Internat. J. Math. & Math. Sci.
VOL. 13 NO. 2 (1990) 411-414

RESEARCH NOTES

ON THE COEFFICIENT DOMAINS OF UNIVALENT FUNCTIONS

M.M. ELHOSH

Department of Mathematics
The University College of Wales
Aberystwyth, Wales
United Kingdom

(Received March 10, 1987)

ABSTRACT. Coefficient domains for functions whose derivative has positive real part in the interior of an ellipse are given in this paper.

KEY WORDS AND PHRASES. Univalent functions.
1980 AMS CLASSIFICATION CODES. Primary 30C45, 30C50.

1. INTRODUCTION.

Let f(z) be regular and satisfy the condition

$$\operatorname{Re} f'(z) > 0 \tag{1.1}$$

in a domain D. Then it is wellknown (see [1, p. 582], [2]) that f(z) is univalent in D. Let D be the interior of a fixed ellipse $E_o = \{z = \cosh(s_o + i\tau),\ 0 \leq \tau < 2\pi,\ s_o = \tanh^{-1}(b/a),\ a > b > 0\}$ with foci ±1. Let $r_o = a + b$ be the sum of the semi-axes of the ellipse E_o and set $z = \cosh \eta$ where $\eta = s + i\tau$ and $0 < s < s_o$. Then we see from the operator $\partial/\partial z = \partial/\partial\eta$ that (1.1) becomes

$$\operatorname{Re}\sqrt{z^2-1}\, f'(z) > 0 \tag{1.2}$$

for z in $\operatorname{Int}(E_o)$.

In this note we shall study coefficient domains for functions which are regular and satisfy (1.2) in $\operatorname{Int}(E_o)$. In connection with this problem see [3, Problem 6.54], [4, Problem 7.2], [5], [6], and [7] and [8, p.141].

THEOREM. Let $f(z) = \sum_{n=1}^{\infty} a_n T_n(z)$ be regular and satisfy (1.2) in $\mathrm{Int}(E_o)$. Then for $n > 1$ we have the sharp inequalities

$$|a_n| < 2/n \sinh ns_o, \tag{1.3}$$

$$\alpha_n^2 \sinh^2 ns_o + \beta_n^2 \cosh^2 ns_o < 4/n^2. \tag{1.4}$$

The inequality (1.4) shows that the coefficients a_n lie in ellipses with centre at the origin and semi-axes $4/n(r_o^n \pm r_o^{-n})$ where $n=1,2,3,\ldots$.

PROOF OF THE THEOREM. We see from (1.2), setting $z = \cosh(s_o + i\tau)$, $s_o = \log r_o$ and $a_n = \alpha_n + i\beta_n$, that

$$\mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)] = 1 + \sum_{n=1}^{\infty} n(\alpha_n \sinh ns_o \cos n\tau - \beta_n \cosh ns_o \sin n\tau)$$

where $\mathrm{Re}[1 + \sqrt{z^2-1}\, f'(z)] > 0$ in $\mathrm{Int}(E_o)$.

Since this is a Fourier series for fixed s_o; we then see that

$$\int_0^{2\pi} \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)]\, d\tau = 2\pi, \tag{1.5}$$

$$\int_0^{2\pi} \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)] \cos n\tau = n\pi i \alpha_n \sinh ns_o, \tag{1.6}$$

$$\int_o^{2\pi} \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)] \sin n\tau = n\pi i \beta_n \cosh ns_o. \tag{1.7}$$

Using (1.5), (1.6), and (1.7) we obtain

$$|a_n| = |\alpha_n + i\beta_n|$$

$$= \left|\frac{1}{n\pi i} \int_0^{2\pi} \frac{\sinh n(s_o + i\tau)\, \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)]}{\sinh ns_o \cosh ns_o}\, d\tau\right|$$

$$< \frac{1}{n\pi \sinh ns_o} \int_0^{2\pi} \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)]\, d\tau$$

$$< 2/n \sinh ns_o$$

since $|\sinh n(s_o + i\tau)| < \cosh ns_o$. This is (1.3).

We also see form (1.5), (1.6) and (1.7) that

$$|\alpha_n \sinh ns_o + i\beta_n \cosh ns_o| = \left|\frac{1}{n\pi i} \int_0^{2\pi} \mathrm{Re}[1+\sqrt{z^2-1}\, f'(z)]e^{ni\tau}d\tau\right| \leq 2/n.$$

This gives

$$\alpha_n^2 \sinh^2 ns_o + \beta_n^2 \cosh^2 ns_o \leq 4/n^2$$

as required in (1.4) and the proof of the theorem is complete.

Finally we see from [9, Theorems 2 and 6] that

$$f'(z) = -1/\sqrt{z^2-1} + \int_0^{2\pi} (K(z,\bar{\xi})\,/\sqrt{z^2-1})d\psi(\tau')$$

where $\xi = \cosh(s_o + i\tau')$, $z = \cosh(s + i\tau)$, $0 < s < s_o$, $0 \leq \tau' < 2\pi$ and

$0 \leq \tau < 2\pi$ plays the role of the extermal function in this case.

REMARK. Normalizing in the sense of [3, Remark 2] we obtain the analogous results in [2, Theorem 1].

REFERENCES

1. KAPLAN, W. Advanced Calculus, Addison-Wesley (1952).
2. MacGREGOR, T.H. Functions whose derivative has a positive real part. Trans. Amer. Math. Soc. 104 (1962), 532-537.
3. ANDERSON, J.M., BARTH, K.F. and BRANNAN, D.A. Research problems in complex analysis, Bull. London Math. Soc. 9 (1977), 129-162.
4. HAYMAN, W.K. Research problems in function theory, Athlone Press, London (1967).
5. ROYSTER, W.C. Functions having positive real part in an ellipse, Proc. Amer. Math. Soc. 10 (1959), 266-269.
6. ROYSTER, W.C. Coefficient problems for functions regular in an ellipse, Duke Math. J. 26 (1959), 361-372.
7. SCHAEFFER, A.C. and SPENCER, D.C. Coefficient regions for Schlicht functions, Amer. Math. Soc., Colloq. Pub. 35 (1950).
8. SHAFFER, D.B. The Biekerbach conjecture, Contemporary Math. 38 (1985), 139-141.
9. ROYSTER, W.C. A Poisson integral formula for the ellipse and some applications. Proc. Amer. Math. Soc. 15 (1964), 661-670.
10. ELHOSH, M.M. On mean p-valent functions in an ellipse, Proc. Roy. Soc. Edinburgh 92A (1982), 1-11.