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ON FRÉCHET THEOREM IN THE SET OF MEASURE PRESERVING FUNCTIONS OVER THE UNIT INTERVAL

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ABSTRACT. In this paper, we study the Fréchet theorem in the set of measure preserving functions over the unit interval and show that any measure preserving function on [0,1] can be approximated by a sequence of measure preserving piecewise linear continuous functions almost everywhere. Some application is included.

KEY WORDS AND PHRASES. Measure preserving, Borel set, distribution function, spline.
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1. INTRODUCTION.

A theorem of Fréchet states that for every measurable function f(x) which is defined and is finite almost everywhere on the closed interval [a,b], there exists a sequence of continuous function converging to f(x) almost everywhere (Natanson [1]). In certain applications it is important to know whether a Fréchet-type theorem holds when the functions involved are measure preserving functions. In this paper we show that every measure preserving function f on [0,1] can be approximated by a sequence of piecewise linear measure preserving continuous functions almost everywhere. More precisely, given a measure preserving function f on [0,1], there exists a sequence of piecewise linear measure preserving continuous functions converging to f almost everywhere. The next section contains the proof of this assertion. Furthermore, we show that every measure preserving function can be approximated by a sequence of piecewise linear one-to-one measure preserving functions almost everywhere. Also included are some applications of the results.

2. FRECHET THEOREM IN THE SET OF MEASURE PRESERVING FUNCTIONS OVER [0,1].

Let m denote the Lebesgue measure on [0,1]. Let f be a measurable function from a closed set B to B. f is said to be measure preserving on B if for each Borel set $A \subseteq B$, $m(f^{-1}(A)) = m(A)$. This notion can be further generalized. Let μ_1 and μ_2 be probability measures defined on close sets B_1 and B_2, respectively. A measurable function f from B_1 to B_2 is said to be measure preserving from (B_1,μ_1) to (B_2,μ_2) if for every Borel set A of B_2, $\mu_1(f^{-1}(A)) = \mu_2(A)$. We now state the main result of this section.

THEOREM 2.1. For every measure preserving function f over [0,1], there exists a sequence of piecewise linear measure preserving continuous functions converging to f almost everywhere.

To prove this theorem, we need several preliminary lemmas. The following lemma of Riesz can be found in Royden [2].

LEMMA 2.1. Let $\{f_n\}$ be a sequence of measurable functions which converges in measure to the function f. Then there is a subsequence $\{f_{n_i}\}$ which converges to f almost everywhere.

LEMMA 2.2. A measure preserving function f on [0,1] is monotone nondecreasing (nonincreasing) if and only if $f(x) = x$ ($f(x) = 1 - x$).

PROOF. Suppose f is monotone nondecreasing. Since f is measure preserving, f must be strictly increasing and

$$x = m([0,x]) = m(f^{-1}[0,x]) = m([0,f^{-1}(x)]) = f^{-1}(x).$$

The nonincreasing case can be proved similarly.

LEMMA 2.3. If f is a piecewise linear continuous function from [0,1] to [0,1], then f is measure preserving if and only if for $0 < y < 1$ but a finite number of values of y, $\sum_{x_i} \frac{1}{|m_i|} = 1$, where the summation is taken over all the elements x_i of the finite set $\{x_i\colon f(x_i) = y\}$ and m_i is the slope of the line segment on the graph of f through the point (x_i,y).

REMARK 2.1. Those points y for whcih m_i is not well-defined are contained in the exceptional set.

PROOF. Suppose that the graph of f is made up of k line segments with k + 1 endpoints, which are defined according to the partition on [0,1]. Let m_i, $1 \le i \le k$ be the corrsponding slopes. Consider a y, $0 < y < 1$, such that y is not the ordinate of any endpoint. It is easy to see that $f^{-1}(\{y\}) = \{x_i \mid f(x_i) = y\}$ is a finite set. Each point (x_i,y) is an interior point of some line segment lying on the graph of f. Let $\delta > 0$ be small enough such that the interval $[y,y+\delta]$ does not contain the ordinate of any k + 1 endpoints as its interior point. Then

$$f^{-1}([y,y+\delta]) = \bigcup_{x_i \in f^{-1}(\{y\})} [a_i,b_i],$$

where $f([a_i,b_i]) = [y,y+\delta]$ and one of the a_i, b_i is x_i. If f is measure preserving, then

$$m(f^{-1}[y,y+\delta]) = \sum m([a_i,b_i])$$

$$= \sum_{x_i \in f^{-1}(\{y\})} \frac{\delta}{|m_i|} = \delta.$$

Thus $\sum_{x_i \in f^{-1}(\{y\})} \frac{1}{|m_i|} = 1.$

Conversely, if $\sum_{x_i \in f^{-1}(\{y\})} \frac{1}{|m_i|} = 1$ for all $0 < y < 1$ except for y being the ordinate of one of k + 1 endpoints. For an arbitrary interval $[a,b] \subseteq [0,1]$ which does not contain ordinate of any endpoint, we have

$$m(f^{-1}([a,b])) = m(\cup [x_i,x_i']) = \sum m([x_i,x_i'])$$

$$= \sum_{x_i} (b - a)/|m_i|$$

$$= (b - a) \sum_{x_i} \frac{1}{|m_i|} = b - a = m([a,b])$$

where the summation is over $[x_i,x'_i]$ such that

$$f([x_i,x'_i]) = [a,b].$$

For general [a,b], the proof follows by partitioning [a,b] as

$$[a,b] = [a,y_1) \cup [y_1,y_2) \cup \dots \cup [y_{n-1},y_n) \cup [y_n,b],$$

where $y_1,\dots,y_n$ are ordinates of endpoints of line segments lying on the graph of f. Furthermore none of the above subintervals contains ordinates of endpoints as an interior point. Now

$$m(f^{-1}[a,b]) = m(f^{-1}([a,y_1) \cup \dots \cup [y_n,b])$$

$$= m(f^{-1}[a,y_1)) + \dots + m(f^{-1}[y_n,b))$$

$$= (y_1 - a) + \dots + (b - y_n)$$

$$= m([a,b]).$$

Hence f is measure preserving.

PROOF OF THEOREM 2.1. In the first part of the proof, we show that for arbitrarily small numbers $\delta > 0$, $\varepsilon > 0$ we can construct a measure preserving piecewise linear continuous function ϕ such that $m(\{x: |f(x) - \phi(x)| > \delta\}) < \varepsilon$.

Choose a natural number n, $\frac{1}{n} < \delta$ and consider the sets

$$E_i = \{x: \frac{(i - 1)}{n} < f(x) < \frac{i}{n}\},\ i = 1,\dots,n - 1,\ E_n = \{x: \frac{(n - 1)}{n} < f(x) < 1\}.$$

Since f is measure preserving, $m(E_i) = \frac{1}{n}$, $i = 1,\dots,n$. These sets are measurable and pairwise disjoint and $[0,1] = \bigcup_{i=1}^{n} E_i$. For each i, choose a closed set $F_i \subseteq E_i$ such that $m(F_i) > m(E_i) - \frac{\varepsilon}{2n} = \frac{1}{n} - \frac{\varepsilon}{2n}$ and set $F = \bigcup_{i=1}^{n} F_i$. It is clear $[0,1] - F = \bigcup_{i=1}^{n} (E_i - F_i)$, and therefore,

$$m([0,1] - F) < 1 - \sum_{i=1}^{n} (\frac{1}{n} - \frac{\varepsilon}{2n}) = \frac{\varepsilon}{2}.$$

Since $[0,1] - F_i$ is an open set of [0,1], it is equal to the union of countable disjoint open intervals in [0,1]. If $[0,1] - F_i$ is the union of finite disjoint open intervals, F_i is the union of a finite number of disjoint closed intervals. If [0,1] $- F_i$ is the union of an infinite number of open intervals, consider the set $L_i = \{\ell_{ij}\}_j$ of all endpoints of these open intervals. By the Bolzano-Weierstrass Theorem, $L_i' \equiv \{\ell_j'\}$, the set of all limit points of L_i, is nonempty. For a point $\ell_j' \in L_i'$, two cases can be considered.

(i)If there exist two sequences of points in L_i; one converges to ℓ_j' from the right and the other converges to ℓ_j' from the left, then we construct an interval

$$I_{ij} = (a_j,b_j) \text{ with } a_j,b_j \text{ belonging to some open intervals of } [0,1] - F_i,\ a_j < \ell_j' < b_j,\ b_j - a_j < \varepsilon/(2^{j+1}n).$$

(ii) If there exists only one sequence of points of L_i converging to ℓ'_j from the right or from the left, then construct the interval

$$I_{ij} = (\ell_j',b_j) \text{ with } b_j - \ell_j' < \varepsilon/(2^{j+1}n)$$

for the former and the interval

$$I_{ij} = (a_j,\ell_j') \text{ with } \ell_j' - a_j < \varepsilon\ \varepsilon/(2^{j+1}n)$$

for the latter case, where a_j, b_j are elements of $[0,1] - F_i$.

In any of these cases, append the resulting interval to $[0,1] - F_i$. It is clear that $([0,1] - F_i) \cup (\bigcup_j I_{ij})$ is an open set of [0,1] and is the union of a finite number of open intervals of [0,1]. Then

$F_i^* = [0,1] - (([0,1] - F_i) \cup (\bigcup_j I_{ij}))$ is a closed set of [0,1] and is equal to the union of a finite number of closed intervals. Furthermore

$$m(F_i^*) > 1 - \left[1 - (\frac{1}{n} - \frac{\varepsilon}{2n}) + \sum_j \frac{\varepsilon}{2^{j+1}n} \right] > \frac{1}{n} - \frac{\varepsilon}{n}.$$

Thus without loss of generality, suppose that each F_i has the property that F_i is the union of a finite number of disjoint closed intervals of [0,1],

$$F_i = \bigcup_{j=1}^{n_i} [a_{ij},b_{ij}], \text{ where}$$

$$a_{i1} < b_{i1} < a_{i2} < b_{i2} < \dots < a_{in_i} < b_{in_i}, \text{ and}$$

$$\frac{1}{n} > m(F_i) = \sum_{j=1}^{n_i} (b_{ij} - a_{ij}) > \frac{1}{n} - \frac{\varepsilon}{n}, \ i = 1,\dots,n.$$

On F, we define a function $\psi_n(x)$ as follows. Restricted to each F_i, the function $\psi_n(x)$ is linear on each interval $[a_{ij},b_{ij}]$ with the absolute value of the slope equal to $\frac{1}{n(b_{ij} - a_{ij})}$. That is, it linearly maps $[a_{ij}, b_{ij}]$ onto the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$. Note that

$$\sum_{j=1}^{n_i} (b_{ij} - a_{ij}) = m(F_i) < \frac{1}{n} \text{ implies}$$

$$\frac{\frac{1}{n}}{b_{ij} - a_{ij}} > 1 \text{ and } \frac{\sum (b_{ij} - a_{ij})}{\frac{1}{n}} < 1.$$

It is trivial that we can extend ψ_n to the whole interval [0,1] by adding a finite number of line segments to form a piecewise linear function ϕ_n satisfying the slope condition in Lemma 2.3. Then by Lemma 2.3, ϕ_n is measure preserving. Also

$$m(\{x: |f(x) - \phi_n(x)| > \tfrac{1}{n}\}) \le m([0,1] - F) < \varepsilon.$$

Since $\frac{1}{n} < \delta$,

$$m(\{x: |f(x) - \phi_n(x)| > \delta\}) \le \varepsilon.$$

To complete the proof of the theorem, just choose two null decreasing sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of positive numbers. For every n, we construct a measure preserving piecewise linear function ϕ_n such that

$$m(\{x: |f(x) - \phi_n(x)| \ge \delta_n\}) < \varepsilon_n.$$

It is clear that ϕ_n converges to f in measure. In fact for any $\delta > 0$, there is a natural number n_0 such that for all $n \ge n_0$, $\delta_n < \delta$, $\varepsilon_n < \varepsilon$

$$m(\{x: |f(x) - \phi_n(x)| \ge \delta\}) < m(\{x: |f(x) - \phi_n(x)| \ge \delta_n\}) < \varepsilon.$$

By Lemma 2.1, there is a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ converging to the function f almost everywhere.

We remark that with a minor change in the construction of the function ψ_n in the proof of Theorem 2.1, the following result is obtained.

THEOREM 2.2. Let f be a measure preserving function over [0,1]. Then there exists a sequence of one-to-one piecewise linear measure-preserving functions over [0,1] converging to f almost everywhere.

PROOF. The proof is similar to that of Theorem 2.1. The only detail changed is the construction of ψ_n. This time over each F_i we approximate the function f by a one-to-one function from F_i to $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and on each $[a_{ij},b_{ij}]$ it is linear with slope 1 or -1. Then we extend the function to ϕ_n on [0,1] by adding a finite number of line segments with slope 1 or -1 and keep the one-to-one property. It is clear that ϕ_n is measure preserving since the slope condition in Lemma 2.3 is satisfied.

Theorem 2.1. has several interesting applications. One can use it to study a certain dynamic system arising from the so-called tent function (Devaney [3]) mapping from unit interval onto unit interval. To be in line with this paper we given an application arising from probability. Let μ represent a probability measure on an interval B. The distribution function of the measure μ is defined as $F_\mu(x) = \mu(B \cap (-\infty,x])$, for all $x \in B$, and is a right continuous nondecreasing function, $0 \leq F_\mu(x) \leq 1$. If μ does not have any atom point, F_μ is a continuous function on B. For an arbitrary probability measure μ on B, μ is completely defined if and only if F_μ is defined. In the case of μ having no atom point, i.e., F_{μ_1} is continuous, F_μ is a function from B onto [0,1]. In this case, the function F_μ^{-1} is defined by $F_\mu^{-1}(y) = \inf\{x\colon F_\mu(x) \geq y\}$,
for all $0 \leq y \leq 1$. Hence

$$\mu(F_\mu^{-1}([0,y]) = \mu(B \cap(-\infty,F_\mu^{-1}(y)]\} = y = m([0,y]).$$

Then F_μ is a measure preserving function from (B,μ) to $([0,1],m)$.

COROLLARY 2.1. Let μ_1 and μ_2 be probability measures without atom points on closed sets B_1 and B_2, respectively. Let f be a measure preserving function from (B_1,μ_1) to (B_2,μ_2). Then there is a sequence of measure preserving continuous functions from (B_1,μ_1) to (B_2,μ_2) that converges to f almost everywhere.

PROOF. F_{μ_1} and F_{μ_2} are continuous functions on B_1 and B_2, respectively. Then $F_{\mu_2} \circ f \circ F_{\mu_1}^{-1}$ is a measure preserving function on [0,1]. By Theorem 2.1, there is a sequence of measure preserving piecewise linear continuous functions ϕ_n over [0,1] converging to $F_{\mu_2} \circ f \circ F_{\mu_1}^{-1}$ almost everywhere. The sequence of continuous functions $F_{\mu_2}^{-1} \circ \phi_n \circ F_{\mu_1}$ is measure preserving from (B_1,μ_1) to (b_2,μ_2) and converges to the function f almost everywhere, since F_{μ_1} and $F_{\mu_2}^{-1}$ are continuous functions.

Of course, a corresponding corollary to Theorem 2.2 can be formulated for a measure preserving function from (B_1,μ_1) to (B_2,μ_2).

Let $\mathcal{F}_{\mu_1\mu_2}$ be the set of all measure preserving functions from $([0,1],\mu_1)$ to $([0,1],\mu_2)$. There is one further problem one can try to investigate. That is, what are the conditions on μ_1 and μ_2 so that $\mathcal{F}_{\mu_1,\mu_2} \cap \mathcal{F}_{m,m} \neq \phi$, the empty set and the conditions for $\mathcal{F}_{\mu_1,\mu_2} \cap \mathcal{F}_{m,m} = \{x, 1 - x\}$?

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