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ON α-CONVEX FUNCTIONS OF ORDER β WITH M-FOLD SYMMETRY

WANCANG MA

Department of Mathematics
Northwest University
Xian, China

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ABSTRACT. This note is a continuation of the previous work [1,2,3]. First we get a new subordination for α-convex functions of order β when $\alpha=1-2\beta$, which implies the rotation theorem for $(1-2\beta)/m$-convex functions of order β with m-fold symmetry. Then we extend the known results on α-convex functions of order β to the functions with m-fold symmetry. In particular, we give the sharp order of convexity of α-convex functions of order β with m-fold symmetry for $\alpha > 1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].

KEYWORDS AND PHRASES. Subordination, α-convex functions of order β, m-fold symmetry, rotation theorem, order of convexity, distortion theorems.
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1. INTRODUCTION.

Let $J_m(\alpha,\beta)$ be the class of α-convex functions of order β with m-fold symmetry, where $\alpha > 0$, $0 \le \beta < 1$ and $m=1,2,\dots$. That is, it consists of analytic functions $f(z) = z + \sum_{n=1}^{\infty} a_{nm+1} z^{nm+1}$ in the unit disk $D= z:\{|z| < 1\}$ with $f(z)f'(z)/z \neq 0$ and

$$\text{Re}\{(1-\alpha)zf'(z)/f(z)+\alpha(1+zf''(z)/f'(z))\}>\beta. \tag{1.1}$$

In [1], Miller,, Mocanu and Reade studied the class $J(\alpha,0)=J_1(\alpha,0)$. Liu [2] and we [3] discussed the class $J(\alpha,\beta)=J_1(\alpha,\beta)$. Liu got the sharp bounds of $|f(z)|$, $|a_3-\mu a_2^2|$ $(-\infty<\mu<+\infty)$ and $|\arg f'(z)|$ for $\alpha=0,1$. In [3], we obtained a subordination result for $J(\alpha,\beta)$, some distortion theorems, etc.

This note is a continuation of previous work. First we get a new subordination theorem for the class $J(1-2\beta,\beta)$, which implies the rotation theorem for $J_m((1-2\beta)/m,\beta)$. Then we extend known results on $J(\alpha,\beta)$ to the class $J_m(\alpha,\beta)$. In particular, we give the sharp order of convexity of functions in the class $J_m(\alpha,\beta)$ for $\alpha>1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].

2. SUBORDINATION AND DISTORTION PROPERTIES.

At first, we establish a homeomorphic relation betwen $J_m(\alpha,\beta)$ and $J(m\alpha,\beta)$.

LEMMA 1. $f(z)\in J_m(\alpha,\beta)$ if and only if $g(z)\in J(m\alpha,\beta)$, where $g(z)=f(z^{1/m})^m$.

PROOF. If $f(z)\in J_m(\alpha,\beta)$, then $g(z)$ is also analytic in D. It is not difficult to show that $g(z)g'(z)/z\neq 0$ and

$$(1-\alpha)zf'(z)/f(z)+\alpha(1+zf''(z)/f'(z))$$

$$=(1-m\alpha)ug'(u)/g(u)+m\alpha(1+ug''(u)/g'(u))$$

in D, where $u=z^m$. Hence $g(z)\in J(m\alpha,\beta)$. Similarly we can prove

$f(z)=g(z^m)^{1/m}\in J_m(\alpha,\beta)$ if $g(z)\in J(m\alpha,\beta)$. This completes the proof.

It is well known that $G(z)\in J(0,\beta)$ if and only if there is a probability measure $\mu(x)$ on the unit circle $X=\{x:|x|=1\}$ such that

$$G(z)=z\exp\{2(1-\beta)\int_X -\log(1-xz)d\mu(x)\}.$$

This implies, by Lemma 1, that $F(z)\in J_m(0,\beta)$ if and only if there is a probability measure $\mu(x)$ on X such that

$$F(z)=z\exp\{2(1-\beta)m^{-1}\int_X -\log(1-xz^m)d\mu(x)\}. \tag{2.2}$$

Because $g(z)\in J(m\alpha,\beta)$ if and only if there is a $G(z)\in J(0,\beta)$ such that [2]

$$g(z)=\{\alpha^{-1}m^{-1}\int_0^z u^{-1}G(u)^{1/m}\,du\}^{m\alpha},$$

we have for $\alpha>0$ that $f(z)\in J_m(\alpha,\beta)$ if and only if there is a $F(z)\in J_m(0,\beta)$ such that

$$f(z)=\{\alpha^{-1}\int_0^z u^{-1}F(u)^{1/\alpha}du\}^{\alpha}. \tag{2.3}$$

From (2.2) and (2.3), we obtain the following result.

If $f(z)\in J_m(\alpha,\beta)$ and $|z|=r<1$, then

$$e^{-i\pi/m}k_m(\alpha,\beta,re^{i\pi/m})\leq|f(z)|\leq k_m(\alpha,\beta,r), \tag{2.4}$$

where

$$k_m(\alpha,\beta,z)=\begin{cases} z(1-z^m)^{-2(1-\ )/m} & (\alpha=0)\\ \alpha^{-1}\int_0^z u^{-1+1/\alpha}(1-u^m)^{-2(1-\beta)/m\alpha}du & (\alpha>0)\end{cases} \tag{2.5}$$

is the α-convex Koebe function of order β with m-fold symmetry. Specifically we denote $k_1(\alpha,\beta,z)$ by $k(\alpha,\beta,z)$.

In order to state our subordination theorem, we shall make use of the following lemma.

LEMMA 2. Let $\log q(z)$ be a convex univalent function in D and $p_i(z) \prec q(z)$ $(i=1,2,\ldots,n)$. Then for $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i=1$

$$\prod_{i=1}^{n} p_i(z)^{\lambda_i} \prec q(z).$$

PROOF. Since $\log q(z)$ is a convex function and $p_i(z) \prec q(z)$, we have $p_i(z) \neq 0$ and $\log p_i(z) \prec \log q(z)$, which implies

$$\log p_i(D) \subset \log q(D).$$

From the fact that $\log q(D)$ is a convex domain, we get for each $z \in D$

$$\sum_{i=1}^{n} \lambda_i \log p_i(z) \in \log q(D),$$

and then

$$\sum_{i=1}^{n} \lambda_i=1 \log p_i(z) \prec \log q(z),$$

which is equivalent to the desired result.

COROLLARY 1. If $p_i(z) \prec (1-bz)/(1-az)$ $(i=1,2,\ldots,n,\ -1<a,b<1)$, then for $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i=1$ we have

$$\prod_{i=1}^{n} p_i(z)\lambda_i \prec (1-bz)/(1-az).$$

PROOF. For $a=b$, the result is trivial. For $a \neq b$, we know $\log(1-bz) - \log(1-az)$ is a convex function. Hence the required result follows from Lemma 2.

This corollary and some of its applications may be found elsewhere [4].

THEOREM 1. Let $g(z) \in J(1-2\beta,\beta)$ and $0<\lambda<1$, then

$$g'(z)^{\lambda}(g(z)/z)^{1-2\lambda} \prec 1/(1-z). \tag{2.6}$$

In particular, we have

$$g'(z) \prec 1/(1-z)^2, \tag{2.7}$$

$$g(z)/z \prec 1/(1-z). \tag{2.8}$$

PROOF. First we prove (2.8).

If $\beta=\frac{1}{2}$, then (1.1) becomes $\mathrm{Re}\{zg'(z)/g(z)\} > \frac{1}{2}$, which gives

$$zg'(z)/g(z) \prec 1/(1-z).$$

If $\beta < \frac{1}{2}$, we know [3]

$$zg'(z)/g(z) \prec zk'(1-2\beta,\beta,z)/k(1-2\beta,\beta,z)=1/(1-z).$$

In both of these cases, we have

$$zg'(z)/g(z) - 1 \prec z/(1-z).$$

Since $z/(1-z)$ is convex [5],

$$\int_0^z u^{-1}(ug'(u)/g(u)-1)du \prec \int_0^z 1/(1-u)du.$$

That is, $\log g(z) \prec \log 1/(1-z)$, which is equivalent to (2.8).

By using Corollary 1 for $p_1(z)=zg'(z)/g(z)$, $p_2(z)=g(z)/z$, $\lambda_1=\lambda$, and $\lambda_2=1-\lambda$, we obtain (2.6). The proof is completed.

THEOREM 2. Let $f(z) \in J_m((1-2\beta)/m,\beta)$, $0<\lambda<1$ and $|z|=r<1$, then we have the sharp estimates

$$1/(1-r^m) \leq |f'(z)|^{\lambda}|f(z)/z|^{m(1-\lambda)-\lambda} < 1/(1-r^m), \tag{2.9}$$

$$|\lambda \arg f'(z) + (m(1-\lambda)-\lambda)\arg(f(z)/z)| \leq \arcsin r^m \tag{2.10}$$

PROOF. Let $g(z)=f(z^{1/m})^m$, we know $g(z) \in J(1-2\beta,\beta)$ from Lemma 1 and $zf'(z)/f(z)=ug'(u)/g(u)$, where $u=z^m$. Let

$$p(z)=g'(z)^{\lambda}(g(z)/z)^{1-2\lambda}, p_1(z)=f'(z)^{\lambda}(f(z)/z)^{(1-\lambda)m-\lambda},$$

then

$$p_1(z)=(zf'(z)/f(z))^{\lambda}(f(z)/z)^{(1-\lambda)m}=(ug'(u)/g(u))^{\lambda}(g(u)/u)^{1-\lambda}=p(u).$$

From Theorem 1 and the principle of subordination, we have

$p(|u|<R) \subset q(|u|<R)$ for every R $(0<R<1)$, where $q(z)=1/(1-z)$. This implies $p_1(|z|<r) \subset q_1(|z|<r)$ for every r $(0<r<1]$, where $q_1(z)=1/(1-z^m)$, which gives the results. This completes the proof of theorem 2.

The inequality (2.10) contains the following rotation theorem for $J_m((1-2\beta)/m,\beta)$.

COROLLARY 2. If $f(z) \in J_m((1-2\beta)/m,\beta)$ and $|z|=r$, then

$$|\arg f'(z)| < (m+1)\arcsin r^m/m. \tag{2.11}$$

The following subordination is due to Liu [2].

$$g'(z)^{\alpha}(g(z)/z)^{1-\alpha} \prec (1-z)^{-2(1-\beta)} \tag{2.12}$$

whenever $g(z) \in J(\alpha,\beta)$. In [3] we found that if $g(z) \in J(\alpha,\beta)$, then

$$zg'(z)/g(z) \prec zk'(\alpha,\beta,z)/k(\alpha,\beta,z). \tag{2.13}$$

By using a method similar to that used in the proof of theorem 2, we can obtain the following theorems from (2.12) and (2.13). Here we omit most of their proofs. When m=1. most of the following results were given in [2] and [3] respectively.

THEOREM 3. Let $f(z) \in J_m(\alpha,\beta)$, $|z|=r<1$, then we have sharp results

$$r^{1-\alpha}/(1+r^m)^{2(1-\beta)/m} < |f'(z)|^{\alpha} \, |f(z)|^{1-\alpha} < r^{1-\alpha}/(1-r^m)^{2(1-\beta)/m}, \tag{2.14}$$

$$|\arg\{f'(z)^{\alpha}(f(z)/z)^{1-\alpha}\}| \le 2(1-\beta)m^{-1}\arcsin r^m, \tag{2.15}$$

$$\mathrm{Re}\{f'(z)^{\alpha}(f(z)/z)^{1-\alpha}\} > 2^{-2(1-\beta)/m}. \tag{2.16}$$

THEOREM 4. Let $f(z)\in J_m(\alpha,\beta)$, $|z|=r<1$, then we have the sharp inequalities

$$re^{i\pi/m}k_m'(\alpha,\beta,re^{i\pi/m})/k_m(\alpha,\beta,re^{i\pi/m}) \le |zf'(z)/f(z)|$$

$$\le rk_m'(\alpha,\beta,r)/k_m(\alpha,\beta,r), \tag{2.17}$$

$$|\arg\{zf'(z)/f(z)\}| \le \max_{|z|=r} \arg\{zk_m'(\alpha,\beta,z)/k_m(\alpha,\beta,z)\}. \tag{2.18}$$

PROOF. We give an outline of the proof of (2.17). Let

$$p(z)=zf'(z)/f(z), \quad q(z)=zk_m'(\alpha,\beta,z)/k_m(\alpha,\beta,z).$$

We know that $q(z^{1/m})$ is univalent in D [3]. As the proof of theorem 2, we can get

$$p(|z|<r) \subset q(|z|<r) \qquad (0<r<1).$$

Thus for $|z|=r$ we obtain

$$\min_{|z|=r} |q(z)| \le |zf'(z)/f(z)| \le \max_{|z|=r} |q(z)|.$$

We prove $\max_{|z|=r} |q(z)|=q(r)$ and $\min_{|z|=r} |q(z)|=q(re^{i\pi/m})$.

Let $q(z)=1+B_1z^m+B_2z^{2m}+\ldots$, it follows from

$$q(z)+\alpha zq'(z)/q(z)=(1+(1-2\beta)z^m)/(1-z^m)$$

that

$$(1+mn\alpha)B_n=2(1-\beta) + \sum_{k=1}^{n-1} (2-2\beta-B_{n-k})\beta_k.$$

By using the fact that $\mathrm{Re}\,q(z) > \beta$ [3], we have $|B_k| \le 2(1-\beta)$ [6]. Hence we get $B_n>0$ $(n=1,2,\ldots)$ by induction and also $\max_{|z|=r} |q(z)|=q(r)$.

Because the coefficients of q(z) are all real and q(z) is m-fold symmetric, we can obtain $\min_{|z|=r} |q(z)|=q(re^{i\pi/m})$ by proving

$$|q(re^{i\theta})| > q(re^{i\pi/m}) \quad (0<\theta<\pi/m). \tag{2.19}$$

If $\alpha=0$, it is obvious that (2.19) is true for $q(z)=(1+(1-2\beta)z^m)/(1-z^m)$.

If $\alpha>0$, we have

$$q(z)=(z(1-z^m)^{-2(1-\beta)/m})^{1/\alpha}(\alpha^{-1}\int_0^z u^{1/\alpha-1}(1-u^m)^{-2(1-\beta)/m\alpha}du)^{-1},$$

$$\left|\int_0^{re^{i\theta}} u^{1/\alpha-1}(1-u^m)^{-2(1-\beta)/m\alpha}du\right|$$

$$< \int_0^r t^{1/\alpha-1}(1-2t^m\cos m\theta+t^{2m})^{-(1-\beta)/m\alpha}dt,$$

which implies that

$$|q(re^{i\theta})| > \frac{\alpha(r(1-2r^m\cos m\theta+r^{2m})^{-(1-\beta)/m})^{1/\alpha}}{\int_0^r t^{1/\alpha-1}(1-2t^m\cos m\theta+t^{2m})^{-(1-\beta)/m\alpha}dt}.$$

$$q(re^{i\pi/m})=\alpha(r(1+r^m)^{-2(1-\beta)/m})^{1/\alpha}\Big/\int_0^r t^{1/\alpha-1}(1+t^m)^{-2(1-\beta)/m\alpha}dt.$$

Let $I(\theta)=$

$$(1-2r^m\cos m\theta+r^{2m})^{-(1-\beta)/m\alpha}\int_0^r t^{1/\alpha-1}(1-2t^m\cos m\theta+t^{2m})^{-(1-\beta)/m\alpha}dt.$$

We can verify $I'(\theta) > 0$ $(0<\theta<\pi/m)$, which implies the desired result. The proof of (2.17) is now complete.

From (2.4) and (2.17), we get the following distortion result.

COROLLARY 3. If $f(z)\in J_m(\alpha,\beta)$, $|z|=r<1$, then

$$k_m'(\alpha,\beta,re^{i\pi/m}) < |f'(z)|<k_m'(\alpha,\beta,r).$$

From (2.13), we can also obtain the sharp order of starlikeness for functions in $J_m(\alpha,\beta)$.

THEOREM 5. Let $f(z)\in J_m(\alpha,\beta)$. Then $f(z)\in J_m(0,s_m(\alpha,\beta))$, that is, f(z) is starlike of order $s_m(\alpha,\beta)$, where

$$s_m(\alpha,\beta) = \min_{0<\theta<2\pi/m} \mathrm{Re}\{e^{i\theta}k_m'(\alpha,\beta,e^{i\theta})/k_m(\alpha,\beta,e^{i\theta})\} > \beta.$$

Miller, Mocanu and Reade [1] proved that f(z) is a convex function if $f(z)\in J(\alpha,0)$ and $\alpha > 1$. By making use of theorem 5, we get the following sharp order of convexity, which is analogous in sharpness to a result in [1].

COROLLARY 4. If $f(z)\varepsilon J_m(\alpha,\beta)$ and $\alpha>1$, then

$f(z)\varepsilon J_m(1,\beta/\alpha+(1-1/\alpha)s_m(\alpha,\beta))$, that is, $f(z)$ is convex of order

$\beta/\alpha+(1-1/\alpha)s_m(\alpha,\beta)$ $(>\beta)$.

By using the method we used in [3], we can easily get the following covering theorem from (2.4).

THEOREM 6. Let $w=f(z)\varepsilon J_m(\alpha,\beta)$. Then we have the sharp result $f(D)\supset\{w:|w|<d_m(\alpha,\beta)\}$, where

$$d_m(\alpha,\beta)=\begin{cases}2^{-2(1-\beta)/m} & (\alpha=0)\\ F(1/m\alpha,2(1-\beta)/m\alpha,1+1/m\alpha;-1) & (\alpha>0)\end{cases}$$

and F is the hypergeometric functtion.

Finally, we note a coefficient inequality, which can be deduced from (2.1) and a similar result on $J(\alpha,\beta)$ given in [2].

THEOREM 7. Let $f(z)=z+a_{m+1}z^{m+1}+a_{2m+1}z^{2m+1}+\ldots\varepsilon\ J_m(\alpha,\beta)$, then we have the sharp inequalities

$$|a_{2m+1}-\lambda a_{m+1}^2|\le$$

$$\begin{cases}\dfrac{(1-\beta)^2}{m(1+2m\alpha)}\left\{\dfrac{2m+6m^2\alpha+m^3\alpha^2-(2\lambda-1)(1+2m\alpha)}{m(1+m\alpha)^2}+\dfrac{\beta}{1-\beta}\right\} & -\infty<\lambda\le a;\\ (1-\beta)/(m(1+2m\alpha)) & a\le\lambda\le b;\\ \dfrac{(1-\beta)^2}{m(1+2m\alpha)}\{(m-1+2\lambda)(1+2m\alpha)/(m(1+m\alpha)^3)-\beta/(1-\beta)\} & b\le\lambda<+\infty,\end{cases}$$

where

$$a=\frac{1}{2}+\frac{1}{2}m^2\alpha/(1+2m\alpha),\quad b=\frac{1}{2}+\frac{1}{2}m^2\alpha/(1+2m\alpha)+\frac{1}{2}m(1+m\alpha)^2/((1+2m\alpha)(1-\beta)).$$

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