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ORTHODOX Γ-SEMIGROUPS

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ABSTRACT. Let $M = \{a,b,c,\dots\}$ and $\Gamma = \{\alpha,\beta,\gamma,\dots\}$ be two non-empty sets. M is called a Γ-semigroup if $a\alpha b \in M$, for $\alpha \in \Gamma$ and $b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a,b,c \in M$ and for all $\alpha,\beta \in \Gamma$. A semigroup can be considered as a Γ-semigroup. In this paper we introduce orthodox Γ-semigroups and extend different results of orthodox semigroups to orthodox Γ-semigroups.

KEY WORDS AND PHRASES. Semigroup, Γ-semigroup, Orthodox Γ-semigroup, Inverse Γ-semigroup .
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1. INTRODUCTION.

Let A and B be two non-empty sets, M the set of all mappings from A to B, and Γ a set of some mappings from B to A. The usual mapping product of two elements of M can not be defined. But if we take f, g from M and α from Γ then the usual mapping product $f\alpha g$ can be defined. Also we find that $f\alpha g \in M$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for $f,g,h \in M$ and $\alpha,\beta \in \Gamma$.

If M is the set of mxn matrices and Γ is a set of some nxm matrices over the field of real numbers, then we can define $A_{m,n}\alpha_{n,m}B_{m,n}$ such that

$$(A_{m,n}\alpha_{n,m}B_{m,n})\beta_{n,m}C_{m,n} = A_{m,n}\alpha_{n,m}(B_{m,n}\beta_{n,m}C_{m,n}) \text{ where } A_{m,n}, B_{m,n}, C_{m,n} \in M \text{ and}$$

$\alpha_{n,m}, \beta_{n,m} \in \Gamma$. An algebraic system satisfying the associative property of the above type is a Γ-semigroup (Saha [1]).

DEFINITION 1.1. Let $M = \{a,b,c,\dots\}$ and $\Gamma=\{\alpha,\beta,\dots\}$ be two non-empty sets. M is called a Γ-semigroup if (i) $a\alpha b \in M$ for $\alpha \in \Gamma$ and $a,b \in M$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a,b,c \in M$ and for all $\alpha,\beta \in \Gamma$.

A semigroup can be considered a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the given binary relation in S to S ∪ 1 by defining 11 = 1 and 1a = a1 = a for all a in S. It can be shown that S ∪ 1 is a semigroup with identity element 1.

Let Γ = {1}. Putting ab = a1b it can be shown that the semigroup S is a Γ-semigroup where Γ = {1}. Since every semigroup is a Γ-semigroup in the above sense, the main thrust of our work is to extend different fundamental results of semigroups to Γ-semigroups. In Sen and Saha [2] and Saha [1,3,4] we have extended some results of semigroups to Γ-semigroups. In this paper we want to introduce orthodox Γ-semigroups and we want to extend results of Hall [5] and Yamada [6] to Γ-semigroups.

2. PRELIMINARIES.

We recall the following definitions and results from [1], [2], [3] and [4].

DEFINITION 2.1. Let M be a Γ-semigroup. A non-empty subset B of M is said to be a Γ-subsemigroup of M if $B\Gamma B \subset B$.

DEFINITION 2.2. Let M be a Γ-semigroup. An element $a \in M$ is said to be regular if $a \in a\Gamma M\Gamma a$, where $a\Gamma M\Gamma a = \{a\alpha b\beta a: b \in M, \alpha,\beta \in \Gamma\}$. A Γ-semigroup M is said to be regular if every element of M is regular.

EXAMPLE 2.1. Let M be the set of 3x2 matrices and Γ be a set of some 2x3 matrices over a field. We show that M is a regular Γ-semigroup. Let A ε M, where $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$

Then we choose B ε Γ according to the following cases such that ABABA = ABA = A.

CASE 1. When the submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular, then $ad - bc \neq 0$. e,f may both be 0 or one of them is 0 or both of them are non-zero.

Then $B = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 \end{pmatrix}$ and we find ABA = A.

CASE 2. $af - be \neq 0$. Then $B = \begin{pmatrix} \frac{f}{af-be} & 0 & \frac{-b}{af-be} \\ \frac{-e}{af-be} & 0 & \frac{a}{af-be} \end{pmatrix}$ and ABA = A.

CASE 3. $cf - de \neq 0$. Then $B = \begin{pmatrix} 0 & \frac{f}{cf-de} & \frac{-d}{cf-de} \\ 0 & \frac{-e}{cf-de} & \frac{c}{cf-de} \end{pmatrix}$ and ABA = A.

CASE 4. When the submatrices are singular. Then either $\begin{cases} ad - bc = 0 \text{ or} \\ cf - be = 0 \end{cases}$

$\begin{cases} ad - bc = 0, \\ af - de = 0 \end{cases}$

If all the elements of A are 0, then the case is trivial. Next we consider at least one of the elements of A is non-zero, say $a_{ij} \neq 0$, $i = 1,2,3$ and $j = 1,2$. Then we take the b_{ji}th element of B as $(a_{ij})^{-1}$ and the other elements of B are zero and we find that ABA = A. Thus A is regular. Hence M is a regular Γ-semigroup.

DEFINITION 2.3. Let M be a Γ-semigroup. An element $e \in M$ is said to be an idempotent of M if there exists an $\alpha \in \Gamma$ such that $e\alpha e = e$. In this case we shall say e is an α-idempotent.

DEFINITION 2.4. Let M be a Γ-semigroup and $a \in M$. Let $b \in M$ and $\alpha, \beta \in \Gamma$. b is said to be an (α, β) inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. In this case we shall write $b \in V_\alpha^\beta(a)$.

DEFINITION 2.5. A regular Γ-semigroup M is called an inverse Γ-semigroup if $|V_\alpha^\beta(a)| = 1$, for all $a \in M$ and for all $\alpha, \beta \in \Gamma$, whenever $V_\alpha^\beta(a) \neq \phi$. That is, every element a of M has a unique (α, β) inverse whenever the (α, β) inverse of a exists.

THEOREM 2.1. Let M be a Γ-semigroup. M is an inverse Γ-semigroup if and only if (i) M is regular and (ii) if e and f are any two α-idempotents of M then $e\alpha f = f\alpha e$, where $\alpha \in \Gamma$.

LEMMA 2.2. Let M be a regular Γ-semigroup and let M' be a Γ'-semigroup. Let (f,g) be a homomorphism from (M,Γ) onto (M',Γ'). Then M' is a regular Γ'-semigroup.

3. ORTHODOX Γ-SEMIGROUP.

DEFINITION 3.1. A regular Γ-semigroup M is called an orthodox Γ-semigroup if for e an α-idempotent and f a β-idempotent then $e\alpha f$, $f\alpha e$ are β-idempotents and $e\beta f$, $f\beta e$ are α-idempotents.

EXAMPLE 3.1. Let A = {1,2,3} and B = {4,5}. M denotes the set of all mappings from A to B. Here members of M will describe the images of the elements 1,2,3. For example the map 1→4, 2→5, 3→4 will be written as (4,5,4) and (4,4,5) denotes the map 1→4, 2→4, 3→5. Again a map from B→A will be in the same fashion. For example (1,2) denotes 4→1, 5→2. Now, M = {(4,4,4), (4,4,5), (4,5,4), (4,5,5), (5,5,5), (5,4,5), (5,4,4), (5,5,4) }. Let Γ = {(1,1), (1,2), (1,3), (2,1), (2,2), (3,3)} be a set of some mappings from B to A. Let $f,g \in M$ and $\alpha \in \Gamma$. Under the usual mapping composition, $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in M$. Also, we can easily show that $(f\alpha g)\beta h = f\alpha(g\beta h)$, for all $f,g,h \in M$ and $\alpha, \beta \in \Gamma$. One can easily verify that M is a regular Γ-semigroup. Here

(4,4,4) (1,1) (4,4,4) (1,1) (4,4,4) = (4,4,4)
(4,4,5) (1,3) (4,4,5) (1,3) (4,4,5) = (4,4,5)
(4,5,4) (1,2) (4,5,4) (1,2) (4,5,4) = (4,5,4)
(4,5,5) (1,2) (4,5,5) (1,2) (4,5,5) = (4,5,5)
(5,5,5) (1,1) (5,5,5) (1,1) (5,5,5) = (5,5,5)
(5,4,5) (1,2) (5,4,4) (1,2) (5,4,5) = (5,4,5)
(5,4,4) (1,2) (5,4,5) (1,2) (5,4,4) = (5,4,4)

$$(5,5,4)\ (1,2)\ (5,4,5)\ (1,3)\ (5,5,4) = (5,5,4)$$

Here (4,4,5) is (1,3) idempotent, and (5,4,4) is (2,1) idempotent, but (4,4,5) (1,3) (5,4,4) = (5,5,4) is not an idempotent. Hence this regular Γ-semigroup is not an orthodox Γ-semigroup.

EXAMPLE 3.2. Let Q* denote the set of all non-zero rational numbers. Let Γ be the set of all positive integers. Let $a \in Q^*$, $\alpha \in \Gamma$ and $b \in Q^*$. $a\alpha b$ is defined by $|a|\ \alpha b$. For this operation Q* is a Γ-semigroup. Let $\frac{p}{q} \in Q^*$. Now $\left|\frac{p}{q}\right| |q| \left|\frac{1}{p}\right|\ 1\ \frac{p}{q} = \frac{p}{q}$. Hence this is a regular Γ-semigroup. Here $\frac{1}{q}$, $|q| \in \Gamma$ is a $|q|$-idempotent. These are the only idempotents of Q*. Now $\frac{1}{q}|q|\frac{1}{p}$ is a $|p|$-idempotent. Hence Q* is an orthodox Γ-semigroup.

THEOREM 3.3. Every inverse Γ-semigroup is an orthodox Γ-semigroup.

PROOF. Let M be an inverse Γ-semigroup. Let e be a α-idempotent and f be a β-idempotent. Now $e\alpha f \in M$. Since M is an inverse Γ-semigroup, let $x \in V_\delta^\gamma(e\alpha f)$. Then $e\alpha f\delta x\gamma e\alpha f = e\alpha f$, $x\gamma e\alpha f\delta x = x$. Let $g = f\delta x\gamma e\alpha f$. Then $g\alpha g = g$. Also, let $h = f\delta x\gamma e$. Then, $f\delta x\gamma e\alpha f\beta f\delta x\gamma e\alpha f\delta x\gamma e\alpha f = f\delta x\gamma e\alpha f\delta x\gamma e\alpha f = f\delta x\gamma e\alpha f = g$. This shows that $g\beta h\alpha g = g$. Similarly $h\alpha g\beta h = h$. Hence $g \in V_\alpha^\beta(h)$. Also $e\alpha f \in V_\alpha^\beta(h)$. Since M is an inverse Γ-semigroup, $g = e\alpha f$. Hence $e\alpha f$ is a β-idempotent. Similarly we can show that $f\alpha e$ is β-idempotent, and both $e\beta f$ and $f\beta e$ are α-idempotents.

EXAMPLE 3.4. In example 3.2 we have shown that Q* is an orthodox Γ-semigroup. Now $(1/q) \in V_p^1(q/p)$. Also $(-1/q) \in V_p^1(q/p)$. Hence Q* is not an inverse Γ-semigroup.

THEOREM 3.5. A regular Γ-semigroup M is an orthodox Γ-semigroup if an only if for any α-idempotent $e \in M$, where $\alpha \in \Gamma$, if $V_\alpha^\beta(e) \neq \phi$ and $V_\beta^\alpha(e) \neq \phi$, then each member of $V_\alpha^\beta(e)$ and $V_\beta^\alpha(e)$ is a β-idempotent.

PROOF. Suppose M is an orthodox Γ-semigroup. Let e be an α-idempotent of M and let $x \in V_\alpha^\beta(e)$. Then $e\alpha x\beta e = e$ and $x\beta e\alpha x = x$. Now $e\alpha x$ is a β-idempotent and $x\beta e$ is an α-idempotent. Then $x=(x\beta e)\alpha(e\alpha x)$ is a β-idempotent. Next let $y \in V_\beta^\alpha(e)$. Then $e\beta y\alpha e = e$ and $y\alpha e\beta y = y$. Now $y\alpha e$ is a β-idempotent and $e\beta y$ is an α-idempotent. Then $y = (y\alpha e)\alpha(e\beta y)$ is a β-idempotent. Conversely suppose that M satisfies the given conditions. Let e be an α-idempotent and f be a β-idempotent. Consider $e\alpha f$. Now $e\alpha f \in M$, and since M is regular there exists $x \in M$ and $\gamma, \delta \in \Gamma$ such that $e\alpha f\gamma x\delta e\alpha f = e\alpha f$ and $x\delta e\alpha f\gamma x = x$. Let $g = f\gamma x\delta e$. Then $g\alpha g = f\gamma(x\delta e\alpha f\gamma x)\delta e = f\gamma x\delta e = g$. Now, $e\alpha f\beta f\gamma x\delta e\alpha e\alpha f = e\alpha f\gamma x\delta e\alpha f = e\alpha f$ and $f\gamma x\delta e\alpha e\alpha f\beta f\gamma x\delta e = f\gamma x\delta e\alpha f\gamma x\delta e = f\gamma x\delta e$. Hence $e\alpha f \in V_\alpha^\beta(f\gamma x\delta e)$. Then by the given condition $e\alpha f$ is β-idempotent. Dually we can prove that $f\alpha e$ is β-idempotent. Similarly, it is easy to see that $e\beta f$ and $f\beta e$ are α-idempotents.

THEOREM 3.6. A regular Γ-semigroup M is an orthodox Γ-semigroup if and only if for $a,b \in M$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^{\alpha_2}(a)$, and $b' \in V_{\beta_1}^{\beta_2}(b)$ we have

$$b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b) \text{ and } b'\alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a\beta_2 b).$$

PROOF. Let us assume that M is an orthodox Γ-semigroup. Let $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$. Then $a\alpha_1a'\alpha_2a = a$, $a'\alpha_2a\alpha_1a' = a'$, $b\beta_1b'\beta_2b = b$, and $b'\beta_2b\beta_1b' = b'$. Now $a'\alpha_2a$ is an α_1-idempotent and $b\beta_1b'$ is a β_2-idempotent. Hence $(a'\alpha_2a)\alpha_1(b\beta_1b')$ is a β_2-idempotent, and $(b\beta_1b')\beta_2(a'\alpha_2a)$ is an α_1-idempotent. $(a'\alpha_2a)\beta_2(b\beta_1b')$ is an α_1-idempotent and $(b\beta_1b')\alpha_1(a'\alpha_2a)$ is a β_2-idempotent.

$$\begin{aligned} a\alpha_1b\beta_1b'\beta_2a'\alpha_2a\alpha_1b &= a\alpha_1a'\alpha_2a\alpha_1b\beta_1b'\beta_2a'\alpha_2a\alpha_1b\beta_1b'\beta_2b \\ &= a\alpha_1a'\alpha_2a\alpha_1b\beta_1b'\beta_2b \text{ (since } a'\alpha_2a\alpha_1b\beta_1b' \text{ is } \beta_2\text{- idempotent)} \\ &= a\alpha_1b. \end{aligned}$$

$$\begin{aligned} b'\beta_2a'\alpha_2a\alpha_1b\beta_1b'\beta_2a' &= b'\beta_2b\beta_1b'\beta_2a'\alpha_2a\alpha_1b\beta_1b'\beta_2a'\alpha_2a\alpha_1a' \\ &= b'\beta_2b\beta_1b'\beta_2a'\alpha_2a\alpha_1a' \text{ (since } b\beta_1b'\beta_2a'\alpha_2a \text{ is } \alpha_1\text{- idempotent)} \\ &= b'\beta_2a'. \end{aligned}$$

Hence $b'\beta_2a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1b)$. Similarly it can be shown that $b'\alpha_1a' \in V_{\beta_1}^{\alpha_2}(a\beta_2b)$. Conversely, assume that the given conditions hold in M. Let e be an α-idempotent and f be a β-idempotent of M. Now $f \in V_\beta^\beta(f)$ and $e \in V_\alpha^\alpha(e)$. Then by the given conditions (i) $e\alpha f \in V_\alpha^\beta(f\beta e)$ and (ii) $e\beta f \in V_\alpha^\beta(f\alpha e)$. From (i) we get $e\alpha f\beta f\beta e\alpha e\alpha f = e\alpha f$. Then $e\alpha f\beta e\alpha f = e\alpha f$. Thus $e\alpha f$ is a β-idempotent. From (ii) we get $f\alpha e\alpha e\beta f\beta f\alpha e = f\alpha e$. Then $f\alpha e\beta f\alpha e = f\alpha e$. So, $f\alpha e$ is β-idempotent. Again $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$. Then by the given conditions we get (iii) $f\beta e \in V_\beta^\alpha(e\alpha f)$ and (iv) $f\alpha e \in V_\beta^\alpha(e\beta f)$. From (iii) we get $f\beta e\alpha e\alpha f\beta f\beta e = f\beta e$. Then $f\beta e\alpha f\beta e = f\beta e$. Hence $f\beta e$ is α-idempotent. From (iv) we get $e\beta f\beta f\alpha e\alpha e\beta f = e\beta f$. So, $e\beta f\alpha e\beta f = e\beta f$. Thus $e\beta f$ is α-idempotent. Hence M is an orthodox Γ-semigroup.

THEOREM 3.7. A regular Γ-semigroup M is an orthodox Γ-semigroup if and only if for $a,b \in M$, $V_\alpha^\beta(a) \cap V_\alpha^\beta(b) \neq \phi$ for some $\alpha, \beta \in \Gamma$. This implies that $V_\gamma^\delta(a) = V_\gamma^\delta(b)$ for all $\gamma, \delta \in \Gamma$.

PROOF. Suppose M is an orthodox Γ-semigroup. For $a,b \in M$, let there exist $\alpha, \beta \in \Gamma$ such that $V_\alpha^\beta(a) \cap V_\alpha^\beta(b) \neq \phi$. Let $\gamma, \delta \in \Gamma$. First let us show that $V_\gamma^\delta(a) \subset V_\gamma^\delta(b)$. Let $a' \in V_\alpha^\beta(a) \cap V_\alpha^\beta(b)$, and $a^* \in V_\gamma^\delta(a)$. Then $a\alpha a'\beta a = a$, $a'\beta a\alpha a' = a'$, $b\alpha a'\beta b = b$, $a'\beta b\alpha a' = a'$, $a\gamma a^*\delta a = a$, $a^*\delta a\gamma a^* = a^*$. We can easily show that

$$(a^*\delta a)\alpha(a'\beta b)\mathcal{R}(a^*\delta a) \ . \tag{3.1}$$

Now $a^*\delta a$ is γ-idempotent, and $a'\beta b$ is α-idempotent. Hence $(a^*\delta a)\alpha(a'\beta b)$ is γ-idempotent. Then from (3.1) we get

$$(a^*\delta a)\alpha(a'\beta b)\gamma(a^*\delta a) = a^*\delta a \ . \tag{3.2}$$

Now $a'\beta a = a'\beta a\gamma a^*\delta a = a'\beta a\gamma a^*\delta a\alpha a'\beta b\gamma a^*\delta a = a'\beta a\alpha a'\beta b\gamma a^*\delta a = a'\beta b\gamma a^*\delta a$.

Hence $b\alpha a'\beta a = b\alpha a'\beta b\gamma a^*\delta a = b\gamma a^*\delta a$. (3.3)

Again we can show that $(b\alpha a')\beta(a\gamma a^*)\mathcal{L}(a\gamma a^*)$. Now $(b\alpha a')\beta(a\gamma a^*)$ is δ-idempotent. Hence $(a\gamma a^*)\delta(b\alpha a')\beta(a\gamma a^*) = a\gamma a^*$.

Then $a\alpha a' = a\gamma a^*\delta a\alpha a' = a\gamma a^*\delta b\alpha a'\beta a\gamma a^*\delta a\alpha a' = a\gamma a^*\delta b\alpha a'\beta a\alpha a' = a\gamma a^*\delta b\alpha a'$

$a\alpha a'\beta b = a\gamma a^*\delta b\alpha a'\beta b = a\gamma a^*\delta b$. (3.4)

Now, $b\gamma a^*\delta b = b\gamma a^*\delta a\gamma a^*\delta b$

$= b\gamma a^*\delta a\alpha a'\beta b$ by (3.4)

$= b\alpha a'\beta a\alpha a'\beta b$ by (3.3)

$= b\alpha a'\beta b = b$

and $a^*\delta b\gamma a^* = a^*\delta a\gamma a^*\delta b\gamma a^*\delta a\gamma a^*$

$= a^*\delta a\gamma a^*\delta b\alpha a'\beta a\gamma a^*$ by (3.3)

$= a^*\delta a\alpha a'\beta b\alpha a'\beta a\gamma a^*$ by (3.4)

$= a^*\delta a\alpha a'\beta a\gamma a^* = a^*\delta a\gamma a^* = a^*$.

Hence $a^* \in V_\gamma^\delta(b)$. Thus $V_\gamma^\delta(a) \subset V_\gamma^\delta(b)$. Similarly $V_\gamma^\delta(b) \subset V_\gamma^\delta(a)$. Thus $V_\gamma^\delta(a) = V_\gamma^\delta(b)$.

Conversely, assume that the given condition holds in M. Let e be α-idempotent and f be β-idempotent. Consider $e\alpha f$. Since M is regular, there exists $\gamma, \delta \in \Gamma$ and $x \in M$ such that $e\alpha f\gamma x\delta e\alpha f = e\alpha f$ and $x\delta e\alpha f\gamma x = x$. Let $g = f\gamma x\delta e$. Then $g\alpha g = g$. Hence $f\gamma x\delta e \in V_\alpha^\alpha(f\gamma x\delta e)$. Let $h = e\alpha f\gamma x\delta e$. Then $h\alpha h = h$. Also, $f\gamma x\delta e \in V_\alpha^\alpha(e\alpha f\gamma x\delta e)$. Hence $V_\alpha^\alpha(g) \cap V_\alpha^\alpha(h) \neq \phi$. Then $V_\alpha^\theta(g) = V_\alpha^\theta(h)$ for any $\theta \in \Gamma$. But $e\alpha f\beta f\gamma x\delta e\alpha e\alpha f = e\alpha f$ and $f\gamma x\delta e\alpha e\alpha f\beta f\gamma x\delta e = f\gamma x\delta e$. Hence $e\alpha f \in V_\alpha^\beta(f\gamma x\delta e)$. Then $e\alpha f \in V_\alpha^\beta(e\alpha f\gamma x\delta e)$. This implies that $e\alpha f\beta e\alpha f\gamma x\delta e\alpha e\alpha f = e\alpha f$. So $e\alpha f\beta e\alpha f = e\alpha f$. Hence $e\alpha f$ is β-idempotent. Similarly, it can be proved that $f\alpha e$ is a β-idempotent and both $e\beta f$ and $f\beta e$ are α-idempotents. Let M be a regular Γ-semigroup and $a,b \in M$, $a' \in V_\alpha^\beta(a)$, and $b' \in V_\gamma^\delta(b)$. Then $e = a'\beta a$ is α-idempotent and $f = b\gamma b'$ is δ-idempotent. Let $\theta \in \Gamma$. Suppose $x \in V_{\alpha_1}^{\beta_1}(e\theta f)$. Then $e\theta f = e\theta f\alpha_1 x\beta_1 e\theta f$ and $x = x\beta_1 e\theta f\alpha_1 x$.

Let $g = f\alpha_1 x\beta_1 e$. Now $g\theta g = f\alpha_1 x\beta_1 e\theta f\alpha_1 x\beta_1 e = f\alpha_1 x\beta_1 e = g$. Hence g is θ-idempotent.

Also $g\alpha e = g = f\delta g$ and $e\theta g\theta f = e\theta f$. Now,

$$\begin{aligned}(a\theta b)\gamma(b'\delta g\alpha a')\beta(a\theta b) &= a\theta(b\gamma b')\delta g\alpha(a'\beta a)\theta b = a\theta f\delta g\alpha e\theta b \\ &= a\theta f\delta g\theta b = a\theta g\theta b = a\alpha a'\beta a\theta g\theta b\gamma b'\delta b \\ &= a\alpha e\theta g\theta f\delta b = a\alpha e\theta f\delta b = a\theta b.\end{aligned}$$

Similarly we can show that $(\beta'\delta g\alpha a')\beta(a\theta b)\gamma(b'\delta g\alpha a') = b'\delta g\alpha a'$. Hence $b'\delta g\alpha a' \in V_\gamma^\beta(a\theta b)$. Thus we have the following lemma.

LEMMA 3.8. Let M be a regular Γ-semigroup and $a,b \in M$. If $a' \in V_\alpha^\beta(a)$, $b' \in V_\gamma^\delta(b)$ and $\theta \in \Gamma$, then there exists a θ-idempotent $g \in M$ and $b'\delta g\alpha a' \in V_\gamma^\beta(a\theta b)$.

4. INVERSE Γ-SEMIGROUP CONGRUENCE.

DEFINITION 4.1. Let M be a Γ-semigroup. A congruence on M is defined as an equivalence relation ρ on the set M stable under left and right Γ-operations. That is, for every $a,b,c \in M$, $(a,b) \in \rho$ implies $(c\alpha a, c\alpha b) \in \rho$ and $(a\alpha c, b\alpha c) \in \rho$ for all $\alpha \in \Gamma$. A left (right) congruence on M is an equivalence relation on M, stable under left (right) Γ-operation.

Let M be a Γ-semigroup. Let ρ be a congruence on M. We define $(a\rho)\alpha(b\rho) = (a\alpha b)\rho$ for all $a\rho, b\rho \in M/\rho$ and for all $\alpha \in \Gamma$. It can easily be seen that the definition is well defined and M/ρ is a Γ-semigroup. Let us now characterize the minimum inverse Γ-semigroup congruence on an orthodox Γ-semigroup.

DEFINITION 4.2. Let M be an orthodox Γ-semigroup. A congruence ρ on M will be called an inverse Γ-semigroup congruence if M/ρ is an inverse Γ-semigroup.

THEOREM 4.1. Let M be an orthodox Γ-semigroup. Then the relation ρ defined by $\rho = \{(a,b) \in M\times M: V_\alpha^\beta(a) = V_\alpha^\beta(b)$ for all $\alpha,\beta \in \Gamma\}$ is the minimum inverse Γ-semigroup congruence on M.

PROOF. From the definition of ρ it is clear that ρ is an equivalence relation. To prove that ρ is a congruence relation, assume that $(a,b) \in \rho$, $c \in M$ and $\theta \in \Gamma$. Then $V_\alpha^\beta(a) = V_\alpha^\beta(b)$ for all $\alpha,\beta \in \Gamma$. Hence there exists $\alpha,\beta \in \Gamma$ such that $V_\alpha^\beta(a) = V_\alpha^\beta(b) \neq \phi$. Let $a^* \in V_\alpha^\beta(a) = V_\alpha^\beta(b)$, $c^* \in V_\gamma^\delta(c)$. Then by Lemma 3.8 $c^*\delta g\alpha a^* \in V_\gamma^\beta(a\theta c)$ and $c^*\delta g\alpha a^* \in V_\gamma^\beta(b\theta c)$, for some θ-idempotent $g = g\theta g \in M$. Hence $V_\gamma^\beta(a\theta c) \cap V_\gamma^\beta(b\theta c) \neq \phi$. Then by Theorem 3.7, $V_\alpha^\beta(a\theta c) = V_\alpha^\beta(b\theta c)$ for all $\alpha,\beta \in \Gamma$, so that

$(a\theta c, b\theta c) \in \rho$. Similarly we can show that $(c\theta a, c\theta b) \in \rho$. Hence ρ is a congruence on M. Suppose now $e = e\alpha e$ and $f = f\alpha f$ are two idempotents of M. Then $e\alpha f$ and $f\alpha e$ are α-idempotents and $e\alpha f \in V_\alpha^\alpha(e\alpha f)$ and $e\alpha f \in V_\alpha^\alpha(f\alpha e)$. Hence $V_\alpha^\alpha(e\alpha f) \cap V_\alpha^\alpha(f\alpha e) \neq \phi$. Consequently $V_\gamma^\delta(e\alpha f) = V_\gamma^\delta(f\alpha e)$ for all $\gamma, \delta \in \Gamma$. Thus we find that $(e\alpha f, f\alpha e) \in \rho$. Hence from Theorem 2.1 and Lemma 2.2 we find that M/ρ is an inverse Γ-semigroup. Finally, suppose that ρ_1 is a congruence on M such that M/ρ_1 is an inverse Γ-semigroup. If $(a,b) \in \rho$ then $V_\alpha^\beta(a) = V_\alpha^\beta(b)$ for all $\alpha, \beta \in \Gamma$. There exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a = a$, $x\beta a\alpha x = x$, $b\alpha x\beta b = b$ and $x\beta b\alpha x = x$.
Then $(a\rho_1)\alpha(x\rho_1)\beta(a\rho_1) = a\rho_1, (x\rho_1)\beta(a\rho_1)\alpha(x\rho_1) = x\rho_1, (b\rho_1)\alpha(x\rho_1)\beta(b\rho_1) = b\rho_1$, $(x\rho_1)\beta(b\rho_1)\alpha(x\rho_1) = x\rho_1$. Hence $a\rho_1, b\rho_1 \in V_\beta^\alpha(x\rho_1)$. But M/ρ_1 is an inverse Γ-semigroup. Hence $|V_\beta^\alpha(x\rho_1)| = 1$. Then $a\rho_1 = b\rho_1$, so that $(a,b) \in \rho_1$. Hence $\rho \subset \rho_1$. This completes the proof.

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