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SOLVABILITY OF A FOURTH-ORDER BOUNDARY VALUE PROBLEM WITH PERIODIC BOUNDARY CONDITIONS II

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ABSTRACT. Let $f : [0,1]\times R^4 \to R$ be a function satisfying Caratheodory's conditions and $e(x) \in L^1[0,1]$. This paper is concerned with the solvability of the fourth-order fully quasilinear boundary value problem

$$\frac{d^4u}{dx^4} + f(x,u(x),u'(x),u''(x),u'''(x)) = e(x), \quad 0 < x < 1,$$

with $u(0)-u(1) = u'(0)-u'(1) = u''(0-)-u''(1) = u'''(0)-u'''(1) = 0$. This problem was studied earlier by the author in the special case when f was of the form $f(x,u(x))$, i.e., independent of $u'(x)$, $u''(x)$, $u'''(x)$. It turns out that the earlier methods do not apply in this general case. The conditions need to be related to both of the linear eigenvalue problems $\frac{d^4u}{dx^4} = \lambda^4 u$ and $\frac{d^4u}{dx^4} = -\lambda^2 \frac{d^2u}{dx^2}$ with periodic boundary conditions.

KEY WORDS AND PHRASES. fully quasilinear, fourth order boundary value problem, periodic boundary conditions, Leray Schauder continuation theorem, Fredholm operator of index zero, compact perturbation, family of homotopy equations.

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1. INTRODUCTION

Fourth-order boundary value problems arise in the study of the equilibrium of an elastic beam under an external load. There arise several different fourth-order boundary value problems depending on how the beam is supported at the end points [4]. Such problems have been studied extensively in recent times. (See e.g. [1-11].) The purpose of this paper is to study the fourth-order fully quasilinear boundary value problem with periodic boundary conditions

$$\frac{d^4u}{dx^4} + f(x,u(x),u'(x),u''(x),u'''(x)) = e(x), \quad 0 < x < 1, \tag{1.1}$$

$$u(0) - u(1) = u'(0) - u'(1) = u''(0) - u''(1) = u'''(0) - u'''(1) = 0, \tag{1.2}$$

where $f : [0,1]\times R^4 \to R$ is a function satisfying Caratheodory's conditions and $e(x) \in L^1[0,1]$. The boundary value problem (1.1)-(1.2) was studied earlier by the author in [6],[7] in the case where f in equation (1.1) is independent of u',u'',u''', that is, f is of the form $f(x,u(x),u'(x),u''(x),u'''(x)) \equiv f(x,u(x))$. However, the methods of [6],[7] do not apply to the more general boundary value problem under consideration in this paper. One needs to show that the set of solutions of the family of homotopy equations for the boundary value problem (1.1)-(1.2) is, *a priori*, bounded in $C^3[0,1]$, while the methods of [6],[7] can at best give *a priori* bounds in $C^2[0,1]$. It turns out that the conditions on the nonlinearity $f(x,u,u',u'',u''')$ are related to both of the following linear eigenvalue problems:

$$\frac{d^4u}{dx^4} = \lambda^4 u\,, \quad 0 < x < 1\,, \tag{1.3}$$

and

$$\frac{d^4u}{dx^4} = -\lambda^2 \frac{d^2u}{dx^2}\,, \quad 0 < x < 1\,,$$

with u satisfying the periodic boundary conditions (1.2) . (1.4)

We use the classical spaces $C[0,1]$, $C^k[0,1]$, and $L^k[0,1]$ of continuous, k-times continuously differentiable, or measurable real-valued functions, the k-th power of whose absolute value is Lebesgue integrable. We also use the space $W^{k,1}(0,1)$ defined by

$$W^{k,1}(0,1) = \left\{ u : [0,1] \to R \mid \frac{d^ju}{dx^j} \text{ absolutely continuous on } [0,1] \text{ for } j=0,1,\ldots,k-1 \right\},$$

with the norm $\|u\|_{W^{k,1}}$ for $u \in W^{k,1}(0,1)$ defined by

$$\|u\|_{W^{k,1}} = \sum_{j=0}^{k} \|u^{(j)}\|_{L^1}\,.$$

2. MAIN RESULTS

Let X,Y denote the Banach spaces $X = C^3[0,1]$, $Y = L^1(0,1)$ with their usual norms, and let H denote the Hilbert-space $L^1(0,1)$. Let Y_2 be the subspace of Y defined by

$$Y_2 = \{u \in Y \mid u = \text{constant a.e. on } [0,1]\}\,,$$

and let Y_1 be the closed subspace of Y such that $Y = Y_1 \oplus Y_2$. (Here and in the following, the symbol $\oplus$ denotes the direct sum.) We note that for $u \in Y$ we can write

$$u(x) = \left[u(x) - \int_0^1 u(t)dt \right] + \int_0^1 u(t)dt\,,$$

for $x \in [0,1]$. We define the canonical projection operators $P : Y \to Y_1$; $Q : Y \to Y_2$ by

$$P(u) = u(x) - \int_0^1 u(t)dt\,,$$

$$Q(u) = \int_0^1 u(t)dt\,,$$

for $u \in Y$. Clearly, $Q = I - P$ where I denotes the identity mapping on Y, and the projection operators P,Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P \mid X : X \to X_1$, $Q : X \to X_2$ are continuous. Similarly, we obtain $H = H_1 \oplus H_2$ and continuous projections $P \mid H : H \to H_1$, $Q \mid H : H \to H_2$. In the following, we shall not distinguish between P, $P \mid X$, $P \mid H$ (respectively, Q, $Q \mid X$, $Q \mid H$) but depend on the context for the proper meaning.

For $u \in X$, $v \in Y$, let $(u,v) = \int_0^1 u(x)v(x)dx$ denote the duality pairing between X and Y. We note that for $u \in X$, $v \in Y$, so $u = Pu + Qu$, $v = Pv + Qv$, we have

$$(u,v) = (Pu,Pv) + (Qu,Qv) .$$

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \Big\{ u \in W^{4,1}(0,1) \mid u(0) = u(1),\ u'(0) = u'(1),$$

$$u''(0) = u''(1),\ u'''(0) = u'''(1) \Big\} . \tag{2.1}$$

For $u \in D(L)$,

$$Lu = \frac{d^4u}{dx^4} . \tag{2.2}$$

Now, for $u \in D(L)$ we see, using integration by parts, that

$$(Lu,u'') = \int_0^1 \frac{d^4u}{dx^4} u''(x)dx = -\int_0^1 (u'''(x))^2 dx . \tag{2.3}$$

For a given $h \in L^1(0,1)$ with $\int_0^1 h(x)dx = 0$, where $h \in Y_1$, we notice that there exists a unique $u \in X_1 \cap D(L)$ such that $Lu = h$. Indeed, the unique u is given by

$$u(x) = C_1 + C_2x + C_3x^2 + C_4x^3 + \frac{1}{6}\int_0^x (x-t)^3 h(t)dt , \tag{2.4}$$

where C_2, C_3, C_4 are obtained from the following three linearly independent equations:

$$C_2 + C_3 + C_4 = -\frac{1}{6}\int_0^1 (1-t)^3 h(t)dt ,$$

$$2C_3 + 3C_4 = -\frac{1}{2}\int_0^1 (1-t)^2 h(t)dt ,$$

$$6C_4 = -\int_0^1 (1-t)h(t)dt . \tag{2.5}$$

C_1 is computed (uniquely) from the requirement that $u \in X_1$ i.e., $\int_0^1 u(t)dt = 0$. Accordingly, the linear mapping $K : Y_1 \to X_1$ defined, for $h \in Y_1$, by $Kh = u$, where u is given by (2.4)-(2.5), is a bounded mapping. It is easy to see, using the Arzela-Ascoli theorem, that $K : Y_1 \to X_1$ is a compact mapping; i.e., K maps bounded sets in Y_1 into relatively compact sets in X_1. Further for $u \in D(L)$, $Lu \in Y_1$, $KLu = Pu$ and for $h \in Y_1$, $Kh \in D(L)$, $LKh = h$.

DEFINITION 1. *A function $f : [0,1] \times R^k \to R$ is said to satisfy Caratheodory's conditions if the following conditions are satisfied:*

(i) *for a.e. $x \in [0,1]$, the function $\underset{\sim}{y} \in R^k \to f(x,\underset{\sim}{y}) \in R$ is continuous;*

(ii) *for every $\underset{\sim}{y} \in R^k$, the function $x \in [0,1] \to f(x,\underset{\sim}{y}) \in R$ is measurable;*

(iii) *for every $r > 0$, there is a real-valued function $g_r(x) \in L^1[0,1]$ such that for a.e. $x \in [0,1]$, $|f(x,\underset{\sim}{y})| \le g_r(x)$ whenever $\|\underset{\sim}{y}\| \le r$.*

If the function $g_r(x)$ in condition (iii) is required to be in $L^2(0,1)$, we say that the function f satisfies L^2-Caratheodory conditions.

Next, let $f:[0,1]\times R^4 \to R$ be a given function satisfying Caratheodory's conditions. We define a (nonlinear) mapping $N: X \to Y$ by setting

$$(Nu)(x) = f(x,u(x),u'(x),u''(x),u'''(x)), \quad x \in [0,1], \tag{2.6}$$

for $u \in X$. We see that $KPN: X \to X_1$ is a well-defined compact mapping and $QN: X \to X_2$ is a bounded mapping.

For $e(x) \in Y = L^1(0,1)$, the boundary value problem (1.1),(1.2) reduces to the functional equation

$$Lu + Nu = e, \tag{2.7}$$

in X with $e \in Y$ given.

THEOREM 1. *Let $f:[0,1]\times R^4 \to R$ satisfy Caratheodory's conditions. Assume that there exist real numbers a,A,r,R with $a \le A$ and $r < 0 < R$ such that*

$$f(x,u,v,w,y) \ge A, \tag{2.8}$$

for a.e. $x \in [0,1]$, all $(v,w,y) \in R^3$, and all $u \ge R$. Further assume that

$$f(x,u,v,w,y) \le a, \tag{2.9}$$

for a.e. $x \in [0,1]$, all $(v,w,y) \in R^3$, and all $u \le r$. Suppose that there exist functions $a(x),b(x),c(x),d(x)$ in $L^\infty(0,1)$ and a function $\alpha(x) \in L^1(0,1)$ such that

$$f(x,u,v,w,y)w \le a(x)w^2 + b(x)|uw| + c(x)|vw| + d(x)|yw| + \alpha(x)|w|, \tag{2.10}$$

for a.e. $x \in [0,1]$ and all $(u,v,w,y) \in R^4$ with

$$4\pi^2\sqrt{3}\,\|a\|_\infty + (2\pi + \sqrt{3})\|b\|_\infty + 2\pi\sqrt{3}\,\|c\|_\infty + 8\pi^3\sqrt{3}\,\|d\|_\infty < 16\sqrt{3}\,\pi^4. \tag{2.11}$$

Suppose, further, there exists an L^2-Caratheodory function $\beta: [0,1]\times R^3 \to R$ and a function $\gamma(x) \in L^1(0,1)$ such that

$$|f(x,u,v,w,y)| \le \beta(x,u,v,w)|y| + \gamma(x), \tag{2.12}$$

for all $(u,v,w,y) \in R^4$ and a.e. $x \in [0,1]$. Then the boundary value problem (1.1)-(1.2) has at least one solution for each given $e(x) \in L^1(0,1)$ with

$$a \le \int_0^1 e(t)dt \le A. \tag{2.13}$$

Proof. Define $f_1: [0,1]\times R^4 \to R$ by

$$f_1(x,u,v,w,y) = f(x,u,v,w,y) - \frac{1}{2}(A+a),$$

and $e_1 \in L^1(0,1)$ by $e_1(x) = e(x) - \frac{1}{2}(A+a)$ so that for a.e. $x \in [0,1]$ and all $(v,w,y) \in R^3$, we get, using (2.8),(2.9),

$$f_1(x,u,v,w,y) \ge \frac{1}{2}(A-a) \ge 0 \text{ if } u \ge R, \tag{2.14}$$

$$f_1(x,u,v,w,y) \le \frac{1}{2}(a-A) \le 0 \text{ if } u \le r, \tag{2.15}$$

and

$$\frac{1}{2}(a-A) \le \int_0^1 e_1(t)\mathrm{d}t \le \frac{1}{2}(A-a). \tag{2.16}$$

Also, $f_1(x,u,v,w,y)$ satisfies (2.10) with $\alpha(x)$ replaced by $\alpha_1(x) = \alpha(x) + \frac{1}{2}|A+a|$. Clearly, the boundary value problem (1.1),(1.2) is equivalent to

$$\frac{d^4u}{dx^4} + f_1(x,u(x),u'(x),u''(x),u'''(x)) = e_1(x), \quad 0 < x < 1,$$

$$u(0) = u(1), \; u'(0) = u'(1), \; u''(0) = u''(1), \; u'''(0) = u'''(1). \tag{2.17}$$

Let $N: X \to Y$ be defined by

$$(Nu)(x) = f_1(x,u(x),u'(x),u''(x),u'''(x)), \quad x \in [0,1], \tag{2.18}$$

for $u \in X$. Then $KPN: X \to X_1$ is a well-defined compact mapping, $QN: X \to X_2$ is bounded, and the boundary value problem (2.17) is equivalent to the functional equation

$$Lu + Nu = e_1, \tag{2.19}$$

in X with $e_1 \in Y$. Letting $\tilde{e}_1 = KPe_1$, $\bar{e}_1 = Qe_1$, we see that (2.19) is equivalent to the system of equations

$$Pu + KPNu = \tilde{e}_1,$$

$$QNu = \bar{e}_1, \tag{2.20}$$

$u \in X$.

Now, (2.20) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = \tilde{e}_1 + \bar{e}_1, \tag{2.21}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can therefore apply the version given in [12] (Theorem 1, Corollary 1), [13] (Theorem IV.4), or [14] of the Leray-Schauder continuation theorem which ensures the existence of a solution for (2.21) if the set of solutions of the family of equations

$$Pu + (1-\lambda)Qu + \lambda QNu + \lambda KPNu = \lambda\tilde{e}_1 + \lambda\bar{e}_1 \tag{2.22}$$

is, *a priori*, bounded in X by a constant independent of $\lambda \in (0,1)$. Notice that (2.22) is equivalent to the system of equations

$$Pu + \lambda KPNu = \lambda\tilde{e}_1,$$

$$(1-\lambda)Qu + \lambda QNu = \lambda\bar{e}_1, \quad \lambda \in (0,1). \tag{2.23}$$

Let $u \in X$ be a solution for (2.23) for some $\lambda \in (0,1)$. The second equation in (2.23) can be written as

$$(1-\lambda)\int_0^1 u(t)\mathrm{d}t + \lambda\int_0^1 f_1(t,u(t),u'()t),u''(t),u'''(t))\mathrm{d}t = \lambda\int_0^1 e_1(t)\mathrm{d}t.$$

Thus, if $u(t) \ge R$ for $t \in [0,1]$, then using (2.14),(2.16), we have

$$0 < (1-\lambda)R + \frac{\lambda}{2}(A-a) \le \frac{\lambda}{2}(A-a),$$

which implies

$$0 < (1-\lambda)R \le 0$$

a contradiction.

Similarly, $u(t) \le r$ for $t \in [0,1]$ also leads to a contradiction. Hence, there must exist a $\tau \in [0,1]$ such that

$$r < u(\tau) < R . \tag{2.24}$$

Now, for $x \in [0,1]$ we have

$$u(x) = u(\tau) + \int_\tau^x u'(t)dt .$$

It follows that

$$\|Qu\| = \left|\int_0^1 u(x)dx\right| \le |u(\tau)| + \left|\int_0^1\int_\tau^x u'(t)dtdx\right|$$

$$\le \max(R,-r) + \frac{1}{\sqrt{3}}(\tau^{3/2} + (1-\tau)^{3/2})\|u'\|_2$$

$$\le \max(R,-r) + \frac{1}{\sqrt{3}}\|u'\|_2$$

$$\le \max(R,-r) + \frac{1}{4\pi^2\sqrt{3}}\|u'''\|_2 , \tag{2.25}$$

since $\max\{\tau^{3/2} + (1-\tau)^{3/2} \mid \tau \in [0,1]\} = 1$ and $\|u'\|_2 \le \frac{1}{2\pi}\|u''\|_2$, $\|u''\|_2 \le \frac{1}{2\pi}\|u'''\|_2$, in view of the Wirtinger's inequalities. Next we apply L to the first equation in (2.23) to get

$$Lu + \lambda PNu = \lambda Pe_1 . \tag{2.26}$$

Adding the second equation in (2.23) to (2.26), we get

$$Lu + (1-\lambda)Qu + \lambda Nu = \lambda e_1 ,$$

which can be written as

$$\frac{d^4u}{dx^4} + (1-\lambda)Qu + \lambda f_1(x,u(x),u'(x),u''(x),u'''(x)) = \lambda e_1(x) , \tag{2.27}$$

for $x \in (0,1)$ with u satisfying the boundary conditions (1.2). Next we multiply (2.27) by $u''(x)$ and then integrate the resulting equation over [0,1]. This gives

$$0 = \int_0^1 \frac{d^4u}{dx^4}u''(x)dx + \lambda\int_0^1 f_1(x,u(x),u'(x),u''(x),u'''(x))u''(x)dx - \lambda\int_0^1 e_1(x)u''(x)dx.$$

$$\le -\int_0^1 (u'''(x))^2dx + \lambda\int_0^1 [a(x)(u''(x))^2 + b(x)|u(x)u''(x)| + c(x)|u'(x)u''(x)|$$

$$+ d(x)|u'''(x)u''(x)| + \alpha_1(x)|u''(x)|]dx + \lambda\int_0^1 |e_1(x)||u''(x)|dx$$

$$\le -\|u'''\|_2^2 + (\|a\|_\infty \cdot \|u''\|_2 + \|b\|_\infty\|u\|_2 + \|c\|_\infty\|u'\|_2 + \|d\|_\infty\|u'''\|_2)\|u''\|_2$$

$$+ (\|\alpha_1\|_1 + \|e_1\|_1)\|u''\|_\infty$$

$$\le -\|u'''\|_2^2 + \left[\frac{1}{2\pi}\|a\|_\infty + \frac{1}{8\pi^3}\|b\|_\infty + \frac{1}{4\pi^2}\|c\|_\infty + \|d\|_\infty\right]\frac{1}{2\pi}\|u'''\|_2^2$$

$$+ \|b\|_\infty \|Qu\|\cdot\|u''\|_2 + (\|\alpha_1\|_1 + \|e_1\|_1)\|u'''\|_2$$

$$\le -\|u'''\|_2^2 + \frac{1}{16\pi^4}\left[4\pi^2\|a\|_\infty + \|b\|_\infty + 2\pi\|c\|_\infty + \|d\|_\infty\right]\|u'''\|_2^2$$

$$+\frac{1}{2\pi}\|b\|_\infty\cdot\left[\max(R,-r) + \frac{1}{4\pi^2\sqrt{3}}\|u'''\|_2\right]\cdot\|u'''\|_2 + (\|\alpha_1\|_1 + \|e_1\|_1)\|u'''\|_2 ,$$

in view of (2.10), (2.25) and the Wirtinger's inequalities

$$\|Pu\|_2 \le \frac{1}{2\pi}\|u'\|_2 , \quad \|u'\|_2 \le \frac{1}{2\pi}\|u''\|_2 , \quad \|u''\|_2 \le \frac{1}{2\pi}\|u'''\|_2 . \tag{2.28}$$

It follows from (2.11) that there exists a constant C, independent of $\lambda \in (0,1)$, such that

$$\|u'''\|_2 \le C . \tag{2.29}$$

It is easy to see from (2.25), (2.28) and from (1.2) that there exists a constant C_1, independent of $\lambda \in (0,1)$ such that

$$\|u\|_{C^2[0,1]} \le C_1 . \tag{2.30}$$

We next use (2.27), (2.12), (2.29), (2.30) to obtain a constant C_2, independent of $\lambda \in (0,1)$, such that

$$\left\|\frac{d^4u}{dx^4}\right\|_1 \le C_2 .$$

Finally, since $u''(0) = u''(1)$, we see that there must exist a $\xi \in (0,1)$ such that $u'''(\xi) = 0$. Hence, for $x \in [0,1]$,

$$|u'''(x)| = \left|\int_\xi^x \frac{d^4u}{dx^4}(t)dt\right| \le \left\|\frac{d^4u}{dx^4}\right\|_1 \le C_2 .$$

Thus,

$$\|u'''\|_\infty \le C_2 . \tag{2.31}$$

It follows from (2.30),(2.31) that the set of all possible solutions of (2.23) is, *a priori*, bounded in $X = C^3[0,1]$ by a constant independent of $\lambda \in (0,1)$. □

THEOREM 2. *Let $f : [0,1]\times R^4 \to R$ satisfy Caratheodory's conditions. Suppose that f satisfies conditions (2.8), (2.9), (2.10), and (2.12) of Theorem 1, with $a(x), b(x), c(x)$ in $L^1[0,1]$ and $d(x)$ in $L^2[0,1]$, and*

$$4\pi^2\sqrt{3}\|a\|_1 + 3\sqrt{3}\|b\|_1 + 2\pi\sqrt{3}\|c\|_1 + 24\pi^2\|d\|_2 < 48\pi^2\sqrt{3} .$$

Then the boundary value problem (1.1)-(1.2) has at least one solution for each given $e(x) \in L^1(0,1)$, with

$$a \le \int_0^1 e(t)dt \le A .$$

The proof of Theorem 2 is similar to that of Theorem 1, except now we need to use the following Wirtinger-type inequalities,

$$\|Pu\|_\infty \le \frac{1}{2\sqrt{3}}\|u'\|_2 , \quad \|u'\|_\infty \le \frac{1}{2\sqrt{3}}\|u''\|_2 , \quad \|u''\|_\infty \le \frac{1}{2\sqrt{3}}\|u'''\|_2$$

along with (2.26) and (2.29). We leave the details for the reader, in the interest of brevity.

Our next theorem concerns the boundary value problem

$$-\frac{d^4u}{dx^4}+f(x,u(x),u'(x),u''(x),u'''(x))=e(x)\,,\quad 0<x<1\,,$$

$$\text{with } u \text{ satisfying the boundary conditions (1.2)}\,, \tag{2.32}$$

where $e(x)\in L^1(0,1)$ and $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ satisfies Caratheodory's conditions.

THEOREM 3. *Let $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ satisfy Caratheodory's conditions. Assume that f satisfies conditions (2.8),(2.9),(2.12) of Theorem 1.*

Suppose that there exist functions $a(x),b(x),c(x),d(x),\alpha(x)$ and non-negative numbers a,m,n,p,q with $b(x)\in C^2[0,1]$, $c(x),d(x)\in C^1[0,1]$, $\alpha(x)\in L^1[0,1]$, $b(0)=b(1)$, $b'(0)=b'(1)$, $c(0)=c(1)$, $d(0)=d(1)$, and $a(x)\ge -a$, $b(x)\le m$, $b''(x)\ge -2n$, $c'(x)\le 2p$, $d'(x)\le 2q$ such that

$$f(x,u,v,w,y)w\ge a(x)w^2+b(x)uw+c(x)vw+d(x)wy+\alpha(x)|w|\,, \tag{2.33}$$

for almost a.e. $x\in[0,1]$ and all $(u,v,w,y)\in\mathbb{R}^4$. Suppose further that

$$48\pi^4(a+q)+12\pi^2(m+p)+n(4\pi^2+3)<192\pi^6\,. \tag{2.34}$$

Then the boundary value problem (2.32) has at least one solution for each given $e(x)\in L^1(0,1)$, with

$$a\le\int_0^1 e(t)dt\le A\,.$$

Proof. Define $f_1:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ by $f_1(x,u,v,w,y)=f(x,u,v,w,y)-\frac{1}{2}(A+a)$ and $e_1\in L^1(0,1)$ by $e_1(x)=e(x)-\frac{1}{2}(A+a)$ as in the proof of Theorem 1, so that (2.14), (2.15), (2.16) hold and f_1 satisfies (2.33) with $\alpha(x)$ replaced by $\alpha_1(x)=\alpha(x)-\frac{1}{2}|A+a|$. Further, the boundary value problem (2.32) is equivalent to

$$-\frac{d^4u}{dx^4}+f_1(x,u(x),u'(x),u''(x),u'''(x))=e_1(x)\,,\quad 0<x<1\,,$$

$$\text{with } u \text{ satisfying the boundary conditions (1.2)}\,. \tag{2.35}$$

Define $\tilde{L}:D(L)\subset Y$ by $\tilde{L}u=-Lu=-\frac{d^4u}{dx^4}$, where L is defined by (2.1), (2.2). Take $\tilde{K}:Y_1\to X_1$ as $\tilde{K}=-K$, where K is the linear mapping defined earlier, so that for $u\in D(L)$, $\tilde{L}u\in Y_1$, $\tilde{K}\tilde{L}u=Pu$ and for $h\in Y_1$, $\tilde{K}h\in D(L)$, $\tilde{L}\tilde{K}h=h$. Again, define $N:X\to Y$ by

$$(Nu)(x)=f_1(x,u(x),u'(x),u''(x),u'''(x))\,,\quad x\in[0,1]\,,$$

for $u\in X$, as in the proof of Theorem 1. Proceeding, as in the proof of Theorem 1, it suffices to show that the set of solutions of the family of equations

$$Pu+(1-\lambda)Qu+\lambda QNu+\lambda\tilde{K}PNu=\lambda\tilde{e}_1+\lambda\bar{e}_1\,, \tag{2.36}$$

is, *a priori*, bounded in X by a constant independent of $\lambda\in(0,1)$, where $\tilde{e}_1=\tilde{K}Pe_1$, $\bar{e}_1=Qe_1$. We notice that (2.36) is equivalent to the system of equations

$$Pu+\lambda\tilde{K}PNu=\lambda\tilde{e}_1\,,$$

$$(1-\lambda)Qu+\lambda QNu=\lambda\bar{e}_1\,,\quad \lambda\in(0,1)\,. \tag{2.37}$$

It follows from the second equation in (2.37), as in the proof of Theorem 1, that

$$\|Qu\| = \left|\int_0^1 u(x)dx\right| \le \max(R,-r) + \frac{1}{4\pi^2\sqrt{3}}\|u'''\|_2 . \tag{2.38}$$

Next we get, as in the proof of Theorem 1,

$$-\frac{d^4u}{dx^4} + (1-\lambda)Qu + \lambda f_1(x,u(x),u'(x),u''(x),u'''(x)) = \lambda e_1(x),$$

$$\text{with } u \text{ satisfying the boundary conditions (1.2)}, \tag{2.39}$$

for $x \in (0,1)$, using (2.37). Now we multiply the equation in (2.39) by $u''(x)$ and integrate the resulting equation over $[0,1]$ to obtain

$$0 = -\int_0^1 \frac{d^4u}{dx^4} u''(x)dx + \lambda\int_0^1 f_1(x,u(x),u'(x),u''(x),u'''(x))u''(x)dx - \lambda\int_0^1 e_1(x)u''(x)dx$$

$$\ge \int_0^1 (u'''(x))^2dx + \lambda\int_0^1 [a(x)(u''(x))^2 + b(x)u(x)u''(x) + c(x)u'(x)u''(x)$$

$$+ d(x)u''(x)u'''(x) + \alpha_1(x)|u''(x)|]dx - \lambda\int_0^1 e_1(x)u''(x)dx$$

$$\ge \int_0^1 (u'''(x))^2dx + \lambda\int_0^1 [a(x)(u''(x))^2 + \frac{1}{2}b''(x)(u(x))^2 - b(x)(u'(x))^2$$

$$-\frac{1}{2}c'(x)(u'(x))^2 - \frac{1}{2}d'(x)(u''(x))^2]dx - \lambda\int_0^1 (|\alpha_1(x)| + |e_1(x)|)|u''(x)|dx$$

$$\ge \int_0^1 (u'''(x))^2dx - a\|u''\|_2^2 - n\|u\|_2^2 - m\|u'\|_2^2 - p\|u'\|_2^2 - q\|u''\|_2^2$$

$$-(\|\alpha_1\|_1 + \|e_1\|_1)\|u''\|_\infty$$

$$\ge \|u'''\|_2^2 - \frac{1}{192\pi^6}[48\pi^4(a+q) + 12\pi^2(m+p) + n(4\pi^2+3)]\|u'''\|_2^2$$

$$-n(\max(R,-r))^2 - n(\max(R,-r))\cdot\frac{1}{2\pi^2\sqrt{3}}\|u'''\|_2 - (\|\alpha\|_1 + \|e\|_1)\|u'''\|_2 ,$$

where we have used the Wirtinger's inequalities (2.28) and the estimate (2.38). it follows from (2.34) that there exists a constant C, independent of $\lambda \in (0,1)$, such that

$$\|u'''\|_2 \le C .$$

Finally, there exists a constant C_1, independent of $\lambda \in (0,1)$, such that

$$\|u\|_X = \|u\|_{C^3[0,1]} \le C_1 ,$$

as in the proof of Theorem 1. We have thus verified that the set of solutions of (2.36) is, *a priori*, bounded in $X = C^3[0,1]$ by a constant independent of $\lambda \in (0,1)$. □

Remark 1. If $f : [0,1]\times R^4 \to R$ in Theorem 1 (resp., Theorem 2 and Theorem 3) is independent of y (i.e., $f(x,u,v,w,y) \equiv g(x,u,v,w)$ for some $g : [0,1]\times R^3 \to R$, then we do not need the assumption (2.12) in Theorem 1 (resp., Theorem 2 and Theorem 3). We remark that assumption (2.12) is needed to obtain an *a* *priori* bound for $\left\| \frac{d^4u}{dx^4} \right\|_1$ once an *a priori* bound for $\| u \|_{C^2[0,1]}$ has been obtained. So (2.12) can be replaced by any other assumption that accomplishes this task.

Remark 2. We note that for any given continuous function $g: R \to R$ and any $u \in W^{4,1}(0,1)$ with $u''(0) = u''(1)$, $\int_0^1 g(u'')u'''dx$ and $\int_0^1 g(u'')u'''u''dx$, both vanish. Accordingly, we can add the term $g(u'')u'''$ to the equations studied in Theorems 1, 2, and 3 and obtain existence of solutions of the modified boundary value problems, namely,

$$\pm \frac{d^4u}{dx^4} + g(u'')u''' + f(x,u(x),u'(x),u''(x),u'''(x)) = e(x),$$

with u satisfying the periodic boundary conditions (1.2).

Remark 3. Suppose that $a(x) \equiv -a$, $b(x) \equiv m$, $c(x) \equiv c$, $d(x) \equiv d$, where c and d are some constants in Theorem 3, so that $n = p = q = 0$. Then the conclusion of Theorem 3 remains valid if $4\pi^2 a + m < 16\pi^4$.

Remark 4. We refer the reader to [15] for Wirtinger inequalities used in this paper.

Finally, we remark that the theorems of this paper clearly apply to a wider class of boundary value problems than the theorems studied by the author in [6],[7]. But it is easy to find situations where the results of [6] and [7] apply and the results of this paper do not apply. Accordingly, the results of this paper complement the results of [6] and [7].

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