GL(2,F)$, satisfying the condition that for each matrix $M \in \ell'$ there exits a matrix $N \in \ell'$ such that $A^{-1}(B + MD) = N$ and $A^{-1}B = M_k$. That is, for every matrix $M \in \ell'$ there exists a matrix $N \in \ell'$ such that $N - M_k = A^{-1}MD$. Suppose that the collineation γ maps the d.p.k onto the d.p.k' then $\delta^{-2}\gamma\delta^2 \in G_0$ and maps the d.p. 38 onto the d.p.k'. Observing the action of δ^2 on the d.p.s, it is therefore enough if we take the values of k from $\{i,25,j \mid 1 < i < 11,\ 27 < j < 37\}$. For these values of k the det. structure of $\ell' - M_k$ is same as the det. structure of $A^{-1}\ell'$ D. That is, for these values of k the det. structure of $\ell' - M_k$ must coincide with one of the allied det. structure of ℓ'. This is not possible (see table 3.2). If $\gamma \in G_0$ then γ fixes the d.p. 38 and therefore $\gamma \in G_{0,38}$. Hence the lemma.

THEOREM 4.4. The translation complement G of π is $\langle\alpha,\delta\rangle$ and it is a solvable group of order 600.

PROOF. Since $\langle\alpha,\delta\rangle$ is transitive on the set of all d.p.s of π, $G = \bigcup_{i=0}^{49} G_0\alpha_i$ where α_i is a collineation of π which maps the d.p. 0 onto the d.p. i and it may be taken from $\langle\alpha,\delta\rangle$. The order of G = 50×(order of G_0) = 600. In view of lemma 4.3, $G = \langle\alpha,\delta\rangle$. Notice that $\delta^{-1}\alpha\delta = \alpha^7$ and $G \ \langle\alpha\rangle \ \{e\}$ is a solvable series of G. Therefore G is a solvable group and hence the theorem.

It is interesting to note that the flag transitive group of π itself is the translation complement of π and the two flag transitive planes of order 25 constructed by Foulser [3] also possess this property [4,5].

REFERENCES

1. NARAYANA RAO, M.L., A flag transitive plane of order 49, Proc. Amer. Math. Soc. 32 (1972), 256-262.
2. SHERK, F.A., Indicator sets in an affine space of any dimension, Can. J. Math. 31 (1979), 211-224.
3. FOULSER, D.A., Solvable flag transitive affine groups, Math. Z. 86 (1964), 191-204.
4. SATYANARAYANA, K. and KUPPUSWAMY RAO, K., Full collineation group of the second of Foulser's flag transitive planes of order 25, Houston J. Math. 7 (1981), 537-543.
5. NARAYANA RAO, M.L. and KUPPUSWAMY RAO, K., Full collineation group of Fousler's flag transitive plane of order 25, Houston J. Math. 7 (1981), 115-120.