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AN INEQUALITY OF W.L. WANG AND P.F. WANG

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ABSTRACT. In this note we present a proof of the inequality $H_n/H'_n \le G_n/G'_n$ where H_n and G_n (resp. H'_n and G'_n) denote the weighted harmonic and geometric means of $x_1,\ldots,x_n$ (resp. $1-x_1,\ldots,1-x_n$) with $x_i \in (0, 1/2]$, $i = 1,\ldots,n$.

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1. INTRODUCTION.

Let $p_1,\ldots,p_n$ and $x_1,\ldots,x_n$ be two sequences of positive real numbers with $\sum_{i=1}^{n} p_i = 1$ and $x_i \in (0, 1/2]$, $i=1,\ldots,n$. In what follows we denote by A_n, G_n and H_n (resp. A'_n, G'_n and H'_n), the weighted arithmetic, geometric and harmonic means of $x_1,\ldots,x_n$ (resp. $1-x_1,\ldots,1-x_n$), i.e.

$$A_n = \sum_{i=1}^{n} p_i x_i,\quad G_n = \prod_{i=1}^{n} x_i^{p_i} \quad \text{and } H_n = 1/\sum_{i=1}^{n} p_i/x_i, \tag{1.1}$$

resp. $A'_n = \sum_{i=1}^{n} p_i(1-x_i)$, $G'_n = \prod_{i=1}^{n} (1-x_i)^{p_i}$ and

$$H'_n = 1/\sum_{i=1}^{n} p_i/(1-x_i). \tag{1.2}$$

Setting $p_1=\ldots= p_n=1/n$ in (1.1) and (1.2) we obtain the unweighted arithmetic, geometric and harmonic means of $x_1,\ldots,x_n$ (resp. $1-x_1,\ldots,1-x_n$), designated by a_n, g_n and h_n (resp. a'_n, g'_n and h'_n).

In 1961 E.F. Beckenbach and R. Bellman [1] published a remarkable counterpart of the classical arithmetic mean-geometric mean inequality which is due to Ky Fan, namely

$$g_n/ g'_n \le a_n/a'_n \tag{1.3}$$

with equality holding in (1.3) if and only if $x_1=\ldots= x_n$. Since then Fan's inequality has been subjected to considerable investigations resulting in many proofs, sharpenings and refinements (see Alzer [2] and the references therein). It is natural to ask whether there exists a corresponding inequality for geometric and harmonic means. In 1984 W.L. Wang and P.F. Wang [3] have answered this question. They established the inequality

$$h_n/h'_n < g_n/g'_n \tag{1.4}$$

where the sign of equality is valid if and only if $x_1 = \ldots = x_n$. It is worth mentioning that not only (1.3) but also (1.4) has been originally proved by using Cauchy's method of forward and backward induction.

In the last year , different authors have verified that Fan's inequality holds for weighted mean values, i.e.

$$G_n / G'_n < A_n/A'_n \tag{1.5}$$

with equality if and only if $x_1 = \ldots = x_n$ as in Flanders [4], Levinson [5] and Wang [6-8]. The aim of this note is to show that inequality (1.4) can also be extended to weighted means.

2. AN INEQUALITY FOR WEIGHTED GEOMETRIC AND HARMONIC MEANS.

We establish the following counterpart of (1.5):

THEOREM 2.1. If $x_i \in (0, 1/2]$, $i=1,\ldots,n$, then

$$H_n/H'_n < G_n/G'_n \tag{2.1}$$

with equality holding in (2.1) if and only if $x_1 = \ldots = x_n$.

PROOF. If we set

$$z_i = x_i/(1-x_i), \quad 0 < z_i < 1, \ i = 1,\ldots,n,$$

then (2.1) can be rewritten as

$$\frac{\sum_{i=1}^{n} p_i(1+z_i)}{\sum_{i=1}^{n} p_i(1+1/z_i)} < \prod_{i=1}^{n} z_i^{p_i} . \tag{2.2}$$

Since equality holds in (2.2) if $z_1 = \ldots = z_n$, it remains to show that (2.2) is strict if the numbers $z_1,\ldots,z_n$ are not all equal. We use induction on n. Let $n = 2$; then we have to prove that the function

$$f(z_2) = (p_1 z_1^{p_1-1} + z_1^{p_1}) z_2^{p_2} + p_2 z_1^{p_1} z_2^{p_2-1} - p_2 z_2 - p_1 z_1 - 1$$

is positive for $0 < z_1 < z_2 < 1$.

A simple calculation yields

$$f''(z_2) = p_1 p_2 z_1^{-p_2} z_2^{p_2-3} (p_1+z_1)(z_0-z_2) \text{ with}$$

$$z_0 = (2-p_2)z_1/(p_1+z_1) \in (z_1,1),$$

so

$$f''(z_2) > 0 \quad \text{for} \quad z_1 < z_2 < z_0$$

and

$$f''(z_2) < 0 \quad \text{for} \quad z_0 < z_2 < 1.$$

Since $f(z_1) = f'(z_1) = 0$ and $f'(1) > 0$ we obtain

$$f(z_2) > 0 \quad \text{for } z_1 < z_2 \leq 1.$$

Next we assume that (2.2) is true for $n \geq 2$. Let us put $z = z_{n+1}$ and $p = p_{n+1}$. Without loss of generality we set

$$0 < z_1 \leq \ldots \leq z_n \leq z \leq 1, \; z_1 < z. \tag{2.3}$$

Since

$$\frac{1}{1-p} \sum_{i=1}^{n} p_i = 1$$

we get from the induction hypothesis

$$z^p \prod_{i=1}^{n} z_i^{p_i} \geq z^p \left[\frac{\sum_{i=1}^{n} p_i(1+z_i)}{\sum_{i=1}^{n} p_i(1+1/z_i)} \right]^{1-p}$$

and it remains to prove

$$z^p \left[\frac{\sum_{i=1}^{n} p_i(1+z_i)}{\sum_{i=1}^{n} p_i(1+1/z_i)} \right]^{1-p} > \frac{\sum_{i=1}^{n} p_i(1+z_i) + p(1+z)}{\sum_{i=1}^{n} p_i(1+1/z_i) + p(1+1/z)}. \tag{2.4}$$

We set

$$a = \sum_{i=1}^{n} p_i(1+z_i) \quad \text{and} \quad b = \sum_{i=1}^{n} p_i(1+1/z_i).$$

Then (2.4) can be written as

$$z^p (a/b)^{1-p} > \frac{a + p(1+z)}{b + p(1+1/z)}$$

and this is equivalent to

$$g(a,b,z) = p\ln(z) + (1-p)\ln(a) - (1-p)\ln(b) - \ln(a+p(1+z)) + \ln(b+p(1+1/z)) > 0.$$

Partial differentiation reveals

$$\frac{\partial}{\partial a} g(a,b,z) = \frac{p}{a} [(1-p)(1+z) - a] / [a+p(1+z)]$$

and

$$\frac{\partial}{\partial b} g(a,b,z) = \frac{p}{b} [(p-1)(1+1/z)+b] / [b+p(1+1/z)].$$

From (2.3) we conclude

$$a < (1-p)(1+z) \quad \text{and} \quad b > (1-p)(1+1/z)$$

hence we obtain

$$\frac{\partial}{\partial a} g(a,b,z) > 0 \quad \text{and} \quad \frac{\partial}{\partial b} g(a,b,z) > 0 .$$

Since $1 - p < a \leq b$ we get

$$g(a,b,z) > g(1-p,1-p,z).$$

We define

$$h(p) = g(1-p,1-p,z)$$

then we get

$$h''(p) = (z/(1+pz))^2 - (1/(p+z))^2 \leq 0$$

and because of

$$h(0) = h(1) = 0$$

we have

$$h(p) \geq 0 \text{ for } 0 < p < 1,$$

which completes the proof of inequality (2.4).

REMARK 2.1. We notice that the method used to establish inequality (2.2) for $n = 2$, can also be used to prove (2.4). And the technique applied to establish inequality (2.4) can be used to prove (2.2) for $n = 2$ as well.

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